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NEUTRON PHYSICS

A Course by E. Fermi

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Notes written by
I. Halpern

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A course in neutron physics was given by E. Fermi in the Fall of 1945 as part of the program of the Los Alamos University. The course consisted of some thirty lectures, almost all of which were given by Mr. Fermi. In his absence, R. F. Christy and E. Segre gave several lectures.

These notes are, except for a few very minor rearrangements and insertions, those taken down in class. The division into chapters and sections was made only when the notes were put together for printing, but the chronological order of presentation was everywhere preserved. The homework assignments are given in their proper place.

Mr. B. T. Feld kindly read these notes before they were printed and made some corrections and valuable suggestions. Miss Arva Frazier typed up the notes.

As soon as declassification permits, the University intends to make these notes available for sale, along with the other non-classified notes published by the University.

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NEUTRON SOURCES

(1) (α, n) Reactions

(A) $Li^7(\alpha,n)B^{10}$ will mean $Li^7 + He^4 \longrightarrow B^{10} + n^1 + Q$, where Q is the energy balance. As an example, let us calculate Q.

The masses of the atoms involved in the reaction are available from mass-spectrographic data:

	7.01804	atomic mass units		10.01605
	4.00388			1.00893
left side	11.02192		rt. side	11.02498

As an assignment (and in the future, all remarks enclosed in rectangles are assignments) explain why you can use atomic masses for such calculations, when it is true that nuclear masses alone are involved in the Reaction.

Note that the mass on the right exceeds that on the left by .00306 mass units (the Reaction is Endothermic, i.e. Q is negative)

Now one mass unit is 931 Mev so that

Q = - 2.85 Mev.

Prove that one mass unit equals 931 Mev, using $E = mc^2$. Convert from mass units to gms to Ergs to Mev.

The threshold for the Reaction is, however, somewhat larger than 2.85 Mev. For 2.85 Mev would suffice only if neither end-product particle moved. In order to satisfy the conservation of momentum this cannot be so, for the α particle is moving at a stationary Li^7 to begin with, and so the center of gravity has a momentum which it will still have after the reaction.

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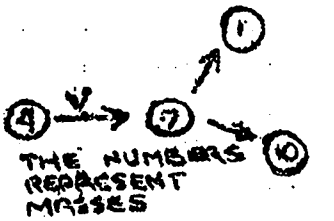
Under this assumption of stationary Li^7, let us calculate the threshold energy for the Reaction.



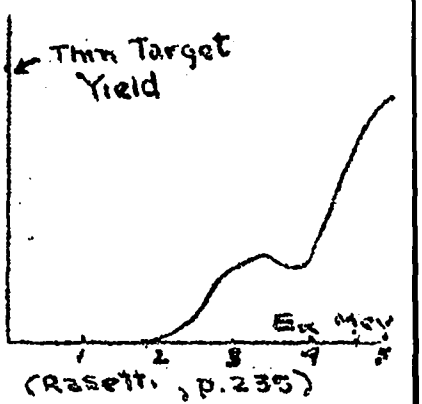
V_c, the velocity of the Center of Gravity is $\frac{4V}{7+4} = \frac{4}{11}V$. The minimum kinetic energy of the Reaction products is then $\frac{1}{2} \times 11 \times (\frac{4}{11}V)^2$

Therefore, for the reaction to go, the minimum velocity V of the α particle must be given by

$$\frac{1}{2} \times 4 \times V^2 = 2.85 \text{ Mev} = \frac{1}{2} \times 11 \times (\frac{4}{11}V)^2$$

or $\frac{7}{11}(\frac{1}{2} \times 4 \times V^2) = 2.85$ Mev (i.e. only $\frac{7}{11}$ of the kinetic energy of the α particle is available for conversion to mass) Hence the threshold is $2.85 \times \frac{11}{7} = 4.5$ Mev

(B) $Be^9(\alpha, n)C^{12}$ is the most probable reaction with Be^9, although $Be^9(\alpha, n)\,3He^4$ also occurs. This is exothermic, ($Q = 5.5$ Mev) and is the basic reaction in one of the most common neutron sources (radium -- beryllium). There is no sharp threshold in this Reaction, the yield curve rising slowly with E_α (because of the Gamow formula for penetration of a Coulomb barrier by charged particles.)



(Rasetti, p.235)

Three natural neutron sources in common use are based on this Reaction

Ra Be (~5 gms Be to 1 gm Ra)
Rn Be (Rn gas in a Be Capule; very light; ~4 day half life)
} These sources emit too many γ's for some uses.
Po Be (relatively low intensity)

To see what α's are emitted in each case, one must turn to the Decay Series of the Uranium family.

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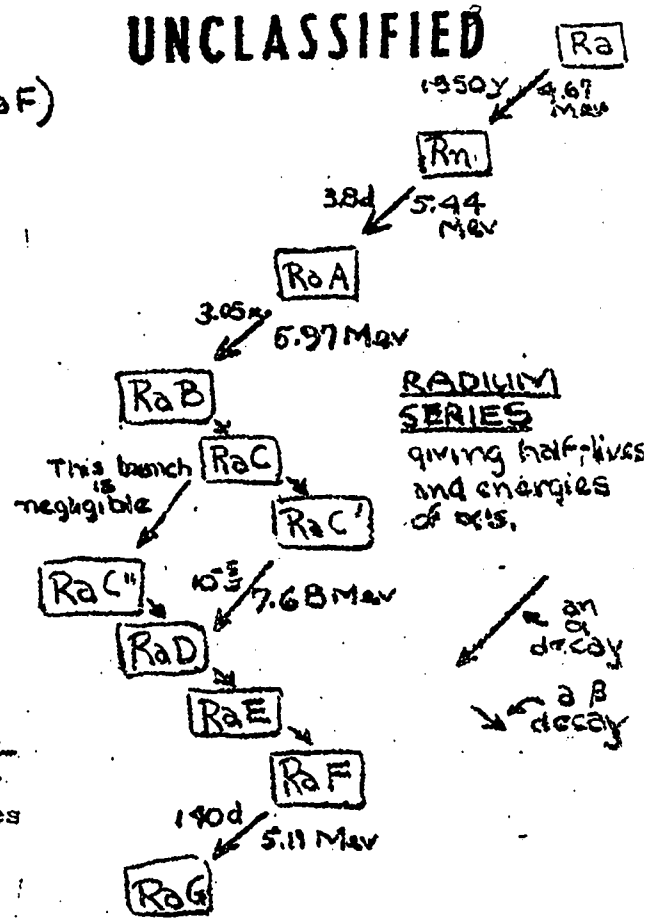
(At least that part from Ra to Po = RaF)

Normal Radium is a mixture of Ra and the products emitting the 4 α's to Ra D. Rn emits 3 α's. Po emits but 1 α and is quite a bit weaker.



The strengths of these natural (α,n) sources are :

$$\left.\begin{matrix}\text{Ra Be}\\ \text{Rn Be}\end{matrix}\right\} 10 \text{ to } 15\times10^{6}\ \frac{\text{neutrons}}{\text{sec. curie}}$$

$$\text{Po Be}\quad 2.8\times10^{6}\ \frac{\text{neutrons}}{\text{sec. curie}}$$

where a curie of a substance disintegrates at the same rate as 1 gm of Radium.

In a Ra - Be source the neuts will have all sorts of energies up to $7.68 + 5.5 \cong 13$ Mev. The spectrum will be fairly continuous for in addition to the various α's involved, there's the matter of penetrating Be crystals, and the fact that C^{12} needn't necessarily be left in the ground state.

(3) $B(\alpha,n)N$ gives $\sim 2\times10^{6}\ \frac{\text{neutrons}}{\text{sec. curie}}$ with Ra α's.

(4) $F(\alpha,n)Na^{22}$

Sometimes you can use Ra α's in BF_3. You'd get about $10^{5}\ \frac{\text{neut.}}{\text{sec. curie}}$

Po α's in BF_3 gives the "Mock Fission Spectrum".

(2) Photo Sources (γ,n)

Be and D alone among the nuclei have low enough (γ,n) thresholds (1.61 Mev & 2.17 Mev respectively) to be useful with natural γ's.

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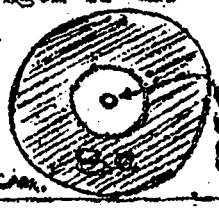
We can get with such sources low energy monochromatic neutrons. Ra γ's on Be, however, give neuts in 2 groups (.12 Mev, .51 Mev) since 2 Ra γ's lie above the threshold.

A practical rule for the calculation of the total number of neuts per second from Ra γ Be is: 1 gm. Ra at 1 cm. from 1 gm. Be gives 3×10^{-4} $\frac{\text{neut.}}{\text{sec.}}$

Neglecting absorption of γ's, what's the strength of the source shown?

sphere of Be
inner radius 1 cm.
outer radius 3 cm.



Other photo sources.

Ms γ D_2O .220 Mev neutrons

Ms γ Be 2 groups .88, .16 Mev neutrons

Using artificially radioactive γ emitters,

Reaction	SbγBe	GaγD_2O	NaγD_2O	LaγBe	NaγBe
Neutron Energies	.030 Mev	.12	.24	.62	.80

(3) Artificial Neutron Sources.

(1) The D-D Reaction: (Bombard heavy ice or heavy paraffin with Deuts.) Based on:

$$D + D \rightarrow He^3 + n$$
$$D + D \rightarrow H^3 + H^1$$

about half and half

exothermic ~3.2 Mev

How many n/sec would one get from a beam of 500 kv Deuts if the current were 1 ma?

Yield for a thick target of Heavy Ice

Energy (Kev)	Yield ($\frac{\text{neut}}{\text{deut}}$)
0	0
50	$.2 \times 10^{-7}$
100	.68
200	3.0
300	6.9
500	19
1000	81
2000	400

Advantages of this Reaction: Reasonable yields at low energies; monochromatic yields.

Disadvantage: Troublesome target

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(B) Protons on Lithium: This Reaction

$$Li^7 + H^1 \rightarrow Be^7 + n - 1.62 \text{ Mev}$$

can give monochromatic neuts down to rather low energies. The threshold is $\frac{8}{7} \times 1.62 = 1.85$ Mev

If the target is bombarded with threshold protons, the neuts come off with a finite energy (~30Kv) for then they move with the speed of the center of gravity. If we raise the proton energy, there is enough energy to give the neuts a velocity <u>with respect to</u> the c. of g. For the net velocity, one adds vectorially the velocity of the center of gravity to the neutron velocity with respect to the center of gravity. For high enough energies, neuts can show resultants of zero velocity, or even backward velocities.

(3) Deuterons on Beryllium: The Cyclotron Source. Yield rises rapidly with energy. For a thick target, 1 Mev deuterons give 10^8 neutrons per second per microamp. 8 Mev neutrons give 10^{10} neutrons per sec. per μamp.

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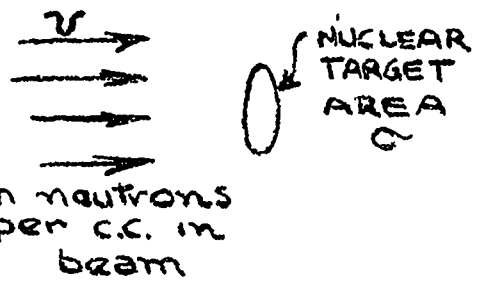
CHAPTER II

Collisions of Neutrons with Nuclei

Examples of such processes are (n,n) which are called scattering processes; (elastic when K.E. of neut before collision = sum of K.E.'s of neutron and recoil nucleus after collision; inelastic when the kinetic energy is not conserved -- some energy may go into nuclear excitation). Reactions where a neutron comes in and other "particles" come out are (n,γ) (n,p) (n,α). A third distinct type is the $(n,2n)$ reaction. A fourth type would be $(n,$ fission$)$

(1) <u>Neutron Cross-Sections as a Function of Energy.</u>

Description of intensities of reactions is usually given in terms of cross-sections. In unit time $nv\sigma$ neutrons will hit a nucleus. Actually the representation is rather fictitious inasmuch as σ is not a well-defined geometrical quantity. We might make more general our notion of σ using semi-classical ideas. We would consider σ as describing the projected area of the nucleus and of a zone including the range of nuclear forces. But even this area would be of the order $3\times10^{-24}\,cm^2$ We would conclude that all σ's must be $\sim 3\times10^{-24}\,cm^2$ But there are observed σ's of the order $10^6\times10^{-24}\,cm^2$ We must adopt a wave mechanical point of view to account for this. A neutron becomes a large body at slow velocity. It can't be considered a point compared to the nucleus but may be much larger. Its linear dimension would be given by $\lambda = \frac{h}{mv}$. At low velocities σ would be restricted (in orders of magnitude) only by $\sigma < \lambda^2$ Let us see therefore how λ depends on E(neutron energy in electron volts)



$$\tfrac{1}{2}mv^2 = E \times 1.6\times10^{-12}\,\text{ergs}$$

$$\therefore\ mv = \sqrt{3.2\times10^{-12}\,mE} \quad \text{and } m = 1.6\times10^{-24}$$

whence $\lambda = \frac{h}{mv} = \frac{2.8\times10^{-9}}{E^{1/2}}$. λ is called the De Broglie wave length. In practical limits E goes from about .001 volts to several million volts. Thermal

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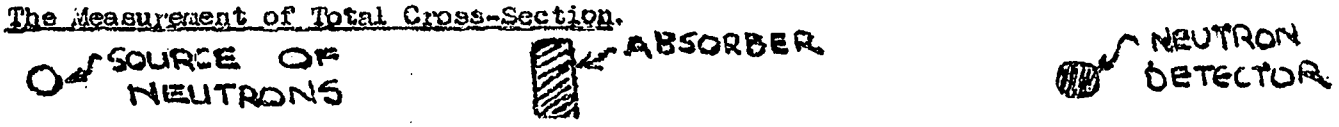
neuts, for example, are about $\frac{1}{40}$ eV which makes $\lambda = 2\times10^{-8}$ cm. (of the order of interatomic distances.) For 1 volt neuts, $\lambda = 3\times10^{-9}$; for 10^6 volts, $\lambda = 2\times10^{-12}$ (of the order of nuclear dimensions). At a high enough energy λ becomes small compared to 10^{-12}. Then we can use the classical corpuscular notion, and for higher energies than this we must use $\sigma < 10^{-24}$ rather than $\sigma < \lambda^2$ to set an upper limit on σ. In keeping with these ideas one could define:

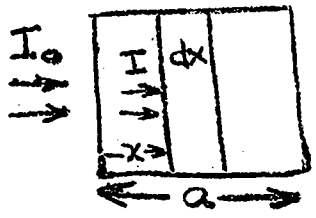
$$\sigma \equiv \frac{\text{PROCESSES PER NUCLEUS PER UNIT TIME}}{n v}$$

and we've seen σ will be less than the largest of the two quantities 3×10^{-24}, λ^2

(2) The Measurement of Total Cross-Section.



Without the absorber, the detector measures intensity I_0. We then put in the absorber. The measured intensity is I. We assume that the solid angle of absorber at source and detector is infinitesimally small (one must make sure that neuts scattered even a little bit can't get to the detector.) Our geometry is thus such that any neutron "effected" will not be detected. Take a closer look at the absorber



N IS THE NUMBER OF ABSORBER ATOMS /C.C. THE PROBABILITY THAT A NEUTRON WILL SUFFER A COLLISION IN THE LAYER dx IS $\sigma_t N dx$, WHERE $\sigma_t = \sigma_{total}$, THE CROSS-SECTION FOR ALL NEUTRON PROCESSES.

$dI = -\sigma_t N dx \cdot I$ whence $I =$ constant $\times\ e^{-\sigma_t N x}$ and the constant must be I_0. If the length of the slab is "a", $I = I_0 e^{-\sigma_t N a}$ and σ_t can be calculated from the measurements of I and I_0

$$\sigma_t = \frac{1}{Na} \log \frac{I_0}{I}$$

σ_t may represent the sum,

$\sigma_{elastic} + \sigma_{inelastic} + \sigma_{absorption}$

where $\sigma_{absorption}$ includes all processes in which a neutron disappears, i.e., (n,α) (n,p) (n,γ) etc.

A copper plate 1 cm thick reduces the intensity of thermal neuts by the factor .36. What is the σ_t for copper for thermal neuts? Express result in units of 10^{-24} cm² (Barns).

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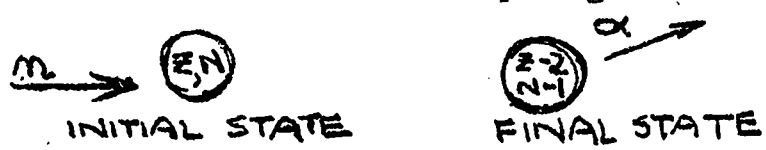
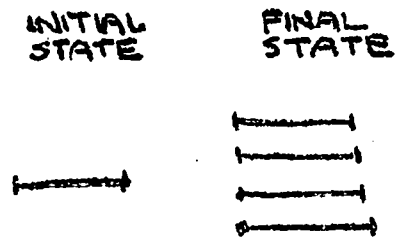
(3) <u>The General Features of Collisions</u>.

As an example, you may be thinking of an (n,α) reaction altho what follows is quite general.





FOR A GIVEN ENERGY OF THE INITIAL STATE, THERE ARE A NUMBER OF POSSIBLE FINAL α ENERGIES.

Final state refers to either the α or the recoil, for if we are given the α momentum and direction, the laws of momentum conservation assign the momentum and direction to the recoil, and from the conservation of energy law, one can determine now the state of excitation of the recoil. Thus a specification of momentum and direction of the α specifies the final state completely.

What is the probability that a transition will occur for which the energy of the α is E? (i.e. between E and E + dE) From Quantum Mechanics, it can be shown that this is given by a product. It is $M \times \rho$ where M is $\frac{4\pi^2}{h} \times |H|^2$ and H is a matrix element depending on the energies of the initial and final states, and $\rho(E)$ is the density of possible final states in the neighborhood of E.

The matrix element is too tough to calculate here, but we can get ρ. Actually the α can come off with any energy and so we'd get a continuum (i.e. $\rho = \infty$) everywhere. However we can put the system in a box of finite volume Ω and this will make the distribution of final energies discrete and ρ (the number of states per unit energy interval) will be finite. The question "How many states per unit energy occur at E?" becomes "How many states of an α particle in a box of volume Ω have energy E?" For the answer to this we turn to statistical mechanics and say, it is proportional to the volume in phase space which corresponds to this energy. If h is the linear dimension of a cell in phase space (a cell can contain one state) then the number of states for which p_x is between p_x and $p_x + dp_x$ etc is

$\frac{\Omega}{h^3} dp_x\, dp_y\, dp_z$ (where the integration over configuration space i.e. over dx, dy, dz has already been done) Hence the number of states

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for which the total momentum is between p and $p+dp$ is proportional to the volume of the spherical shell at $p=(p_x^2+p_y^2+p_z^2)^{1/2}$ in momentum space. This volume is $4\pi p^2 dp$

$$\delta N = 4\pi p^2 dp \times \frac{\Omega}{h^3}$$

REPRESENTS THE NUMBER OF STATES WITH MOMENTUM p TO $p+dp$

FOR PARTICLES, (i.e. α, p, n) $E=\frac{1}{2}Mv^2=\frac{p^2}{2M}$ SO THAT $dE = p/M\,dp$

FOR PHOTONS $E=h\nu$ $p=h\nu/c$ SO THAT $dE=c\,dp$

In general, therefore $dE=v\,dp$ where v is the velocity of the particle or photon. Thus the number of states per unit energy range, $\rho=\frac{\delta N}{\delta E}$ becomes

$$\frac{4\pi\Omega}{h^3}\frac{p^2}{v} \text{ in general or } \frac{4\pi\Omega M p}{h^3} \text{ for PARTICLES, } \frac{4\pi\Omega}{hc^3}\nu^2 \text{ for PHOTONS}$$

Although Ω occurs in ρ, Ω^{-1} will occur in M so that the transition probability is independent of Ω as it must be if our taking of an arbitrary Ω is to be justified. Thus the probability of a transition to a state where the particle or photon has momentum $P_{OUT} \cong M \times \frac{P_{OUT}^2}{v_{OUT}}$. This probability will be proportional to $v_{INCIDENT} \times \sigma$ where σ is the cross-section for neuts of velocity $v_{incident}$ to be "converted" into outgoing particles of velocity v_{OUT}.

$$\sigma = M \times \frac{P_{OUT}^2}{v_{OUT}} \times \frac{1}{v_{IN}}$$

(4) <u>Applying This Result to Several Situations</u>.

As an example note that for elastic scattering ($v_{IN}=v_{OUT}$) we get simply $\sigma = M \times m^2$ (for $p=mv$)

so that for slow neuts (M wouldn't change much in so small an energy range) σ is independent of the velocity of the slow neutrons. The elastic scattering cross-section of slow neutrons does not depend appreciably on their energy. It should be pointed out that for elastic scattering from light nuclei $v_{IN}=v_{OUT}$, only in the center of gravity system and it is in this system that the above is true.

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As a second example, consider one of the so-called absorption processes. Let's say the recoil is heavy so that the light particle gets all the energy. Then

$$\frac{1}{2} m_1 v_{OUT}^2 = \frac{1}{2} m v_{in}^2 + Q$$

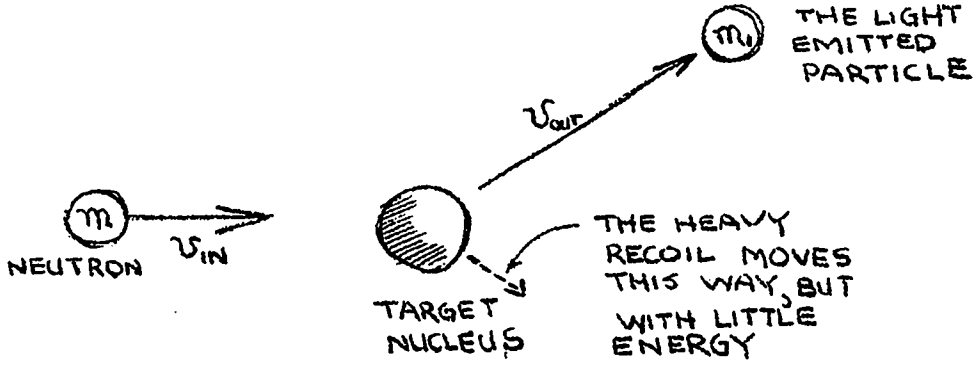
$$\text{or } p_{OUT}^2 = m_1^2 \left[\frac{m}{m_1} v_{in}^2 + \frac{2}{m_1} Q \right]$$

where m = neutron mass; m_1 = light particle mass

Then $$\sigma = M \times \frac{m_1^2}{v_{in}} \left[\frac{m}{m_1} v_m^2 + \frac{2}{m_1} Q \right]^{1/2}$$

Note that when Q is positive (exothermic) and v_{IN} is small, then σ goes as $\frac{1}{v_{IN}}$ [the $\frac{1}{v}$ law]. Obviously a negative Q and very small v_{IN} doesn't make physical sense. The result we get is imaginary. This shows mainly that the formula doesn't cover the case. σ should be zero for such a condition.

Find an example of each of the processes (n,n) [elastic and inelastic] $(n,\gamma)(n,\alpha)$,..... etc. and record the cross-section. (From the literature.)



AN "ABSORPTION" PROCESS

(A NEUTRON IS ABSORBED BY THE TARGET AND ANOTHER TYPE OF PARTICLE IS EMITTED.)

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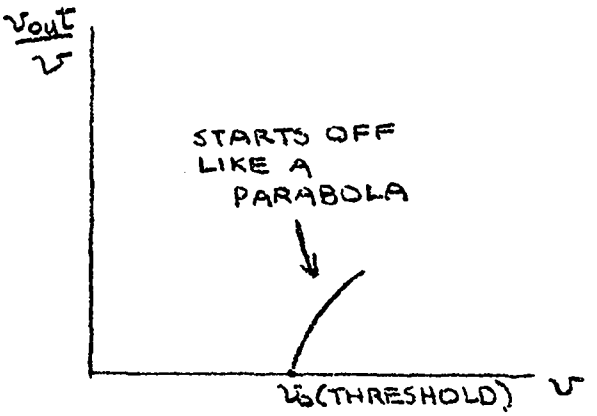
Consider next inelastic scattering (Assume that you're working in the center of gravity system. In this case the neutron mass should be replaced by its reduced mass $\frac{M M_A}{M+M_A} = \mu$. If M_A is large, the reduced mass is practically the neutron mass and the following results will hold approximately in the lab system.) If the first excitation level of the nucleus is W, and the neutron has $\frac{1}{2}\mu v_{in}^2 < W$ then no inelastic scattering can take place. (μ mind you, is the reduced mass). But if $\frac{1}{2}\mu v_{in}^2 > W$ then, if the nucleus is excited to this level, W

$$\frac{1}{2}\mu v_{in}^2 - W = \frac{1}{2}\mu v_{out}^2$$

substituting in the cross-section formula

$$\sigma = M \times \frac{1}{v_{in}} \frac{(\mu v_{out})^2}{v_{out}} = M \mu^2 \frac{v_{out}}{v_{in}}$$

At the right is a plot of the factor $\frac{v_{out}}{v_{in}}$



writing $W \equiv \frac{1}{2}\mu v_0^2$, $v_{out} = (v^2 - v_0^2)^{1/2}$

which is $\left[(v+v_0)(v-v_0)\right]^{1/2}$

showing that at the beginning v_{out} goes as the square root of the excess velocity. There would be a small cross-section for the production of slow neuts by this method since $\sigma \sim v_{out}$.

Next, we take up (n,γ). Here $p = \frac{h\nu}{c}$, $v_{out} = c$ so that $\sigma = M \times \frac{1}{v_{in}} \frac{h^2\nu^2}{c^3}$ Let's say a neutron hits a nucleus (Z,N) so that the compound nucleus is $(Z,N+1)$ Neutron capture is almost always (exception = helium) exothermic. The energy available for photon emission is $\frac{1}{2}\mu v^2 + W$ where W is the neutron binding energy (the so-called energy balance for this case.) If the first photon corresponds to a drop to the level L (not necessarily ground) we may write $h\nu = \frac{1}{2}\mu v^2 + W_L$ As $v \to 0$, $\frac{h^2\nu^2}{c^3}$ approaches a fixed number [depending on W_L]. But in the expression for σ the $\frac{1}{v}$ part increases rapidly. Slow neutron (n,γ) reactions go as $\frac{1}{v}$ (assuming again sufficient constancy of M).

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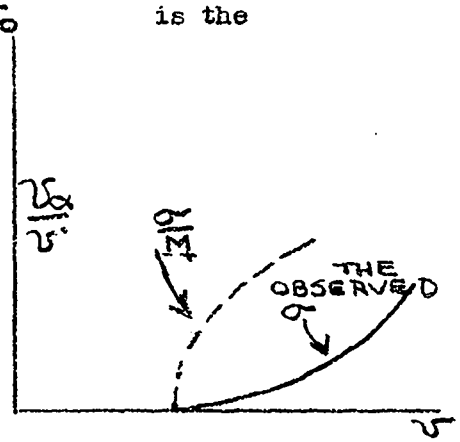
Next consider the (n,α) processes. [The (n,p) processes are of the same type.] $n+A \rightarrow B+\alpha$ These reactions can be endothermic or exothermic (Q negative or positive). The energy equation is

$$\frac{1}{2}\mu v_n^2 = \frac{1}{2}\mu_\alpha v_\alpha^2 - Q \quad \text{(in center of gravity system)}$$

First take $Q > 0$; here v_α is at least $\left(\frac{2Q}{\mu_\alpha}\right)^{1/2}$ Here

$$\sigma = \Gamma 1 \times \frac{1}{v_n} \frac{(\mu_\alpha v_\alpha)^2}{v_\alpha} = \Gamma 1 \frac{\mu_\alpha^2 v_\alpha}{v_n}$$

For low energies, with $Q > 0$ the (n,α) process obeys the $1/v$ law (reasoning similar to that in the (n,γ) situation above.) But for $Q < 0$, things are different. There's a threshold. Write $Q \equiv -\frac{1}{2}\mu v_0^2$ so that $-\frac{1}{2}\mu v_0^2 + \frac{1}{2}\mu v^2 = \frac{1}{2}\mu_\alpha v_\alpha^2$ i.e. $v^2 - v_0^2 = \frac{\mu_\alpha}{\mu} v_\alpha^2$ showing clearly that $v = v_0$ is the threshold velocity. Again we'd expect a parabolic rise in the value $\frac{v_\alpha}{v}$. But the observed σ is quite different as can be seen. The reason is the sharp variation of $\Gamma 1$ in this case. Slow charged particles have very much trouble getting out of Gamow barriers.



(5) <u>Some Examples of the Foregoing Reaction Types.</u>

$B^{10} + n \longrightarrow Li^7 + He^4 + 2.99$ Mev (Normally Li^7 is left excited so that only about 2.5 Mev are available for kinetic energy.) In view of the discussion on page 10 this reaction should go as $\frac{1}{v}$ for slow neuts. It does. In unseparated boron (B^{10} & B^{11} mixed) σ is 705 barns for room temperature neutrons. Since the velocity of such neuts is $2.2 \times 10^5 \frac{cm}{sec}$, the cross-section goes as $\frac{705 \times 2.2 \times 10^5}{v}$ It might be noted that for pure B^{10} the room temperature cross-section is 3800 barns. This reaction is rather important inasmuch as several neutron detectors are based on it. For example

We have a boron layer of $.05 \frac{gm.}{cm^2}$ penetrated by a one ev. neutron beam. What is the reduction in intensity of such a beam?

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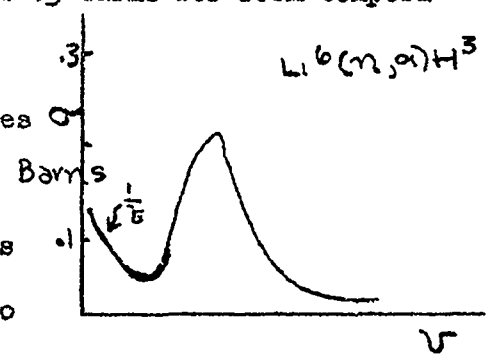
There are many variations of this scheme. You might build the electrodes for more multiplication, i.e. cylindrically symmetrical. One might use argon in the chamber and put boron (in some compound) on one of the electrodes. If the layer is thin, the escaping Li^7 or He^4 might get through to the argon to cause the necessary ionization.

$Li^6 + n \rightarrow H^3 + He^4 + 4.5$ Mev (The most convenient way to make H^3) For ordinary lithium (Li^6 & Li^7 mixed) the cross-section is 65 barns for room temperature neuts. And the $\frac{1}{v}$ law is followed for slow neuts. (up to about .1 Mev) The $\frac{1}{v}$ law holds to higher energies for light nuclei (where the energy levels are spaced far apart) than for heavy nuclei with their close packed levels (which would cause sharp variations in σ) It is well to keep in mind that the $\frac{1}{v}$ law holds for <u>relative</u> velocities. That is, when v becomes small for the neuts, the thermal agitation of the target nuclei must be considered in the application of the $\frac{1}{v}$ law. Let us calculate the number of slow neutrons captured in some material per unit time as a function of the velocity of the neutrons if σ obeys the $\frac{1}{v}$ law. Consider those atoms with velocity $\vec{V}$. What is the number of captures per unit time due to atoms of velocity $\vec{V}$? Since σ goes as $\frac{A}{v_{relative}}$ and the number of "meetings" $\frac{\text{per}}{\text{sec}}$ goes as $N_{\vec{v}} \times v_{rel.}$, the capture probability which is proportional to the product goes as $N_{\vec{v}} A$ where A may be considered constant ($N_{\vec{v}}$ is the concentration of atoms of velocity $\vec{V}$). The total capture probability becomes $\sum_{\vec{v}} A N_{\vec{v}} = AN$, a constant. That is, the number of captures per unit time is constant and independent of $v_{relative}$ whenever σ goes as $\frac{1}{v}$.



What is the average life of a neut in a chunk of lithium. (or in BF_3 at standard conditions)?

$n + N^{14} \rightarrow C^{14} + H^1 + 0.6$ Mev Although Q is positive, it is small and the Gamow factor plays a part, so that the cross-section is only 1.7 barns for room temperature neutrons.

What is the mean life of a neutron in air?

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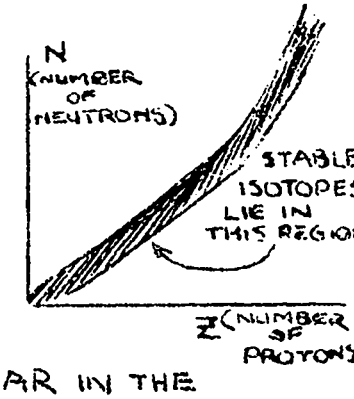
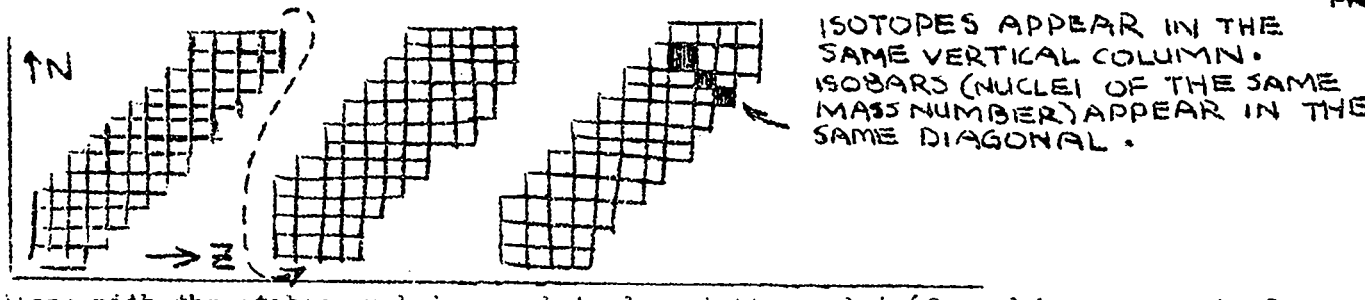
CHAPTER III

The Chart of Stable Isotopes and What it Implies about Nuclear Reactions Involving Neutrons

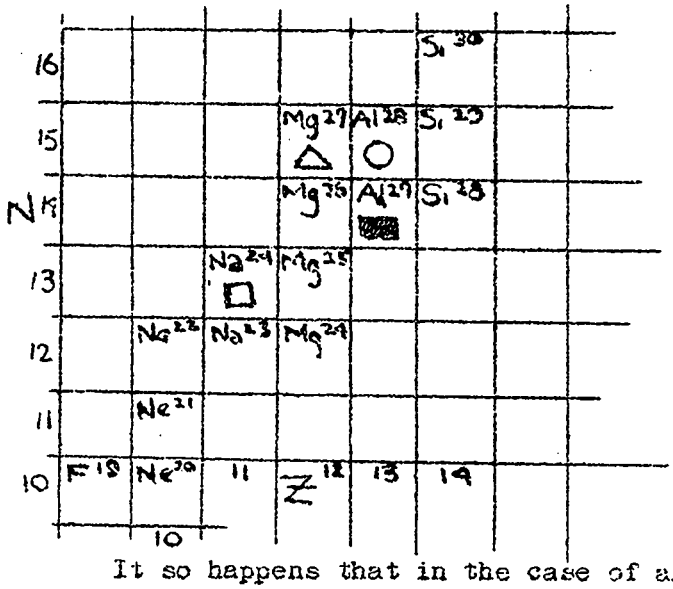


(1) <u>The Segre Isotope Chart.</u> In this chart a nucleus is identified by the number of protons and the number of neutrons it contains. The plot looks somewhat as shown at the Right.

For convenience the strip of stable isotopes is usually broken up and plotted as below.



Along with the stable nuclei are plotted unstable nuclei (formed by some sort of bombardment.) As an example, let us plot the part of the diagram around aluminum.



Al^{27} is marked with a ■

THE END PRODUCTS OF THE FOLLOWING 3 NEUTRON REACTIONS ON Al^{27} ARE ALSO MARKED —

○ REPRESENTS THE PRODUCT IN $Al^{27}(n,\gamma)Al^{28}$

△ REPRESENTS THE PRODUCT IN $Al^{27}(n,p)Mg^{27}$

□ REPRESENTS THE PRODUCT IN $Al^{27}(n,\alpha)Na^{24}$

It so happens that in the case of aluminum these 3 neutron reactions (and these are the 3 most common ones) all yield unstable end products. These are usually β emitters. Thus in the (n,p) reaction discussed, we end up where we started except for the fact that the neutron has become a proton + an electron

$$Al^{27} + n \rightarrow Mg^{27} + H^{1} \quad \text{followed by} \quad Mg^{27} \rightarrow Al^{27} + e^{-}$$

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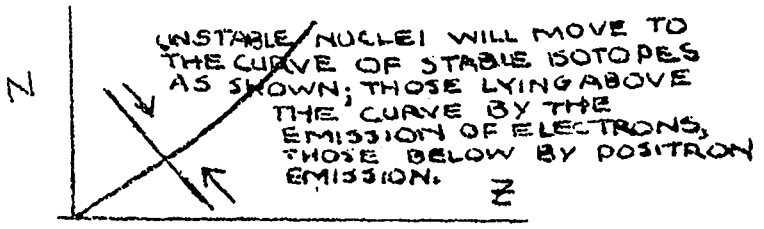
The net effect can be written $n \rightarrow p + e^-$. There exist also (p,n) reactions for which the net result is $p \rightarrow n + e^+$. Consider Al^{27} again and note that by "substituting" neuts for protons and vice versa we obtain a set of nuclei for which $Z+N=27$. These will lie on a line of slope -1 on the chart.

Most nuclei in this set are, of course, unstable.



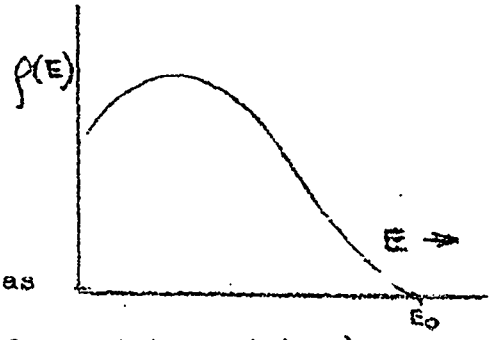
Actually the unstable nuclei below the stable region need not emit positrons. They may move in the right direction by K-electron capture. This has been observed.

The 3 unstable nuclei considered in connection with Al^{27} are all β emitters as would be expected since they all are above the stable curve. They all possess characteristic half lives. These are

Al^{28} 2.3 min Mg^{27} 10 min Na^{24} 15 hours

If Al^{27} is bombarded by neuts of assorted energies, all 3 products may appear and some sort of chemical separation would be used to separate the activities. Here we to isolate one of the activities by chemical means or by using neutrons of proper energy, we could examine the energy spectrum of the emitted β particles, we would find a continuous spectrum. This was (and still is) a somewhat unexpected result. In order to explain this spectrum (there is no particular difficulty, for example, with α spectra which are discrete). But here one usually assumes that the total equation is (take radioactive Mg^{27} as an example): $Mg^{27} \rightarrow Al^{27} + e^- + \nu$ (where ν stands for Pauli's neutrino.)

Find 5 examples each of (n,α) (n,p) (n,γ) processes. Record the half lives of the radioactive product nuclei.



The neutrino takes the extra energy, i.e. $E_\nu + E_{e^-} = E_0$ where E_0 is a constant

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which gives the energy of the β decay, E_0 is the maximum energy on the graph above. For Mg^{27}, this E_0 is 1.8 Mev. Thus the neutrino is here invoked to help conserve energy. Before such a scheme is acceptable, one should show that the maximum β energy (and not, say the average β energy) is the energy lost per nucleus on β decay. Let us assume this in the case of Mg^{27} and see that it leads to observed results $Mg^{27} \rightarrow Al^{27} + e^- + 1.8$ Mev. But now, we know the reaction $Al^{27} + n \rightarrow Mg^{27} + H^1 + Q$, and adding these we find , $.75 \text{ Mev} = Q + E_0$ (.75 is the difference in mass of the neutron and hydrogen atom.) Thus $Q = -1.05$ Mev. Thus the observed β-spectrum of Mg^{27} implies (if our assumptions are correct, i.e. if the maximum energy in the β-spectrum corresponds to the total energy of the reaction of β emission) that the (n, p) reaction on Al^{27} is endothermic with $Q = -1.05$ Mev. And this is observed, for the process has a threshold corresponding to this Q. If the observed Q were, say, -1.2 Mev, this would imply that the energy given out upon β-decay exceeds the maximum of the β spectrum. To satisfy energy conservation one would assign a non-negligible rest mass to the neutrino.

(2) <u>Isotope Weights and (n, γ) Reactions.</u>

One might crudely reason that all such reactions are exothermic by about 8 Mev, because we could write

$$A + n \rightarrow (A+1) + Q$$

as the energy equation. Assuming (and this is quite crude) that the mass of a nucleus containing protons and neutrons of total number A is equal to A, we find

$Q = .009$ millimass units or about 8 Mev.

Actually the true atomic weight differs appreciably from A in many cases. It has, in fact, been found convenient to define the packing fraction

$$\frac{\text{Atomic Weight} - A}{A}$$

This packing fraction appears in the expression for the energy of formation of a nucleus per particle. Per particle the energy of formation = .0085 — packing

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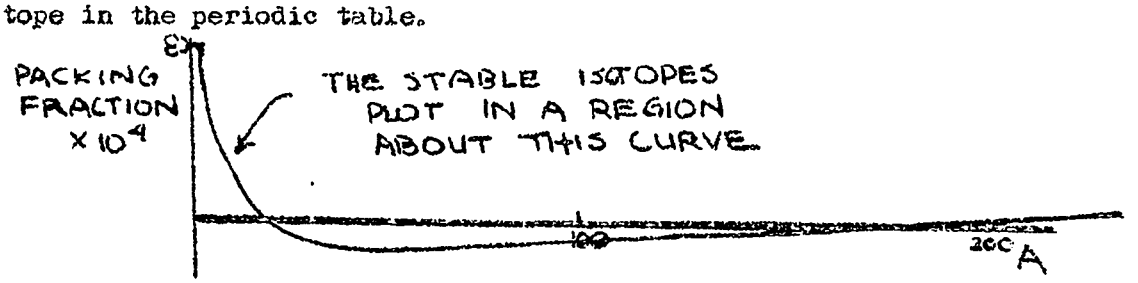
fraction. This can be seen from the following considerations. Total energy of formation is $Z(1.0081) + N(1.0089)$ - atomic weight. Assuming that $Z \sim N$ this can be written approximately

$$(Z+N)(1.0085) - \text{Atomic Weight}$$

Dividing by A gives the above result. For our present purposes the implication of all this is that (n,γ) reactions are not always exothermic with Q = 8 Mev, but that in order to get Q one must know the packing fractions of the nuclei involved.

Look up 10 nuclear masses in the low end of the chart and calculate the binding energy of the neutron to each. Take H^1 H^2 Li^6 Li^7 Be^9 B^{10} B^{11} C^{12} C^{13}...

We have seen that neutron binding energies are of the order 8 Mev through the periodic table, there is an irregular variation of the binding energy with A but in addition there is a regular variation that should be mentioned. This variation involves the packing fraction which is not wholly irregular with respect to the position of the isotope in the periodic table.



Let us then use this observed packing fraction curve to determine any regular variation of binding energy with A. Write

$$P.F. = \frac{M-A}{A} = f(A)$$

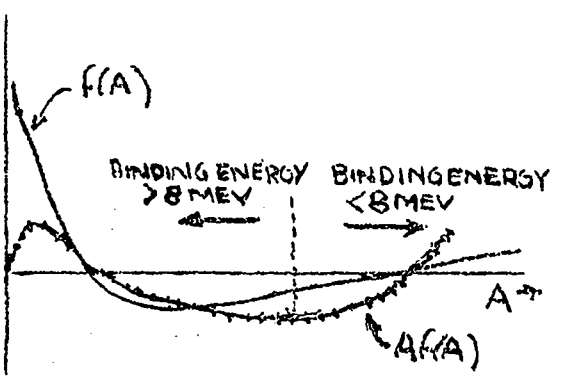
then $M(A) = A[1 + f(A)]$ similarly $M(A+1) = (A+1)[1 + f(A+1)]$

The binding energy of a neutron (or proton) is roughly

$-(A+1)[1+f(A+1)] + A[1+f(A)] + 1.0085$ (where 1.0085 is the average mass of neutron & proton). This is $.0085 - [(A+1)f(A+1) - Af(A)]$

which can be written $.0085 - \frac{d}{dA}[A\,f(A)]$

The function $Af(A)$ looks as shown.



From the curve, note that at very low A the binding energy is erratic (it doesn't mean much to consult the curve here). It soon becomes > 8 Mev

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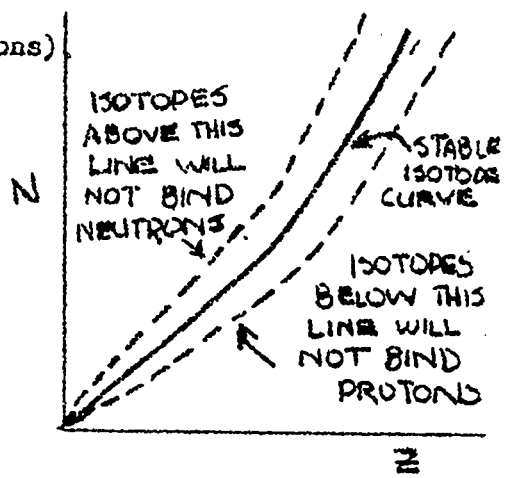
(the slope of the curve is negative). At fairly high A, the binding energy becomes less than 8 Mev.

Look up the actual mass defect curve. Find the average binding energy for nuclei in the following neighborhoods
Fe Ag Pb U

At any rate, the (n,γ) reaction will involve energies of the order 8 Mev. (The only "stable" isotopes that we feel do not bind neutrons at all are He^4 and the neutron itself. Tritium also would probably not bind a third neutron, but this would be true of all the unstable nuclei with enough excess neutrons) In the curve at the right we refer the nuclear binding of neutrons to the stable isotope curve. The closer an isotope is to the top dashed curve the more likely it will emit neutrons (as does a fission fragment). Isotopes only slightly above the stable curve will emit β particles in order to become stable.



It should be pointed out that this 8 Mev rule for (n,γ) reactions is just a rough guide and would often be misleading. For example, opposite process, or (γ,n) reactions, on deuterium and Be^9 show thresholds of 2.2 Mev and 1.7 Mev respectively showing that the neutron binding energies can be rather lower than 8 Mev.

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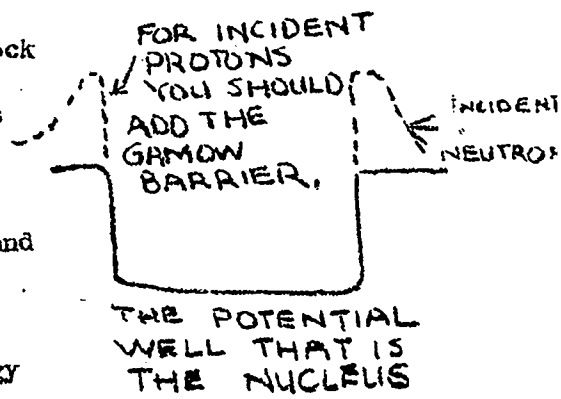
Models of Nuclei and of Nuclear Reactions.

(1) The Compound Nucleus.

We have seen that for an (n,γ) process the cross-section near the origin (i.e. low energy) goes as $\frac{1}{v}\left(\frac{h\nu}{c}\right)^2 \cdot \frac{1}{c}$. About 100 (n,γ) reactions are known and the $\frac{1}{v}$ law adds for most only very close to $v=0$. This is because σ is complicated by the M values. These M values show very sharp variations of resonance levels. An explanation of resonance etc. has to be made in terms of some model. Our picture of a nuclear reaction is somewhat as follows. A neutron when hitting a nucleus will not knock out the first particle it hits. It will distribute its energy among the various members of the nucleus. The combination of neutron + old nucleus is called a compound nucleus. Whether a particle comes out, depends on the probability of concentrating the necessary escape energy on one particle in the course of the interactions within the nucleus. According to this picture an (n,γ) process can be represented



$A+n \longrightarrow (A+n)^*$, where * means excited

$(A+n)^* \longrightarrow (A+n) + h\nu$ (takes a relatively long time)

One usually says that the compound nucleus is "relatively stable". This means that it takes many times the time it would take a nuclear particle to cross the nucleus for a compound nucleus to concentrate the necessary energy on a constituent so that the latter escapes. The crossing time is of the order $\frac{10^{-12}}{10^9} = 10^{-21}$ seconds. The time of existence of a compound nucleus can be inferred from $\hbar \sim \Delta E\, \Delta t$ and a knowledge of widths of resonance levels. Sometimes Δt is of the order 10^{-14} seconds. This is 10^7 times the crossing time.

Find all the information in the literature you can about indium and gold resonances for neutrons.

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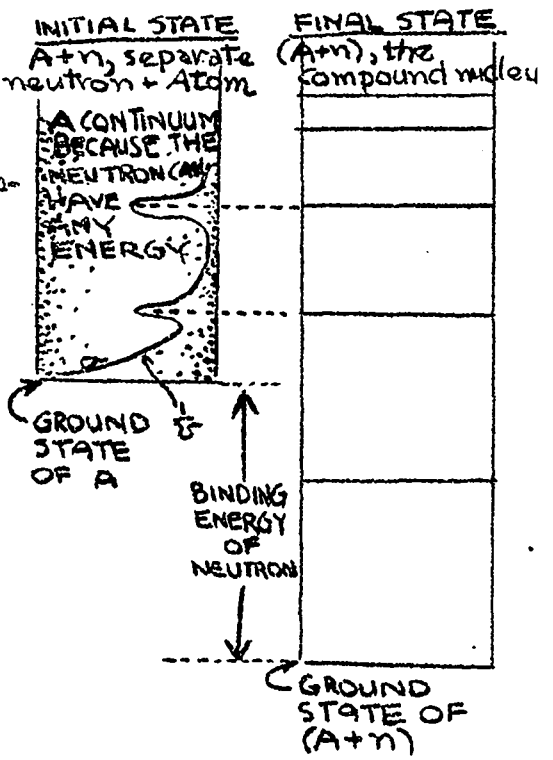
(2) Neutron Resonances and Spacing of Nuclear Levels.

Having seen that the mechanism of a nuclear collision involves the formation of a relatively stable "compound nucleus", let us again consider the (n,γ) process in order to get some more information about nuclei. The cross-section for such a process, we have shown, will follow the $\frac{1}{v}$ law for v small, and will show resonances at higher v. These resonances can be related to the energy levels of the compound nucleus, as is indicated at the right. Neutrons will be absorbed according to the $\frac{1}{v}$ law from very low velocities until they have enough energy, such that the sum of their energy + the binding energy of a neutron to A becomes equal to the energy of some excited state of $(A+n)$, the compound nucleus. At this neutron energy, the absorption cross-section will show a peak.



If the first such resonance peak occurs on the average, for, say, nuclei of medium weight, when neutrons have energy of the order of magnitude E, we would conclude that the energy level spacing of medium weight nuclei is of order E (in the region of excitation corresponding to the binding energy of a neutron). For medium weight nuclei such "first capture resonances" occur at the order of 10 volts. Thus the energy level spacing at $\sim$ 8 Mev

Find from the literature the first resonances for nuclei $A=100$ to $A=150$
Make an estimate of the level spacing here. A IS THE MASS NUMBER.

excitation is of the order 10 volts. Experiments dealing with such nuclei that are near their ground states indicate, however, that the spacing between energy levels is of the order 10^5 & 10^6 volts at low energies.

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(3) Two Models of the Nucleus.

In constructing a nuclear model we thus have at least two observations to account for (1) Why are relatively stable compound nuclei formed in nuclear collisions? (2) Why does energy level spacing decrease so rapidly with excitation energy in medium or large sized nuclei? (The above discussion does not apply too well to very light nuclei.) The first problem is dealt with essentially as follows, no matter what the model. A nucleus is considered to be a mass of close packed particles bound by short range forces. The interactions with neighbors are strong. Whereas in the case of the electrons around the atom, these interactions between neighbors are weak and the collision of a fast moving particle with such an electron can be considered approximately as a two body problem, in the case of nuclear collisions this is not so. The strong interactions act as shock absorbers, and if a particle succeeds in colliding with a nuclear particle, the energy of the collision is quickly shared by the nuclear constituents. This energy is distributed and redistributed as the nuclear particles move about and it is only when, by chance, enough energy is concentrated on a single particle that we have a "disintegration" and this particle leaves the nucleus. It is for this reason that a compound nucleus can exist for so long a time without disintegrating.

In answer to the second question, we shall consider two models. First let us think of the nucleus as a mechanical system with A members, each haveing 3 degrees of freedom. This system will sustain vibrations and there will be 3A characteristic frequencies for there are 3A degrees of freedom in the system. With a frequency of vibration ν_1 there is associated energy $h\nu_1$. If the system is vibrating with frequencies ν_1 and ν_5 at once, it has energy $h\nu_1 + h\nu_5$. In general it will have energy

$$a_1 h\nu_1 + a_2 h\nu_2 + a_3 h\nu_3 + \cdots\cdots + a_{3A} h\nu_{3A}$$

where the a's are integers. If $\nu_1 \neq \nu_2 \neq \nu_3$ etc., it is easy to see that at high energies there would be more levels per unit energy. This argument was qualitative and quite inexact, but the basic idea is reasonable. A system of many degrees of freedom will pack its energy levels at high energy.

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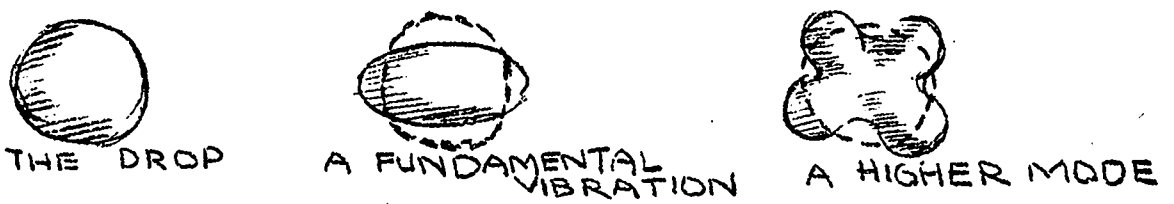
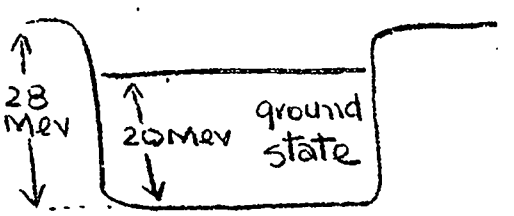
This type of consideration can be refined somewhat. Bohr suggested that a nucleus be thought of as a drop. Both nuclei and drops are held together by short range forces (shorter than the dimensions of the drop or nucleus) and the volumes of each are proportional to the mass. So that nuclei are not very dissimilar to drops.

One can ask about the possible modes of vibration of a drop.



This can be calculated under various assumptions in a not obviously incorrect manner. All theories lead to an expression for energy level density of the form $e^{-f(E)}$ but to say too much about $f(E)$ would not really be justified. It can be said, however, that results based on the drop model are in the right direction.

The second model of the nucleus is one quite different. We consider a nucleus as a gas of protons and neutrons in a well. We speak of a temperature T of this gas. At $T=0$ the nucleus is in the ground state (but some particles are moving rather fast nonetheless, because of the Pauli exclusion principle). If energy is fed in to the gas, some particles get faster and T goes up. It can be shown that for a degenerate gas of the type here considered the energy is proportional to T^2 rather than T. In fact,



$$U = \frac{1}{4}\pi^2\left(\frac{A}{\zeta}\right)\tau^2$$

where U is the energy, A the number of particles in the nucleus, ζ the energy in the ground state ($\sim$ 19 Mev) and $\tau = KT$ If U is written $a\tau^2$ then the Entropy $S = \frac{\partial U}{\partial \tau} = 2a\tau = 2\sqrt{aU} = \frac{\pi}{\sqrt{\zeta}}\sqrt{AU}$ It is customary to write $S = \log P + \text{const.}$ where P is the probability of the state in question. Statistically P is the number

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of states per unit energy interval. We can thus write:

Number of states per unit energy $= C e^{\frac{\pi}{\sqrt{5}}\sqrt{AU}}$

(Thus $-\frac{\pi}{\sqrt{5}}\sqrt{AU}$ corresponds to the $f(E)$ of the drop model) Applying this to a medium nucleus with 8 Mev excitation $(A \sim 100, \zeta \sim 19)$ we find as the number of states per unit energy Ce^{20} or $C \cdot 10^8$. That is, the density of states at 8 Mev is 10^8 times what it is at the ground state. This is somewhat too high.

In conclusion it should be warned that these points of view shouldn't be taken for more than they are worth. They serve to indicate the trends. Essentially the idea is that a nucleus is a system with many degrees of freedom. Almost any attack would show the logarithmic variation we have found. For more details see "Nuclear Physics. Part B" – Bethe §53 Reviews of Mod. Phys. 9 April 1937

Make a table for level density for the following nuclei at the given excitations (in Mev) using the gas model

Be Fe Ag Au 4, 6, 8, 10

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CHAPTER V

The Scattering of Neutrons

(1) The Breit-Wigner Formula.

We have already spoken of a general formula for the cross-section of a nuclear process. (page 8) In the case that resonance levels of a compound nucleus are far apart and we are interested in the cross-section in the neighborhood of one of the resonance levels, the expression for the cross-section of an (a,b) process can be shown to take the form

$$\sigma(a,b) = \pi \lambda\!\!\!^{-}_a{}^2 \frac{\Gamma_a \Gamma_b}{(E_a - E_r)^2 + \Gamma^2/4}$$

This is the Breit-Wigner formula. $\lambda\!\!\!^{-}_a$ and E_a are the wavelength $\left(\frac{\hbar}{m v_a}\right)$ and energy the incident particle. E_r is the resonance level energy and Γ is the width of the resonance peak at half its maximum value. Actually the above expression for σ should be multiplied by factors depending on the spins of the initial particles and the compound nucleus. But, for simplicity let us consider these factors incorporated in the Γ_a and Γ_b . Γ_a & Γ_b are the partial widths of the resonance peak and are associated with the probability of emitting "a" and "b" particles, respectively. Their exact form is rather complicated but since the probability of emission of a particle b of momentum p (p. 9) is proportional to $\rho = \frac{P_{b\,out}^2}{v_{out}}$, so is Γ_b . The relative probability that the outgoing particle will be an "a" particle is $\frac{\Gamma_a}{\Gamma}$ etc. In order that these be true relative probabilities $\sum_{i=a,b\ldots} \Gamma_i = \Gamma$.

It should also be remarked that $\Gamma_i \times \tau_i \cong \hbar$ where τ_i is average time of emission of a particle i after the formation of the compound nucleus. Thus the probability that no particle i has been emitted from the nucleus up to the time t is $\prod_i e^{-t/\tau_i} = e^{-t \sum_i 1/\tau_i}$, $\left(\tau = [\sum 1/\tau_i]^{-1}\text{ is}\right)$ and the average life of the compound nucleus. It will be seen that τ , as well as the individual τ_i's , will satisfy the Heisenberg relation $\Gamma \times \tau \cong \hbar$.

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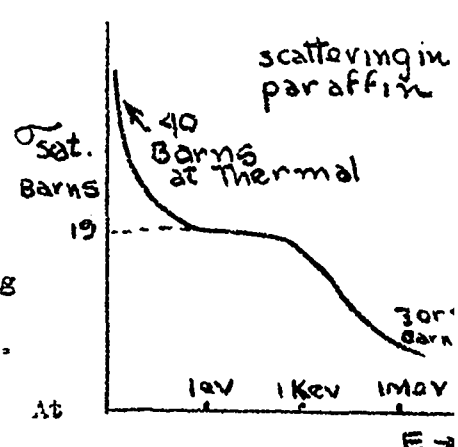
The Breit-Wigner formula can be applied as it was given, to any nuclear collision involving the formation of a compound nucleus provided that the resonance levels are not so close together that they appreciably distort this one-level formula. It can be applied, for example, to (n,γ) processes. Let us do a typical problem. For resonance capture in Indium, $E_r = 1.44$ e.v. and $\sigma(n,\gamma)_{res} = 26{,}000$ barns. Γ is .05 ev. Find the number of neutrons emitted for each γ emitted, given the information that experimentally only γ's and neutrons are observed to be emitted (i.e. $\Gamma = \Gamma_n + \Gamma_\gamma$) and that more γ's are emitted than neutrons ($\Gamma_\gamma > \Gamma_n$). From $\sigma_{res} = \pi \lambda^2_{n\,res} \frac{\Gamma_n \Gamma_\gamma}{\Gamma^2}$, $\Gamma_n \Gamma_\gamma$ can be evaluated and knowing the value of $\Gamma_n + \Gamma_\gamma$ and that $\Gamma_\gamma > \Gamma_n$ both Γ_γ and Γ_n can be evaluated. The result is that $\Gamma_n = .0008$ ev and $\Gamma_\gamma = .05$ ev, whence $\frac{\Gamma_n}{\Gamma_\gamma} = .015$. For Gold, $E_r = 4.8$ ev and σ_{res} is also $\sim$ 26,000 barns and $\Gamma = .07$ ev. From this data, $\Gamma_n = .0035$ and $\Gamma_\gamma \sim .07$. For silver, $E_r = 5.5$ ev, $\sigma_{res} = 7200$ barns, $\Gamma = .19$ ev. These capture reactions are very useful in methods of slow neutron detection.

(2) <u>Some General Considerations on the Scattering of Neutrons.</u>

From the Breit-Wigner formula, the elastic scattering cross-section for low energy neutrons becomes $\sim \pi \lambda^2_n \frac{\Gamma_n^2}{E_r^2}$. Now λ_n goes as $\frac{1}{v_n}$ and Γ_n goes as $\frac{p^2_{out}}{v_{out}}$ or as v_n. Hence the cross-section is independent of energy. Thus one would expect that σ should be fairly constant for scattering of neutrons with energies less than the first resonance energy. The curve for scattering in hydrogen (paraffin) is given here. At low energies the cross-section does level out as predicted at about 19 barns. But then for very low velocities there is a sudden increase in σ. This is explained as follows. Since Γ_n is proportional to m^2, σ must go as m^2. In the application of the B.W. formula this m is the reduced mass of the problem and the



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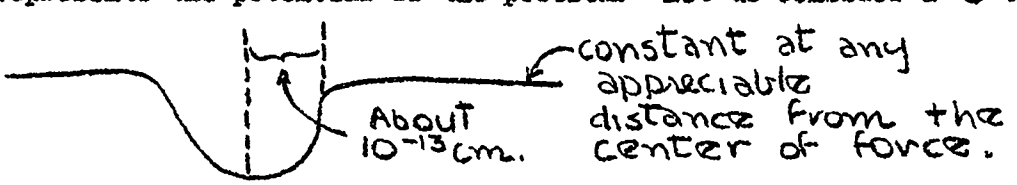
velocities are the relative velocities involved. For neutrons scattered from hydrogen, m is therefore half the neutron mass, except where the hydrogen is bound in a molecule and the relative neutron-proton velocity provides too little energy to free the hydrogen or excite the molecule. Then the hydrogen effectively has infinite mass and the reduced mass is the neutron mass itself. Since σ goes as m^2, the cross-section for bound hydrogen is $\sim$ 4 times that of free hydrogen. For slow neutrons on chilled paraffin, the relative velocities are low and σ approaches $\sim$ 80 barns.

(3) <u>Scattering by a Potential.</u>

The use of the B.W. formula for scattering involves the calculation of Γ's and it is often simpler to use the following approach to the problem of scattering. We consider what happens to the wave function of a particle scattered by a potential representing a nucleus. One advantage of this approach stems from the fact that quite exact information on σ is available even if the potential is but crudely known. A moving particle will satisfy the Schrödinger equation $\Delta\psi + \frac{2m}{\hbar^2}(E-U)\psi = 0$ where U represents the potential of the problem. Let us consider a U that looks so:



If we consider only s-scattering (zero angular momentum), then the Shrödinger equation in spherical coordinates is simply $r\psi'' + 2\psi' + \frac{2m}{\hbar^2}(E-U)r\psi = 0$

or $\frac{d^2}{dr^2}(r\psi) + \frac{2m}{\hbar^2}(E-U)(r\psi) = 0$ Thus for s scattering $r\psi$ (let us call it u) obeys a rather simple one-dimensional differential equation $u'' + \frac{2m}{\hbar^2}(E-U)u = 0$ which one would solve (given a U) under boundary conditions $\psi = 0$ at infinity and finite throughout.

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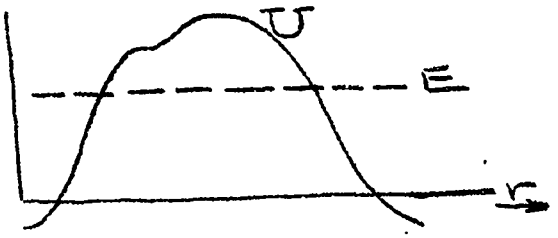
Given an arbitrary potential, what can be said about u at sight?

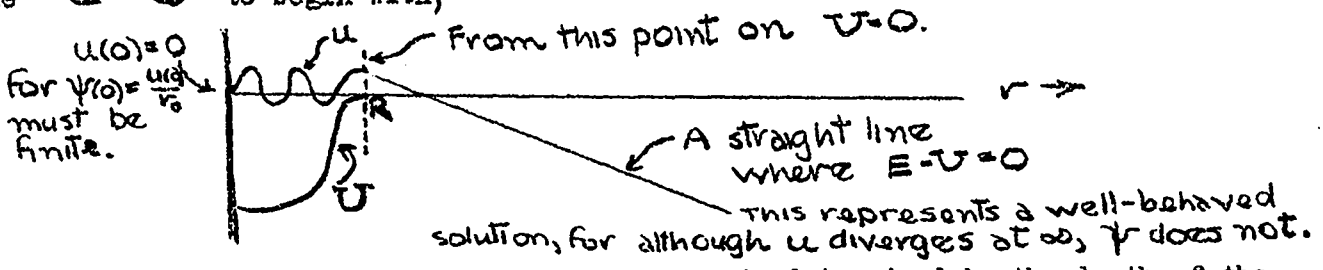


Aside from the $\frac{2m}{\hbar^2}$, the equation reads $u'' = -(E-U)u$.

Whenever $(E-U) > 0$, u'' and u have opposite sign. This means that the function u curves toward the r axis. The concavity is toward the r axis. On the other hand $(E-U) < 0$ implies that the concavity is away from the r axis. In case $E-U=0$ there is no curvature of the function u. In a qualitative way one can thus draw u for a U of any shape. [Of course when an analytical expression for U is available, one can integrate the equation, but in any case our rough consideration can quickly furnish information about the general shape.]

Consider the three situations, $E=0$, $E>0$, $E<0$.

Take $E=0$ to begin with,



The number of bumps in the region of the nucleus is determined by the depth of the well. The deeper the well the sharper the curvature and the more changes of direction u exhibits within the nucleus.

For $E>0$, the equation to be satisfied past R is $u'' + \frac{2m}{\hbar^2}Eu = 0$. The solutions are $\cos\sqrt{\frac{2mE}{\hbar^2}}\,r$ and the sine. This must be matched to the curve within the nucleus (u and its first derivative must be continuous at the junction.) The wave length of these cosine and sine functions, it should be remarked, is the De Broglie wave length of the particle being described.

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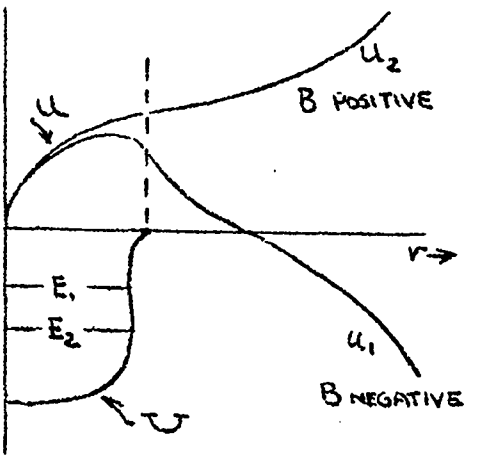
For $E<0$, the equation at distances outside the range of action R becomes $u'' = \frac{2m}{\hbar^2}(-E)\,u$. The solution is well known

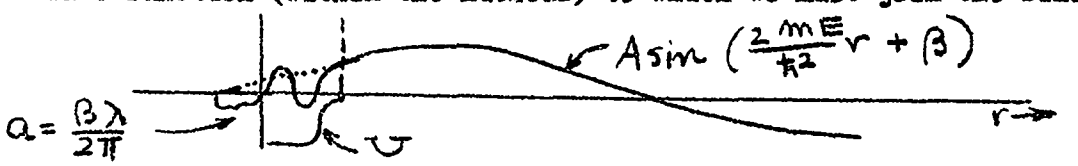
$$u = e^{\pm\sqrt{\frac{2m}{\hbar^2}(-E)}\,r}$$

$-E$ is a positive quantity.

A linear combination of these two exponentials would give a catenary that grows rapidly with r. Such solutions would have to be discarded because they are not well behaved at infinity. Say the energy is E_1, and the curvature is such that the matching exponential solution is as in u_1. For a lower energy E_2, the curvature will be such that u_2 is the solution. Now between E_1 & E_2 there will be a solution that does not diverge. That is B is neither negative nor positive in $u = Ae^{-\sqrt{\frac{2m}{\hbar^2}(-E)}\,r} + Be^{+\sqrt{\frac{2m}{\hbar^2}(-E)}\,r}$, but is exactly zero. From this we see that there are a discrete group of energies giving allowable (non-diverging) ψ functions in the case that $E<0$.



Consider again the case where $E>0$, for this corresponds to the case of an incident particle. We shall take up the scattering of such a particle in the nuclear center. (See Mott & Massey for details.) The sine function that represents the wave outside the nucleus will not necessarily seem to come (when extrapolated) from the origin. It will have a phase shift. This phase shift will depend on the wave function (within the nucleus) to which we must join the sine.



It can be shown (See Mott) that for a scattering the cross-section for a particle depends on the phase shift as follows

$$\sigma_{scat} = \frac{4\pi\hbar^2 \sin^2\beta}{m^2 v^2}$$

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When does one have only s scattering (i.e. the incident particle comes in without angular momentum.)? If a particle would pass a nucleus at a distance b (if it were uneffected by the nuclear force) and has velocity v at a great distance from the nucleus, then its angular momentum would be mvb. But in quantum mechanics the angular momentum is quantized. $mvb = \hbar l$ Then $b = \frac{\hbar}{mv} l$ where l takes on integer values. The region $b = 0$ to $b = \frac{\hbar}{mv} \cdot 1$ gives the region in which we have s scattering. We would have p scattering for b between $\frac{\hbar}{mv}$ and $2\frac{\hbar}{mv}$. If, however, $R < \frac{\hbar}{mv}$ there would obviously be no p scattering (particles passing at "p-distances" from the nucleus would not be aware of its existence.) In this case there would be only s scattering. This condition can be stated, "there is only s scattering if the neutrons are slow."

For very low velocities, the above scattering formula is therefore seen to hold and takes on a very simple form. λ is very big and if a is the distance of the shift of the sine at the origin, the phase shift is $\beta = 2\pi \frac{a}{\lambda}$. "$a$" is pretty much the same for all low velocity particles. For very low velocity, λ is big enough so that β may be substituted for $\sin \beta$. In this case

$$\sigma_{sc} = \frac{4\pi\hbar^2}{m^2 v^2} \cdot \frac{4\pi^2 a^2}{\lambda^2} = 4\pi a^2$$

Given $R = .282 \times 10^{-12}$ (the classical electron radius) and a depth of a rectangular potential well of 10.8 Mev (the singlet state of the deuteron) and of 19.7 Mev (triplet state), answer the following questions concerning the scattering of neutrons by protons.

(1) Are there any bound states in each of these cases?

(2) What is the value of "a"?

(3) What is the average cross-section for low velocity neutrons in hydrogen?

It can be shown that the formula $\sigma_{scat} = \frac{4\pi\hbar^2}{m^2 v^2} \sin^2 \beta$ reduces, in the case of the existence of bound states, to $\sigma_{scat} = \frac{4\pi\hbar^2}{M(\epsilon + \frac{E}{2})}$ where

ϵ is the binding energy of the bound neutron and E the incident kinetic energy.

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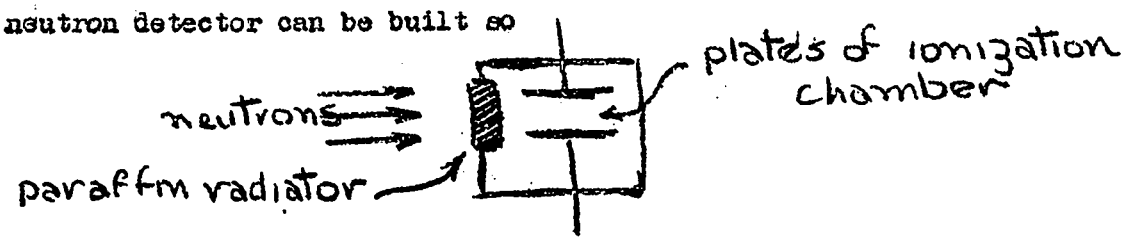
Even when a virtual state exists, this formula is valid, but here ϵ is negative and $|\epsilon|$ is used in place of ϵ in the formula. In the case of the deuteron, the triplet state has a bigger ϵ than the singlet. In fact the ϵ for the singlet (spin of neutron opposite to that of the proton) is so small that it is difficult to decide whether it is positive or negative. In any case the observed scattering cross-section of neutrons in hydrogen would be

$$\sigma = \frac{4\pi\hbar^2}{m}\left(\frac{3}{4}\frac{1}{\epsilon_1+\frac{E}{2}} + \frac{1}{4}\frac{1}{|\epsilon_2|+\frac{E}{2}}\right)$$

The $3:1$ weighting of states follows from the fact that there are $2S+1$ states of spin S, and in our case, $S=0$ (singlet) and $S=1$ (triplet) are the two possible values of S. One can determine whether the "singlet" deuteron state is real or virtual by scattering neutrons from para and ortho hydrogen. It can be shown that the scattering for real states is about 180° out of phase with that for virtual states. In para-hydrogen, the spins of the 2 atoms in the molecule are antiparallel and if one were to scatter slow neutrons from parahydrogen, and if the singlet state were virtual, the 2 atoms would scatter out of phase and the scattering cross-section would be low. For ortho hydrogen (the atoms have spins parallel) the scattering amplitudes for the 2 atoms would always be in phase and the scattering cross-section would be greater. This has been observed and the singlet deuteron state is believed to be virtual.

Before leaving the subject of the scattering of neutrons in hydrogen, it should be mentioned that this scattering is the basis of an important method of neutron detection. One exposes a thin paraffin layer to the neutron flux. The neuts get scattered in the paraffin and in each scattering a proton recoils with energy of the same order as the neutron energy. One can detect fast protons with ionization chambers. Hence a fast neutron detector can be built so



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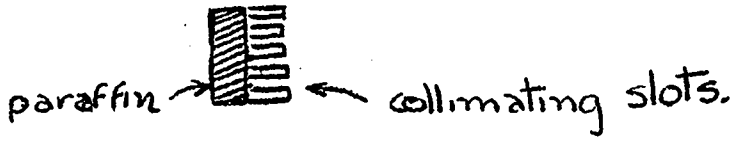
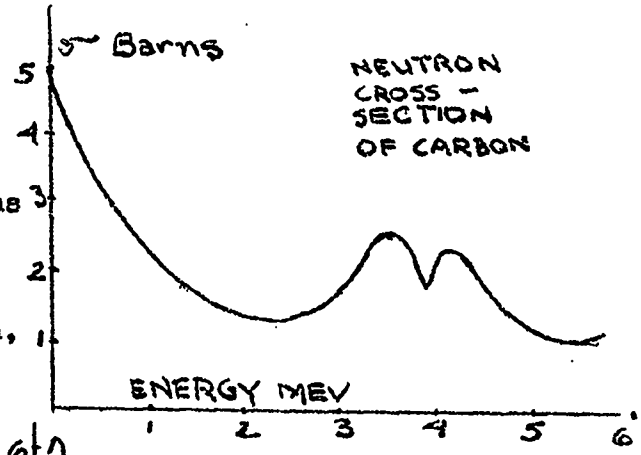
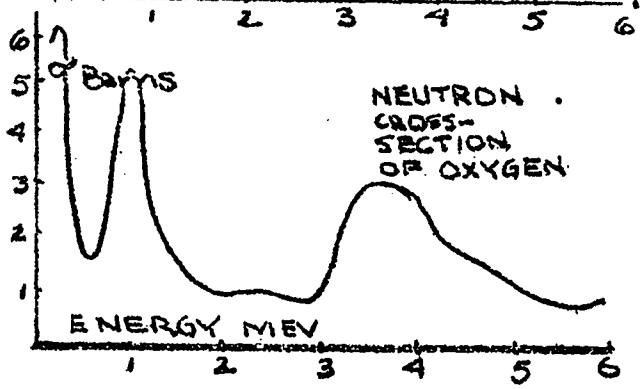
In order to measure neutron energies, a collimating device can be used to allow only forward scattered (maximum energy) protons into the chamber



The pulse in the chamber can be calibrated in terms of proton energy and in this way the original neutron energies can be ascertained.

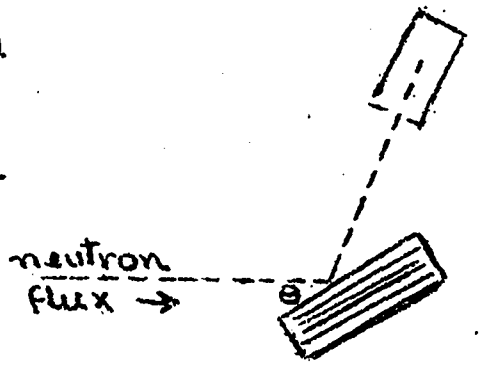


It should be remarked that in general neutron scattering cross-sections show complicated variations with energy, as these examples show. These irregularities are related to involved resonance phenomena, which would not be covered by the simple theory of this section. Incidentally, one usually measures total cross-sections. Since the absorption cross-section is very small compared to the scattering cross-section, such measurements give, in effect, the scattering cross-section.



(4) <u>The Scattering of Neutrons by Crystals.</u>

If neutrons have energy of about 1eV or less, they have De Broglie wavelengths of the order of Angstroms or greater. Interatomic distances are measured in Angstroms and it might be expected, therefore, that atoms will interfere in the scattering of slow neutrons. This is actually the case, as we have seen in the discussion of scattering in ortho and para hydrogen of the last section. One might do an experiment of scattering slow neutrons from a crystal, in order to observe these effects, for the regular arrangement of atoms in a crystal would tend to emphasize any interference phenomena. If one uses an apparatus of the type



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illustrated, finding the angular distribution of scattered intensity as a function of angle of incidence, it is found that neutrons obey a Bragg-like formula:

$$n\lambda = 2a \sin\theta$$

where n is the order, λ the neutron wavelength and a the interatomic spacing. For the first order this can be written:

$$v = \frac{h}{2ma\sin\theta}$$

If a beam of neutrons with a continuous spectrum impinged on a crystal, those neutrons of the proper v for the angle θ would be reflected in a sharp beam at an angle equal to the angle of incidence. Neutrons with other velocities would simply be scattered in the material in a normal way. One can easily check that the reflected beam really contains those neutrons whose velocity is given by the Bragg formula, by taking readings with and without a boron absorber in front of the detector for various angles. The boron cross-section as a function of v is well known and from the observed curve for σ vs θ one could easily get one for v vs θ, and this would be found to give the Bragg relation (aside from complications due to higher order reflection, etc.) It can be seen, therefore, that the combination of a crystal and neutron detector can be used to analyze a beam of neutrons for the velocity distribution. In such a way, for example, slowed down neutrons from a tank of water are found to show a Maxwell distribution... It is found that microcrystalline substances scatter much better than regular crystalline substances. The explanation is along the following lines. The neutron arrives at the first microcrystal. If the Bragg condition is not fulfilled the neutron can get through. Otherwise it will be reflected. Then the neutron arrives at the next microcrystal and once again must pass the test of not fulfilling the Bragg condition if it is not to be scattered. Were there but one crystal, it would have but

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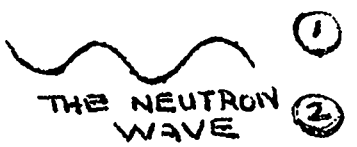
one test to pass. The more microcrystals, the more chance to be scattered. For a single large crystal only those velocity neutrons are removed that satisfy the Bragg relation. For the microcrystalline structure, sooner or later all velocity neuts of the original beam will be removed as the beam moves from crystal to crystal in the microcrystalline structure.

There is one very important difference between x-ray and neutron scattering. If the crystal is composed of 2 isotopes the x-ray scattering is not particularly disturbed, for x-ray scattering depends on extra-nuclear properties of an atom (and these are pretty much the same for 2 isotopes). For neutrons it is the nucleus itself that enters into the scattering process. And nuclei are such that in addition to determining scattering cross-sections, they determine the phases of scattering. Consider a neutron being scattered by 2 isotopes.



For species (1) the scattering intensity is proportional to σ_1. The scattering amplitude for species (1) is $\sqrt{\sigma_1}$. Write

$$\sqrt{\sigma_1} = \frac{\sqrt{\sigma_1}+\sqrt{\sigma_2}}{2} + \frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{2} \qquad \sqrt{\sigma_2} = \frac{\sqrt{\sigma_1}+\sqrt{\sigma_2}}{2} - \frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{2}$$

Thus $\sqrt{\sigma_1}$ & $\sqrt{\sigma_2}$ have a part in common (they scatter the same for this part, the so-called coherent part.) In addition there is an incoherent (opposite phase) part. The coherent part does give rise to interference. The incoherent part gives rise to scattering as if from an unordered assembly of atoms.

For NaCl (using the (001) planes only) make a table of the λ reflected (1st & 2nd order) at various angles θ 1, 2, 3, 4, 5, 10, 20, 30, 40° calculate the ev. for each λ What is the relative intensity of first & second order beams, assuming a Maxwell distribution (T = 300°K) for the neutrons?

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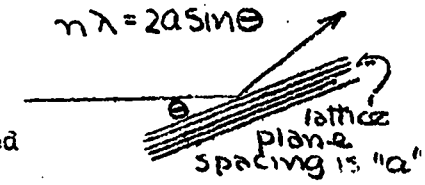
How deep will a beam of neutrons penetrate inside a crystal if the Bragg reflection is satisfied? Let us take finite piece of crystal, say a crystal of M planes with $N \times N$ atoms in each plane. There are then $N \times N \times M$ atoms in the crystal. If there were but one atom and its cross-section were σ the scattered intensity of a beam of intensity I is $I\sigma$. This would be isotropic and at a distance r the intensity would be $\frac{I\sigma}{4\pi r^2}$. This implies that the amplitude of the scattered beam at this point is $\sqrt{\frac{I\sigma}{4\pi r^2}}$ Since this is but an order of magnitude calculation we are doing, let us call the amplitude simply $\frac{\sqrt{I}\sqrt{\sigma}}{r}$. Now let us estimate the amplitude in the reenforced direction. In this direction the amplitudes add. Hence the amplitude is $\frac{\sqrt{I}\sqrt{\sigma}}{r} N^2 M$ This makes the intensity $\frac{I\sigma}{r^2} N^4 M^2$



Now we ask, "What is the width of the angular spread of the reflected beam?" From physical optics this angle $\Theta = \frac{\lambda}{d}$, where d is the dimension of the "mirror". In our case $d = N \times a$ This gives a patch of "light" of size $\left(\frac{\lambda r}{Na}\right)^2$ at the distance r in the right direction. Thus the total energy reflected is

$$I\sigma N^2 M^2$$

under the assumptions a) that the crystal is so small that there is practically no attenuation in the crystal, b) that λ is of the order of magnitude a. Were the crystal a perfect mirror it would remove energy $I(Na)^2$ It could certainly not remove more. Hence certainly

$$I\sigma N^2 M^2 < I(Na)^2$$

or $\sigma M^2 < a^2$ what does this inequality imply? It means that if $M > \frac{a}{\sqrt{\sigma}}$, then the layers beyond $M_0 = \frac{a}{\sqrt{\sigma}}$ do not count. We say the beam doesn't penetrate past M_0 layers. The depth of penetration, then, is $M_0 a$ or $\frac{a^2}{\sqrt{\sigma}}$. a will be of order 3×10^{-8}; $\sqrt{\sigma}$ will be about 2×10^{-12}. This means about 10^4 planes play a vital part in Bragg reflection. This is of the order of microns.

If a heterogeneous (say Maxwellian) beam of neutrons is made to hit a large perfect crystal, those neutrons of the "right" velocity will be weeded out right off,

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(within a micron or so.) But the rest of the neutrons won't be transmitted 100%. There are several reasons for the attenuation of even the "wrong" velocity neutrons.

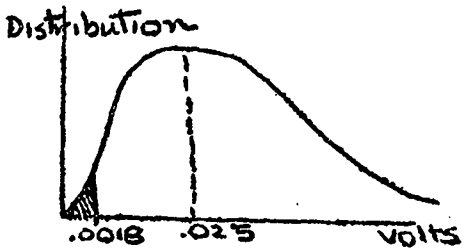
(1) The presence of isotopes presents a random irregularity and this would give incoherent scattering for all velocities.

(2) Another element of irregularity is the random variation of spin direction of the nuclei.

(3) The crystal may not be too regular, but even if it were, the thermal motions of the atoms would complicate matters.

(4) The crystal atoms might absorb as well as scatter neutrons.

How to obtain very slow neutrons using interference phenomena. The Bragg formula can be written $\lambda = 2a \frac{\sin\theta}{n}$ This implies that $\lambda < 2a$ Thus neutrons whose wavelength is larger than twice the maximum crystal spacing do not in any case get Bragg-reflected. All scattering of such neutrons would be due to the above 4 phenomena. Graphite has almost one isotope (99%) and the spin is zero (atoms of even atomic weight have spin zero usually). For graphite $2a = 6.69 \quad 10^{-8}$ cm. This makes the volts of neutrons of this λ equal to .0018 volts [1 volt neutrons have $\lambda = .286 \times 10^{-8}$]. At thermal distributions the peak is at about .025 volts. This implies that a piece of graphite will (in an appreciable distance) weed out almost the entire spectrum of thermal neutrons. Only the very slow neutrons (shaded in diagram) will not be weeded out (they have too large a λ and can be scattered only by the 4 processes mentioned above and these are small for graphite. Thus we can get a very "cold" beam using thermal neutrons and a polycrystalline graphite "filter".



Reflection of neutrons from polished surfaces. If one sends a beam of x-rays on a polished surface at a glancing anle, one observes a total reflection. (Most substances have indices of refraction for x-rays very slightly less than one.) The

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index of refraction of a substance is intimately connected with the scattering properties of the substance. The interference of scattered x-rays and original x-rays produces a change in phase of the original ray. This change of phase can be described most conveniently by assigning an index of refraction that describes the change of wave velocity within the substance. At any rate the same phenomena of total reflection from polished surface occurs for neutrons too. But the index of refraction for neutrons is very close to one. This means that a converging lens for neutrons would have to bulge very much along the axis to do any good if it were made of substances where n is slightly greater than one. For substances in which $n<1$ a converging lens would look like the diverging lenses of optics. These lenses are possible in principle, but because $|n-1|$ is so small, they are not at all practical.

Show that if a continuous distribution of neutrons impinges on a microcrystalline substance where $M < M_0$, that the scattered intensity is of the order of that expected from a non-crystalline substance of the same number of atoms for all energy neutrons. Use the fact that the resolution of a microcrystal reflecting neutrons of wavelength λ according to the Bragg formula is given by $\frac{\delta\lambda}{\lambda} \sim \frac{1}{M}$

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CHAPTER VI

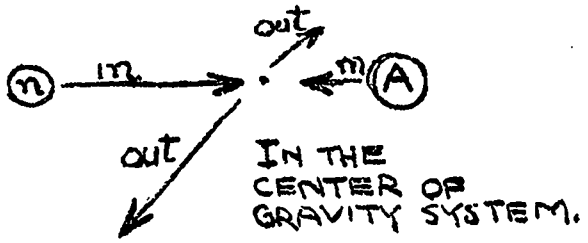
The Slowing Down of Neutrons

(1) The Change of Direction & Energy upon Collision.

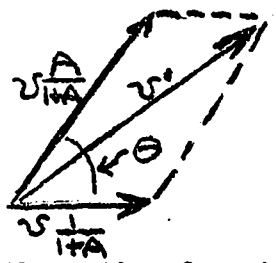
Suppose a beam of neutrons is made to hit some hydrogen. A neutron loses on the order of half its energy per collision. At this rate a million volt neutron becomes thermal in about 24 collisions. (Hydrogen is particularly good for slowing down neutrons, because in addition to its being light, it has a high scattering cross-section for neutrons; $\sim$ 20 barns compared to 3 or 4 barns for other light nuclei)

Assuming isotropic scattering in the center of gravity system, let us investigate the results of a collision of a neutron of velocity v against an atom of weight A at rest. The velocity of the center of gravity is $\frac{v}{1+A}$. The velocities of A and n relative to the center of gravity are $\frac{v}{1+A}$ and $\frac{vA}{1+A}$ respectively.

In a scattering process, the magnitudes of velocities in the center of gravity system don't change. Only the directions change.



Going back to the lab system, we can get v' the resultant velocity of the neutron in the lab system by adding (vectorially) its velocity in the center of gravity system to the velocity of the center of mass in the lab system. From the diagram



$$\frac{1}{2}mv'^2 = \frac{1}{2}m\frac{v^2}{(1+A)^2}\left[A^2+1+2A\cos\theta\right]$$

Thus the ratio of neutron energies before and after collision is

$$\frac{E'}{E} = \frac{A^2+1+2A\cos\theta}{(A+1)^2}$$

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It should be noted that θ is an angle in the c. of g. system. If β is the angle in the lab system between initial and final neutron directions, it is easy to show that

$$\cos\beta = \frac{A\cos\theta + 1}{(A^2 + 1 + 2A\cos\theta)^{1/2}}$$

$\overline{\cos\beta}$ is (unlike $\overline{\cos\theta}$) not equal to zero, but is some positive number showing that colliding particles show a tendency to preserve their direction of motion. In fact,

$$\overline{\cos\beta} = \int_0^\pi \frac{A\cos\theta + 1}{\sqrt{A^2 + 1 + 2A\cos\theta}} \cdot \frac{2\pi\sin\theta\, d\theta}{4\pi} = \frac{2}{3A}$$

showing that this tendency to keep going in the original direction, is greatest for collisions with light particles, as expected.

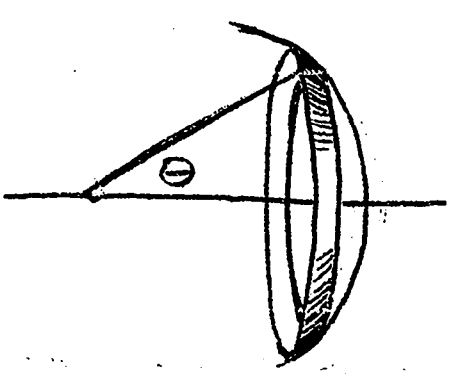
Let us look more closely at the formula for $\frac{E'}{E}$. The maximum and minimum values are:

$$\frac{A^2+1 \pm 2A}{(A+1)^2} \begin{cases} = 1 \text{ for } \theta = 0 \text{ (practically no collision)} \\ = \left(\frac{A-1}{A+1}\right)^2 \text{ for } \theta = \pi \text{ (head-on collision)} \end{cases}$$

For collisions with hydrogen $(A=1)$ the limits are thus 1 and zero. For heavier atoms, it is, of course, impossible to bring the neutron to rest. In fact where A is big, $\left(\frac{A-1}{A+1}\right)^2 \cong 1 - \frac{4}{A}$ neglecting higher terms in $\frac{1}{A}$. Thus for $A = 100$, the biggest possible loss in energy is 4%. For $A = 200$ it is 2% and so on.

We would like to know now what is the relative probability of a neutron's coming off with any energy E' between the limits just described. Let us assume that the scattering is isotropic in the c. of g. system.

What is the solid angle between θ and $\theta + d\theta$ It is $\frac{2\pi\sin\theta\, d\theta}{4\pi} = \frac{\sin\theta}{2} d\theta$



From the formula for E',

$$dE' = -E\frac{2A}{(1+A)^2}\sin\theta\, d\theta = -E\frac{4A}{(1+A)^2}\, dp$$

where dp is the probability $(\frac{1}{2}\sin\theta\, d\theta)$ that the angle is between θ and $\theta + d\theta$

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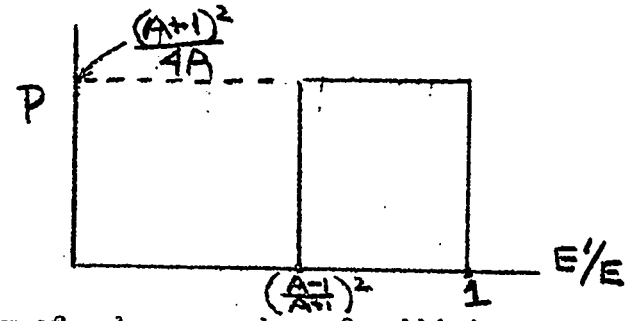
or that the energy E' lies between E' and $E'+dE'$ Thus

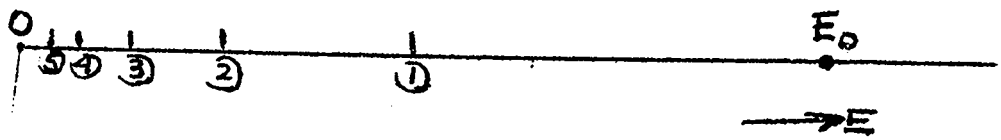
$$dp = \frac{(A+1)^2}{4A}\frac{dE'}{E}$$

Thus the probability of coming off with energy E' is independent of E'!

The curve for the probability of coming off with energy E' as a function of $\frac{E'}{E}$ is shown at the right. For hydrogen the figure would be a square with a corner at the origin.



What would be the influence on neutron energy of a large number of collisions? It is more convenient in this discussion to consider $\epsilon = \log E$. This comes from the result that E'/E is independent of E. In fact since the percent loss in energy is on the average the same, the neutron's energy looks like this after collisions 1, 2, 3, ...



and in each collision it is $\log E$ rather than E that changes by a more or less fixed amount. Let us call $\xi \equiv \overline{\log \frac{E}{E'}}$ and evaluate it.

It is $\int_{min}^{max} \log \frac{E}{E'} \cdot P(E')\,dE$ which is $\int_{\left(\frac{A-1}{A+1}\right)^2 E}^{E} \log \frac{E}{E'} \frac{(A+1)^2}{4A}\frac{dE'}{E}$

This is a fairly simple integral and

$$\xi = 1 - \frac{A\left(1-\frac{1}{A}\right)^2}{2}\log\frac{A+1}{A-1}$$

For carbon, for example, $A = 12$ & ξ is equal to .158 For hydrogen $\xi = 1$. This means that for hydrogen $\frac{E'}{E}$ is on the average $\frac{1}{e}$. To reduce a 1 Mev neutron to thermal energies $\left(\frac{1}{40}\text{ ev}\right)$ by means of hydrogen one would require $\log 4\times10^7 = 17.5$ collisions. About 110 collisions $\left(\frac{17.5}{.158} = 110\right)$ would be required in carbon.

Calculate ξ for H^2 He^4 Be^9 O^{16} U^{238}

How many collisions will be needed to reduce a neutron's energy from 10^6 to 1 volt?

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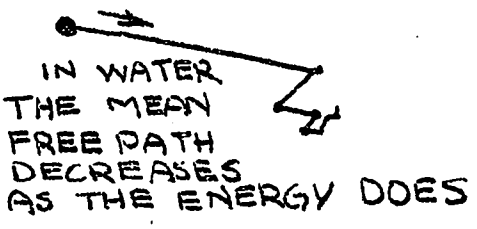
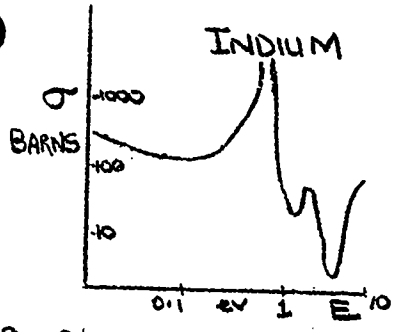
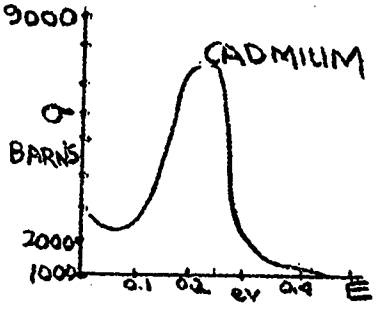
(2) Distribution of Neutrons from a Point Source -- Experimental Methods.

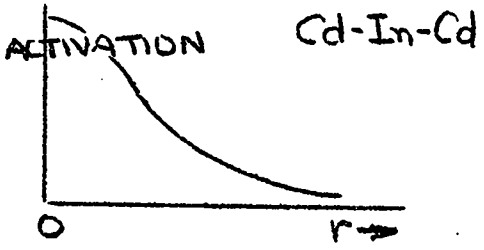
It is because neutrons are slowed down by collisions, that the treatment of most problems in which neutron fluxes are introduced into media requires some knowledge of the nature of the slowing down process. The simplest question to ask is "Given a point source of monoenergetic neutrons, what is the steady-state spatial distribution as a function of energy?" The solution will be basic, for any source distribution can be considered a superposition of point sources. Consider a Ra-Be source in a large tank of water. For hydrogen, the scattering cross-section is particularly large at low energies and so a neutron does most of its traveling on the first one or two of its paths if it starts with 10^6 volts. One could investigate the distribution of neutrons from the source in water by using detectors sensitive to different energy neutrons.

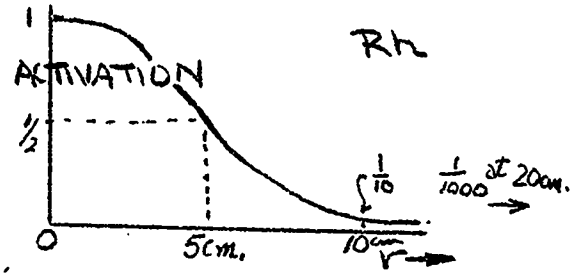


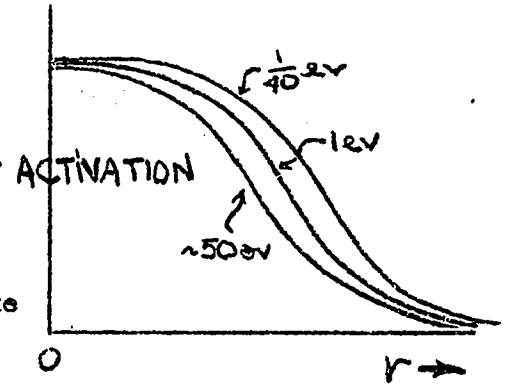
For example one might use an indium foil and get a plot such as the one at the right. The cross-section for indium is shown at the left. To make the indium more useful, we prevent its responding to thermal neuts by surrounding it with cadmium foils. We could use Rh foils (~1 ev resonance energy) too. We could use a Cd-In-Cd sandwich and subtract it from the simple In (unshielded) activation in order to get the distribution of thermal neutrons in space. To detect the spatial distribution of neutrons of about 50 volts one would use iodine. One would get for the detectors mentioned a set of curves like that shown. The curves here are all normalized to unit activation at r=0.











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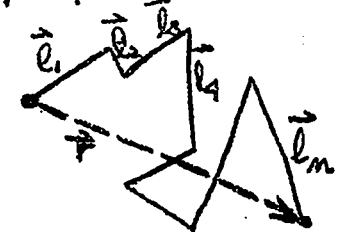
(3) <u>Distribution of Neutrons from a Point Source -- the Calculation of $\overline{r^2}$.</u>

We would now like to solve the following problem. A point source of neutrons of energy E_0 is located in some medium. The neutrons get slowed down by collisions upon leaving the source. Consider all the neutrons of energy E. How far away from the origin or source are they on the average? What is their $\overline{r^2}$?

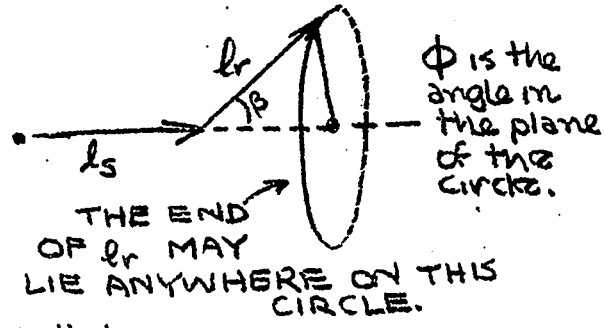


Consider a typical path. The resultant displacement

$$\vec{r} = \vec{l}_1 + \vec{l}_2 + \cdots\cdots\cdots + \vec{l}_n$$

and $$r^2 = l_1^2 + l_2^2 + \cdots\cdots + l_n^2 + 2(\vec{l}_1 \cdot \vec{l}_2 + \vec{l}_1 \cdot \vec{l}_3 + \cdots\cdots + \vec{l}_2 \cdot \vec{l}_3 + \cdots\cdots)$$

We would like to average this expression, and shall do one parameter at a time. First consider all the lengths l fixed and all the angles β fixed, but not fixed as to ϕ (see diagram). Later we shall average over various lengths l and angles β, but for the moment let us do this part of the averaging.



Under these assumptions we should like to prove that

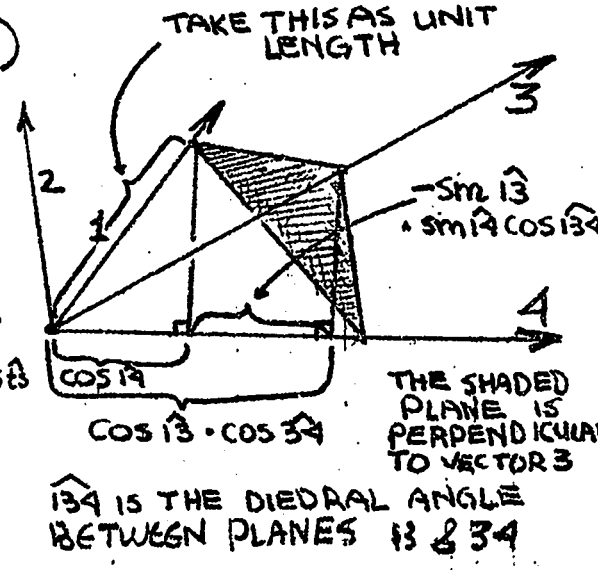
$\overline{\cos \widehat{14}} = \cos \widehat{12} \cos \widehat{23} \cos \widehat{34}$ where $\cos \widehat{rs}$ is the cosine of the angle between $\vec{l}_r$ & $\vec{l}_s$. From the diagram at the right

$$\cos \widehat{14} = \cos \widehat{13} \cos \widehat{34} + \sin \widehat{13} \sin \widehat{14} \cos \widehat{134}$$

On the average $\overline{\cos \widehat{134}} = 0$ (As 4 rotates about 3)

Hence $\overline{\cos \widehat{14}} = \overline{\cos \widehat{13}}\, \overline{\cos \widehat{34}}$ and now we repeat the same scheme and again an average of a diedral angle vanishes

$\overline{\cos \widehat{13}} = \overline{\cos \widehat{12}} \cos \widehat{23}$ But $\cos \widehat{12}$ is β_{12}, a constant. Hence in general, $\overline{\cos \widehat{rs}} = \cos \widehat{rl} \cos \widehat{lm} \cdots \cos \widehat{st}$ where $l_r, l_l, l_m \cdots\cdots$ are successive vectors.



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Now consider the mean free paths. The probability that l_r lies between l_r and $l_r + dl_r$ is $e^{-\frac{l_r}{\lambda_r}}$ From this we can get $\overline{l_r} = \int_0^\infty l_r e^{-\frac{l_r}{\lambda_r}} \frac{dl_r}{\lambda_r}$ and $\overline{l_r^2} = \int_0^\infty l_r^2 e^{-l_r/\lambda_r} \frac{dl_r}{\lambda_r}$ These integrals are of the form $\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n-}}$ and we find $\overline{l_r} = \lambda_r$ the mean free path (this is why λ_r can be called the mean free path) and $\overline{l_r^2} = 2\lambda_r^2$.

Recalling the results of our average over angles and using these last results for the average l's, $\overline{r^2}$ for the set of angles $\beta_{12} = \widehat{12}$, $\beta_{23} = \widehat{23}$ ..... is

$$\overline{r^2} = 2\lambda_1^2 + 2\lambda_2^2 + \ldots\ldots 2\lambda_n^2 + 2\left[\lambda_1\lambda_2\cos\widehat{12} + \lambda_1\lambda_3\cos\widehat{12}\cos\widehat{23} + \ldots\ldots + \lambda_2\lambda_3\cos\widehat{23} + \ldots\ldots\right]$$

In the case of hydrogen this calculation can be completed in an exact way. In general, however, it is now convenient to make some approximations. Let us assume it takes a large number of collisions to produce a small change in energy. This is more true the heavier the atom collided with.

The angles $\widehat{12}$, $\widehat{23}$ etc., may take on all values from zero to π. To determine $\overline{r^2}$ we must average over the various possible angles. Notice that the coefficient of $\lambda_1 m \overline{r^2}$ is

$$2\left[\lambda_2\cos\widehat{12} + \lambda_3\cos\widehat{12}\cos\widehat{23} + \ldots\ldots\right]$$

But since $\overline{\cos\beta} = \frac{2}{3A}$ and since later terms consist of products of an increasing number of such cosines, for A large these terms rapidly become small. One would need but the first few terms in the series to approximate the coefficient of λ_1. Furthermore, the first few λ's will not differ appreciably from λ_1. $\cos\widehat{12}$ is on the average $\frac{2}{3A}$. Call this C. $\cos\widehat{12}\cos\widehat{23}$ is C^2 on the average. The coefficient of λ_1 is therefore $2\lambda_1(C + C^2 + C^3 + \ldots\ldots)$ There will not be much error if we consider this an infinite series. The true $\overline{r^2}$ is then

$$2\lambda_1^2 + 2\lambda_2^2 + \ldots\ldots\ldots\ldots + 2\lambda_1^2\frac{C}{1-C} + 2\lambda_2^2\frac{C}{1-C} + \ldots\ldots = \frac{2}{1-C}\sum_{i=1}^{n}\lambda_i^2$$

This can be written as an integral over the λ's rather than a sum.

$$\overline{r^2} = \frac{2}{1-\frac{2}{3A}}\int_{\log E}^{\log E_0}\frac{\lambda^2(\epsilon)\,d\epsilon}{\xi} = \frac{2}{\xi\left(1-\frac{2}{3A}\right)}\int_E^{E_0}\frac{\lambda^2(E)\,dE}{E} \quad \left[\begin{array}{c}\text{VALID FOR}\\ A \gg 1\end{array}\right]$$

where $\epsilon = \log E$, $\xi = \log\frac{E_0}{E}$

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It was mentioned that for hydrogen the formula for $\overline{r^2}$ could be worked out exactly. The result is

$$\overline{r^2_{E_1}} = 2\lambda^2(0)+2\lambda^2(a)+2\int_0^a \lambda^2(x)\,dx + 2\lambda(0)\int_0^a \lambda(x)\,e^{-\frac{x}{2}}dx + 2\lambda(0)\,\lambda(a)\,e^{-\frac{a}{2}}$$
$$+2\lambda(a)\int_0^a \lambda(x)\,e^{-\frac{a-x}{2}}dx + 2\int_0^a \lambda(u)\,du\int_0^{a-u}\lambda(u+x)\,e^{-\frac{x}{2}}dx$$

where $x \equiv \log_e \frac{E_0}{E}$ $\quad a = \log_e \frac{E_0}{E_1}$

from about 1 volt (chemical binding forces no longer play a part) to ~ 10 KV, σ for neutrons in hydrogen is fairly constant. For slowing down in this region, it can be shown that (because λ is fairly constant).

$$\overline{r^2_{E_1}} = f(E_0) - 6\lambda^2 \log_e E_1$$

This is derived from the complete formula for slowing down in Hydrogen.

Using RaBe as a neutron source in a tank of water, we find the following data experimentally.

Slowing to Rh resonance (~ 1 volt) $\overline{r^2} = 276.6\,cm^2$

" " I " (~ 50 ") $\overline{r^2} = 262.2\,cm^2$

Using the formula for $\overline{r^2_{E_1}}$ that holds from 1ev to ~10KV and subtracting r^2 of I from that of the Rh resonance neutrons

$$\overline{r^2}(Rh) - \overline{r^2}(I) = 6\lambda^2 \log_e \frac{50}{1} = 14.4\,cm^2$$

Since $\log_e 50$ is ~ 3.9, this implies that $\lambda = .61$. In this way, one can get the mean free path in water at energies from 1 to about 10 KV. This is an average mean free path and compares well with actual appropriate averages of differential data.

r	0	1	2	3	4	5	6	7	8	9	10	11	12
A(Rh)	1000	960	890	710	630	500	390	290	210	150	110	88	67

r	13	14	15	16	17	18	19	20
A(Rh)	57	39	29	23	19	16	12	10

r = Distance from point source in cm.
A = Activity Induced in Detector

From this data calculate $\overline{r^2}$. You might have to extrapolate to greater distances. This is best done by using the last 5 or 6 points on a semilog plot.

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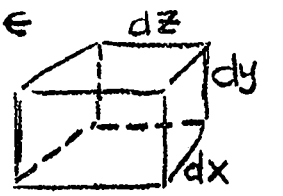
(4) Distribution of Neutrons from a Point Source -- the Age Equation.

In the last section we have discussed one description of the space distribution of neutrons in a tank. Namely $\overline{r^2}$. We shall here discuss another. Call $\epsilon = \log E$ once again and assume that neutrons are fed into a system at energy E_0 within a certain region. We shall be interested in the space distribution of neutrons of various energies. Accordingly define a neutron density $n(x,y,z,\epsilon)d\epsilon$ which is the number of neuts per unit volume at x, y, z in energy range ϵ to $\epsilon + d\epsilon$



Consider a volume element and the neutrons in it in the given energy range. Per unit time this volume will receive neuts in this range from 2 sources. First from diffusion of neuts of this energy from other volumes. Second from the degradation of neutrons of higher energy. Consider first the diffusion. From kinetic theory the diffusion coefficient is

$$D = \frac{\lambda v}{3(1-\overline{\cos\beta})} \quad *$$

(where we recall that $\overline{\cos\beta} = \frac{2}{3A}$)

and is used as follows. Consider the face $dz\,dy$ of our volume element. The net number of neutrons in the energy range $d\epsilon$ diffusing out across this face is

$$d\epsilon \cdot D \left(\frac{\partial n}{\partial x}\right) dy\,dz \qquad \text{per unit time}$$

The number of neutrons diffusing in across the opposite face is

$$d\epsilon \cdot D \left(\frac{\partial n}{\partial x} + \frac{\partial^2 n}{\partial x^2} dx\right) dy\,dz$$

The net number entering the volume across this opposite pair of faces is

$$d\epsilon \; D \frac{\partial^2 n}{\partial x^2} \, dx\,dy\,dz$$

and the total number of neuts in the energy range $d\epsilon$ diffusing into the volume per unit time is

$$D \nabla^2 n \, d\epsilon \qquad \text{per unit volume.}$$

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* $\frac{\lambda}{1-\overline{\cos\beta}}$ is called the transport mean free path.

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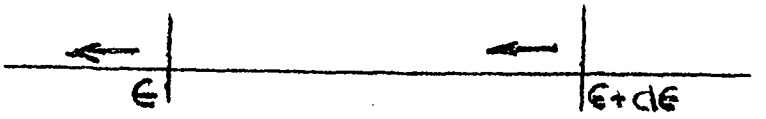
Next consider the other source of neutrons in the energy range $d\epsilon$ at ϵ. $\frac{v}{\lambda}$ would be the number of collisions of a neutron in unit time. $\xi\frac{v}{\lambda}$ gives the degradation of ϵ in unit time. Consider the interval ϵ to $\epsilon + d\epsilon$ on an ϵ axis



$\xi\frac{v}{\lambda}$ IS THE VELOCITY WITH WHICH A NEUTRON PROCEEDS DOWN THE AXIS.

At ϵ the number of neutrons per unit volume moving out of the range in question in unit time is

$$\xi \frac{v(\epsilon)}{\lambda(\epsilon)} n(\epsilon)$$

Similarly per unit volume per unit time

$$\left[\frac{v(\epsilon)}{\lambda(\epsilon)} n(\epsilon)\xi + d\epsilon \frac{\partial}{\partial \epsilon}\left(\frac{vn}{\lambda}\xi\right)\right]$$

neutrons enter the energy range in question at $\epsilon + d\epsilon$.

In the stationary state there is no net change in n at any point. Hence considering both "sources" of neutrons,

$$D\Delta n + \frac{\partial}{\partial \epsilon}\left(\xi \frac{vn}{\lambda}\right) = 0 \qquad \text{where } D = \frac{\lambda v}{3\left(1-\frac{2}{3A}\right)}$$

This is then the slowing down differential equation for the steady state. It is often more convenient to deal with

$$\xi \frac{vn}{\lambda} \equiv q \qquad \text{the so-called slowing down density}$$

rather than with n itself. The reason for this name for q follows from the fact that, as was noticed, $\xi\frac{v}{\lambda}$ is the velocity of a neutron along the ϵ axis. $n\xi\frac{v}{\lambda}$ is then the number of neuts per unit volume per unit time crossing any value ϵ on the ϵ axis. In the steady state if q is integrated overall space the number of neuts crossing any energy value ϵ per unit time is certainly the number of neutrons fed into the system in unit time, a constant, i.e. $\int q(\epsilon)\,dx\,dy\,dz$ = constant for all ϵ. Thus q is in some sense a simpler quantity than n. This is indicated

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somewhat by the differential equation we get if we write our differential equation in terms of q rather than n.

$$\frac{\lambda^2(\epsilon)}{3\xi\left(1-\frac{2}{3A}\right)}\Delta q + \frac{\partial q}{\partial \epsilon} = 0$$

Here the differential operators operate on q only -- no longer on $\lambda(\epsilon), v$ etc. In addition to transforming the dependent variable, let us transform the independent variable ϵ according to

$$\tau \equiv \frac{1}{3\xi\left(1-\frac{2}{3A}\right)}\int_{\epsilon}^{\epsilon_0} \lambda^2(\eta)\, d\eta$$

WHERE τ IS CALLED THE "AGE". IT HAS DIMENSIONS (LENGTH)^2

Then $\frac{\partial q}{\partial \epsilon} = \frac{\partial q}{\partial \tau}\frac{\partial \tau}{\partial \epsilon} = -\frac{\partial q}{\partial \tau}\frac{1}{3\xi\left(1-\frac{2}{3A}\right)}\lambda^2(\epsilon)$ WHENCE $\Delta q = \frac{\partial q}{\partial \tau}$

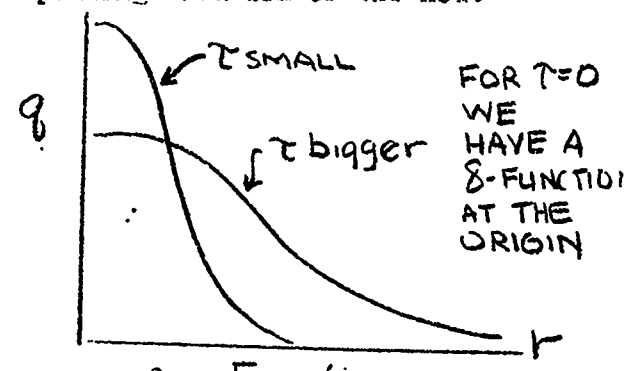
a simple and well known classical equation (recall heat conduction).

Why τ is called "age". The heat conduction equation is $\Delta T = \frac{c\rho}{k}\frac{\partial T}{\partial t}$ and τ is seen to correspond to t, the time in this equation. Perhaps a better reason for the name is that "at birth" the neutron has energy ϵ_0. As time goes on ϵ goes down and the integral τ increases as the "time since birth" or the "age" of the neutron increases.

> Consider a substance where $\lambda =$ constant. What is, in this case, the relation between τ and the actual "time from birth"?

To use our age differential equation we must specify boundary conditions. Let us do the point source problem to begin with. Given a point source of neutrons of energy E_0 in an infinite medium (This corresponds somewhat to our experimental tank of water with the RaBe source in the middle.) what is n as a function of ϵ and position? We first solve the age equation for q. (Let us borrow the corresponding solution of the heat equation, which is rather well-known.)

$$q = \frac{Q}{(4\pi)^{3/2}\tau^{3/2}}e^{-\frac{r^2}{4\tau}}$$



The solution is a Gaussian function of r. Q is the number of neutrons of energy E_0 introduced into the system in unit time, the source strength. The solution clearly shows that as E decreases from E_0 (i.e. as

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τ increases) the space distribution for the energy E gets broader and broader. This is as it should be. Fast neutrons are distributed close to the source. Slow neutrons are considerably spread out.

It was mentioned before that q has the property that $\int q\, dx\, dy\, dz = \text{const.}$ Let us check this with our particular solution

$$4\pi \int_0^\infty q r^2 dr = \frac{4\pi Q}{4\pi \tau^{3/2}} \int_0^\infty e^{-\frac{r^2}{4\tau}} r^2 dr$$

$$= Q \text{, a constant}$$

$$\text{since } \int_0^\infty e^{-\alpha x^2} x^2 dx = \frac{\sqrt{\pi}}{4\alpha^{3/2}}$$

From q calculate $\overline{r^2}$ and see that it is the same as our value of the previous section.

This establishes that Q is truly the source strength as we have asserted.

It might be remarked that this Gaussian distribution for q of a particular τ could have been arrived at without the detailed calculation. Let us consider n instead of q (The space distribution of q and n for the same τ is the same except for a multiplying function of τ). The spatial distribution of n can be arrived at somewhat as follows. Consider the x axis with the source at the origin. We are interested in the distribution n as a function of x for a particular τ. A particular τ means a particular number of collisions since birth, or a particular number of path lengths travelled. If the number of paths is a large number and all mean free paths are small compared to the total distance travelled (and such is the case) then we can use a theorem in the study of large numbers which says, "That if one sums a large number of numbers which are equally likely positive & negative, and whose magnitudes are all less, by far, than the sum of their magnitudes, one finds that the values of these sums (always taken for the same number of small numbers) distribute on a Gaussian curve provided no additional information is known about the numbers summed." (This theorem, incidentally can be used to show why repeated measurements of some physical quantity like a length fall on a Gaussian curve, provided that the errors are what we call random.) In our case the theorem implies that $n_x \sim e^{-\frac{x^2}{K}}$ where K is some measure of the width of the distribution. Similarly for the y and z directions. Hence $n = n_x n_y n_z$ goes as $e^{-\frac{r^2}{K}}$.

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CHAPTER VII

The Distribution of Slow Neutrons in a Medium

(1) The Differential Equation for Slow Neutrons.

To find the distribution of neutrons of various energies in a medium, one usually deals with two separate problems. First there is the slowing down problem, and this was dealt with in the last chapter. But the neutrons do not continue to get slowed down indefinitely, for the nuclei they collide with are not at rest, but have vibrational energies corresponding to their temperature. The neutrons eventually come into thermal equilibrium with these nuclei, and show a Maxwellian distribution corresponding to the temperature of the medium. Clearly the problem of the distribution of these slowed down or thermal neutrons is quite distinct from that of the distributions of the neutrons being slowed down and must be handled by separate methods.

In approaching this second problem we ask, "Given a source of thermal neuts, what can be said about their distribution in a medium in a stationary state?" We seek a differential equation as our description. Let $n(x, y, z)$ be the density of thermal neuts at x, y, z. Consider a unit volume. There are 3 mechanisms by which the number of neutrons in this volume change with time. First the diffusion. This gives a net contribution to $\frac{\partial n}{\partial t}$ which is $D\Delta n$. Second, some neutrons are captured or absorbed*. This number will be proportional to n. We write $-\frac{n}{\theta}$ Lastly we have those neutrons generated in the volume by the slowing down to thermal of fast neutrons. This term is q_T the value of q at this place for τ corresponding to thermal energies. Thus the desired differential equation is

$$D\Delta n - \frac{n}{\theta} + q_T = \frac{\partial n}{\partial t} \quad \frac{\text{neutrons}}{\text{c.c. sec}}$$

which is zero in the steady state. Here we must put in the proper space dependence

*This term should perhaps also have been considered in the slowing down process. But whereas orders of magnitude are such that in the slowing down, the consideration of absorption is usually a refinement, here it is a necessity.

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of q_T as determined from the age equation before we attempt to solve this equation under the proper boundary condition. Recalling that $D = \frac{\lambda v}{3}$ the equation can be written

$$\Delta n - \frac{3n}{\lambda v \theta} + \frac{3 q_T}{\lambda v} = 0 \quad \text{or} \quad \Delta n - \frac{n}{\ell^2} + \frac{3 q_T}{\lambda v} = 0$$

for the steady state. Here $\ell = \sqrt{\frac{\lambda v \theta}{3}}$ and is called "the diffusion length". $v\theta$ is often written Λ and is the capture mean free path.

For a point source of slow neutrons, the solution is $n = A \frac{e^{-r/\ell}}{r}$ where A is a normalization coefficient. For a point source, the last term in the equation is zero everywhere in space (except that we have a delta-function source at the origin.) In order to evaluate A, consider a small sphere drawn about the source, which we assume emits 1 slow neutron each second. Then for $r \ll \ell$, $n \cong \frac{A}{r}$ and the flux through the sphere is $D \times 4\pi r^2 \times \text{grad}\left(\frac{A}{r}\right)$ which must be equal to one. Since $D = \frac{\lambda v}{3}$ this gives $A = \frac{3}{4\pi \lambda v}$ One can check the now complete solution of the homogeneous equation

$$n = \frac{3}{4\pi \lambda v} \frac{e^{-r/\ell}}{r}$$

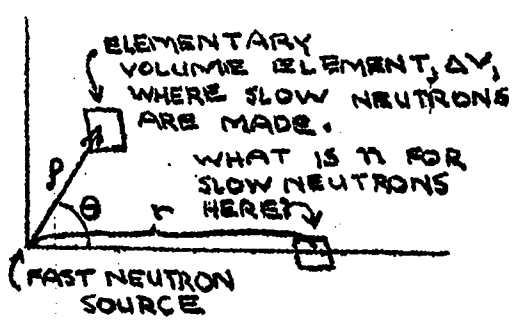
(The name "diffusion length" for ℓ stems from its role in this expression.)

by substituting into

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) n - \frac{3n}{\lambda v \theta} \quad \left(\text{recalling } \ell^2 = \frac{\lambda v \theta}{3}\right)$$

This should, of course, be zero, and it is. The point source solution is particularly important because any source can be represented by a proper assembly of point sources and the corresponding solution would be a superposition of these point source solutions.

To solve this equation for a point source of <u>fast</u> neutrons, one can set up an integral over a distribution of slow neutron point sources all over space that arise from the slowing down of the fast neutrons. We ask "What is the density of slow neuts at a distance r from the point source of fast neutrons?" At any point there are q_T point sources of slow neutrons per unit volume where $q_T = \frac{Q}{(4\pi\tau)^{3/2}} e^{-\frac{\rho^2}{4\tau}}$ where τ



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corresponds to thermal neuts. But for a point source of slow neutrons at (ρ, θ) the slow neutron density at r is

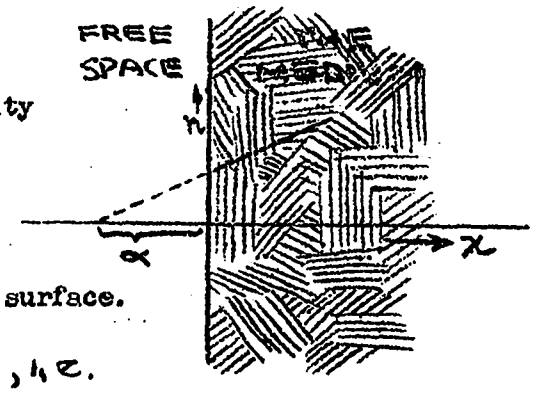
$$n = \frac{3}{4\pi\lambda v} \frac{e^{-\frac{|\vec{\rho}-\vec{r}|}{\ell}}}{|\vec{\rho}-\vec{r}|} \quad \text{, where } |\vec{\rho}-\vec{r}| = \sqrt{\rho^2 + r^2 - 2\rho r\cos\theta}$$

Thus the density of slow neutrons at r is given by the integral

$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\int_{\theta=0}^{\pi} \frac{Q}{(4\pi\tau)^{3/2}} e^{-\frac{\rho^2}{4\tau}} \cdot \frac{3}{4\pi\lambda v} \frac{e^{-\frac{|\vec{\rho}-\vec{r}|}{\ell}}}{|\vec{\rho}-\vec{r}|} \, d\rho\, \rho^2 \sin\theta \, d\theta \, d\phi$$

(2) <u>The Boundary Conditions for the Differential Equation.</u>

In order to solve the slow neutron differential equation, one must know how n or some function of n behaves at the spatial boundaries. Consider a finite convex medium with a neutron source in it and free space everywhere around it. What can be said of q or n at the bounding surface? To a first approximation one can take q or n equal to zero. This is made somewhat plausible by the argument that free space acts as a perfect sink. It absorbs all neutrons and returns none. It therefore acts as so heavy a drain on the neutron density at the boundary that no density can be maintained there. Actually a more refined calculation will show that a more proper boundary condition is the vanishing of n or q at a surface $0.66\,\lambda$ away from the bounding surface (where λ is the neutron mean free path in the medium). An even more refined argument leads to $0.70\,\lambda$ as the distance of this outside surface from the true boundary. We shall consider at least the first refinement.



Consider a plane bounding surface. We shall assume that in the neighborhood of the boundary the neutron density n is a linear function of distance, i.e.

$$n = p(\alpha + x)$$

Let us now calculate the neutron flux Φ at the bounding surface. Certainly it is gradient $n \times$ (the diffusion coefficient), i.e.

$$\Phi = p \cdot \frac{\lambda v}{3} \qquad \text{for } p = \frac{dn}{dx} \quad \text{in this problem.}$$

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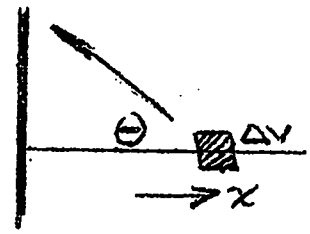
The flux can, however, also be calculated as follows. Consider a unit volume at a distance x from the boundary. There are $n(x)$ neutrons here. The probability that a neutron going off at the angle θ will escape is $e^{-\frac{x}{\cos\theta}} \cdot \frac{1}{\lambda}$



Assuming that neutrons leave this volume in an isotropic fashion, the number of neutrons crossing the boundary per unit time coming from this volume is

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} p(\alpha+x)\frac{v}{\lambda}\, e^{-\frac{x}{\lambda\cos\theta}} \sin\theta\, d\theta \cdot \Delta V$$

Consider that the volume has unit area perpendicular to the x axis and depth dx Then to integrate the above expression over x, would be to find the total amount of neutrons coming each second to the surface from an infinite column of unit cross-sectional area perpendicular to the surface. But this is readily seen to be the flux, or the number of neutrons crossing unit area of the surface per second, provided, of course that everything can be assumed uniform perpendicular to the x direction.

Hence

$$\phi = \frac{1}{2}\int_0^{\frac{\pi}{2}} \int_0^{\infty} p(\alpha+x)\frac{v}{\lambda}\, e^{-\frac{x}{\lambda\cos\theta}} \sin\theta\, d\theta\, dx$$

and writing $\mu = \cos\theta$ this integral is seen to be a rather simple one. The solution gives

$$\phi = \frac{vp}{2\lambda}\left(\frac{\alpha\lambda}{2} + \frac{\lambda^2}{3}\right)$$

Equating this to our previous result for ϕ $\left(= p\frac{\lambda v}{3}\right)$ we get $\alpha = \frac{2}{3}\lambda$

But α is the x intercept. Therefore we have shown that n vanishes at a distance $\frac{2}{3}\lambda$ outside the bounding surface. This is the boundary condition we sought to derive. It is to be noted that this boundary condition really describes n at the <u>boundary</u>. The x intercept is used only to describe this. In particular, the boundary condition does not mean that n vanishes at $x = -\frac{2}{3}\lambda$ and is negative beyond that distance. Our conclusion is simply "The density at a bounding surface behaves as though n is a linear function of x vanishing at $x = -\frac{2}{3}\lambda$."

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(3) The Diffusion Length

In order to be able to use the slow neutron differential equation one must know the value of ℓ. To measure ℓ in the case of water, for example, one can use a pile and a tank of water. Neglecting end effects this is essentially a one dimensional problem.

$$\frac{\partial^2 n}{\partial x^2} - \frac{1}{\ell^2} n = 0$$

and the non-diverging solution is $n = A e^{-\frac{x}{\ell}}$ One makes measurements with and without cadmium separating the pile and water tank to ascertain (by a subtraction) n for thermals as a function of x in the water. From the data obtained for n we can evaluate ℓ for water. It is ~ 2.8 cm. For paraffin we get a result that shows that the absorbing mechanism in the case of water and paraffin is the same, hydrogen capture, (i.e., one need assume no carbon or oxygen absorption to account for the observed ℓ's)

There are other schemes to determine ℓ and related constants for H_2O. Some are described in a paper by Fermi and Amaldi (P.R. 19 899-36). ℓ shows a temperature dependence and it is sometimes of interest to know this temperature dependence. For water it has been measured and $\ell = 2.64 + 0.0061\, T$ ← In degrees Centigrade

Since $\ell^2 = \frac{\lambda \Lambda}{3}$ where Λ is the capture or absorption mean free path and λ is the transport mean free path $\frac{\lambda}{1-\cos\theta}$ (p. 49), we can use our information about ℓ and λ to calculate Λ or vice versa.

Calculate λ from information about ℓ and Λ for water. $\ell = 2.8$ cm and $\Lambda = \frac{1}{\sigma_c n_c}$ where σ_c at 2200 m/sec = .31b. 2200 $\frac{m}{sec}$ corresponds to v in $\frac{1}{2} m v^2 = kT$ for room temperatures. But if there is a Maxwell distribution $\bar{v} = \frac{2}{\sqrt{\pi}} v$, i.e. $\bar{v} = \left(\frac{8kT}{\pi m}\right)^{1/2}$ as one can easily show for water at room temperature, we would want σ_c corresponding to $\bar{v}$ not v. Calculate n_c, Λ and finally λ, the transport mean free path.

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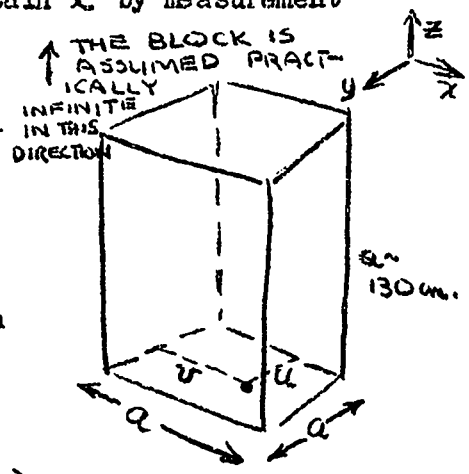
For a substance such as graphite, the procedure of last page does not sufficiently represent a one-dimensional problem for us to obtain ℓ by measurement as simply as for water. (ℓ becomes a length of the order of the dimensions of the medium.) One must solve the 3 dimensional problem. Consider a block of graphite as that shown at the right with a fast neutron source placed at $(u, v, 0)$.



It can be shown that the solution of the slowing down equation

$$\Delta q - \frac{\partial q}{\partial \tau} = 0$$

is, in this case

$$q = \frac{4}{a^2} \frac{Q}{2\sqrt{\pi\tau}} e^{-\frac{z^2}{4\tau}} \sum_{r,s=1}^{\infty} e^{-\frac{\pi^2\tau}{a^2}(r^2+s^2)} \sin\frac{\pi r y}{a} \sin\frac{\pi s x}{a}$$

where Q is the source strength. This solution is obtained by fairly straightforward Fourier methods.

Experimentally it is found that a Ra-Be neutron source has 3 well defined neutron energies. Such a source gives 3 superposed distributions.

	%	τ Indium Resonance	τ Thorium Resonance	τ Iodine Resonance
1)	15.0	130 cm^2	183	54
2)	69.3	340	396	268
3)	15.7	815	867	736

This table gives the value of τ for the 3 components in graphite at different energies, and the % of each component present. Sometimes such data is given in terms of the range r_0 which is equal to $2\sqrt{\tau}$

From the numbers in this table, it is seen that at 30 or 50 centimeters from the source q becomes very small. Then it will be true that the "source" term in the slow neutron diffusion differential equation will be small. The slow neutron density, n, will obey quite closely

$$\Delta n - \frac{1}{\ell^2} n = 0$$ at such distances from the source.

Because of the boundary conditions, one assumes a solution of the form

$$n = \sum_{r,s=1}^{\infty} n_{rs}(z) \sin\frac{\pi r y}{a} \sin\frac{\pi s x}{a}$$

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Substituting into the differential equation, we get an equation on $n_{rs}(z)$

$$\frac{d^2 n_{rs}}{dz^2} - \left[\frac{\pi^2}{a^2}(r^2+s^2) + \frac{1}{\ell^2}\right] n_{rs} = 0$$

This is of the form

$$\frac{d^2 n_{rs}}{dz^2} - \frac{1}{b_{rs}^2} n_{rs} = 0$$

and the solution is well known, $e^{\pm z/b_{rs}}$ of which we omit the diverging solution $e^{+z/b_{rs}}$, so that

$$n = \sum_{r,s=1}^{\infty} e^{-\frac{z}{b_{rs}}} \sin\frac{\pi r y}{a} \sin\frac{\pi s x}{a}$$

where

$$\frac{1}{b_{rs}^2} = \left[\frac{\pi^2}{a^2}(r^2+s^2) + \frac{1}{\ell^2}\right]$$

Because for $r, s > 1$ the solution for n is one quickly damped, experimentally it will suffice to compare activation measurements by slow neuts in such a column with the $r=s=1$ component,

$$n = e^{-\sqrt{\frac{2\pi^2}{a^2}+\frac{1}{\ell^2}}\, z} \sin\frac{\pi y}{a} \sin\frac{\pi x}{a}$$

For a typical graphite sample, we have constants

$\rho = 1.551 \frac{gm}{c.c.}$ $\qquad b_{11} = 28.38$ $\qquad a = 150.49$ cm

(we must add $2 \times \frac{2}{3}\lambda$ to "a" [p.51] so that the effective dimension, $a = 150.49 + \frac{4}{3}\lambda$, which happens to be 153.29 cm ($\lambda =$ 2.1 cm)) This information and some good measurements will allow us to find ℓ. Since ℓ depends on density, by convention, ℓ for all samples of graphite (graphite is tested by measuring its ℓ in this way) is reduced to the value it would have at $\rho =$ 1.60 in order that one may compare different samples of graphite and make statements as to their purity.

Here we briefly record some of the results obtained in this way for 4 common substances.

	ρ	Atoms/c.c.	ℓ	ℓ^2/λ
H_2O	$1.0 \frac{gm}{c.c.}$	$.0334 \times 10^{24}$	2.85 cm.	0.142 cm
D_2O	1.1	.0331	>100	.80
Be	1.8	.1235	31	.87
C	1.62	.0871	50.2	.903

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Actually the 2 microscopic constants in such problems are σ_{scat}, and σ_{abs}. For convenience let us list some of the relations between these and the more easily measured constants.

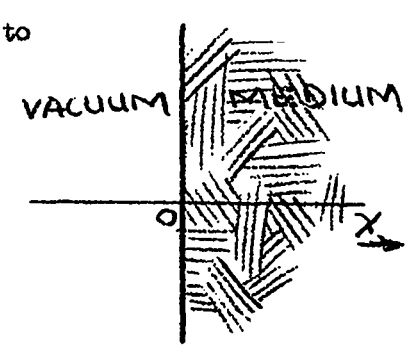
$$\lambda = \frac{1}{\sigma_{sc} \cdot n} \; ; \; v\theta = \Lambda = \frac{1}{\sigma_{ab} \cdot n} \; ; \; l^2 = \frac{\lambda \Lambda}{3} \; ; \; N = \frac{\sigma_{sc}}{\sigma_{ab}} = \frac{3 l^2}{\lambda^2} = \frac{\Lambda}{\lambda}$$

where N is the average number of scattering collisions made per capture.

(4) <u>The Albedo, or the Reflection Properties of Bounding Surfaces for Neutrons.</u>

We have so far in this chapter outlined the methods of obtaining neutron distributions in media due to sources within them. However, neutrons are often introduced into a medium from the outside. It is convenient to define a reflectivity, or as it is called, albedo (whiteness - Latin) for a surface. It is simply the fraction of the incident neutrons eventually returned or "reflected" from the surface. An albedo of one means perfect reflection; an albedo of zero means perfect (black body) absorption. See Fermi's paper on the motions of neutrons in hydrogenous substances "Sul moto dei neutroni (lenti) nelle sostanze idrogenate" in Ric. Sci. 7, 13, 1936.

Let us solve a typical problem. We shall calculate the albedo of an infinite plane surface for slow neutrons. But first it shall be necessary to solve the following problem. "Given a medium bounded at $x=0$ and occupying all of space to the right of this plane, what is the probability that a slow neutron starting at a point d units distant from $x=0$ will escape from the medium, i.e. will reach $x=0$?"



We shall do this problem twice, using two very different approaches. First we shall use the slow neutron diffusion equation and assume that the problem is one dimensional, i.e., that the neutrons move only in the x-direction.

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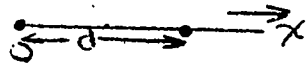
Assume a point source of neutrons at d on the x axis.



We must calculate the flux at the origin for unit source strength (for this is precisely the escape probability). The slow neutron diffusion equation is in this case,

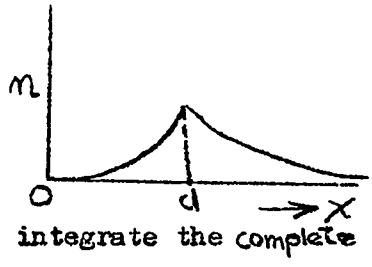
$$\Delta n - \frac{n}{\ell^2} = 0$$

(everywhere but at $x=d$) and since this is a one dimensional problem the solution is

$$n = Ae^{-\frac{x}{\ell}} - Ae^{\frac{x}{\ell}}$$

for $0<x<d$, assuming the boundary condition $n(0)=0$

For $x>d$, $n=Be^{-\frac{x}{\ell}}$ These two solutions must join at d so that n is continuous and the gradient $\frac{dn}{dx}$ has a finite discontinuity. To find the amount of this discontinuity, integrate the complete diffusion equation throughout a small "volume" surrounding the source.



$$\int_{d-\epsilon}^{d+\epsilon} \frac{d^2n}{dx^2}\,dx - \int_{d-\epsilon}^{d+\epsilon} n\,dx = -\int_{d-\epsilon}^{d+\epsilon} \frac{q}{D}\,dx \qquad \text{(p. 49)}$$

The first term is $\left.\frac{dn}{dx}\right|_{d-\epsilon}^{d+\epsilon}$, the discontinuity in $\frac{dn}{dx}$. The second term vanishes as $\epsilon \to 0$ for n must for physical reasons be a continuous function of x. The last term is simply $-\frac{Q}{D}$ where Q is the source strength. Thus the conditions on n and $\frac{dn}{dx}$ at d are:

$$Ae^{-\frac{d}{\ell}} - Ae^{+\frac{d}{\ell}} = Be^{-\frac{d}{\ell}}$$

$$-\frac{A}{\ell}e^{-\frac{d}{\ell}} - \frac{A}{\ell}e^{\frac{d}{\ell}} = -\frac{B}{\ell}e^{-\frac{d}{\ell}} + \frac{Q}{D}$$

whence $A = -\frac{\ell Q}{2D}e^{-\frac{d}{\ell}}$, $B = \frac{\ell Q}{2D}\left[1-e^{-\frac{2d}{\ell}}\right]$

The flux at $x=0$ is then $D\cdot\left(\frac{dn}{dx}\right)_{x=0} = Qe^{-\frac{d}{\ell}}$, and for unit source strength it is simply $e^{-\frac{d}{\ell}}$ This gives the probability, $p(d)$ that a slow neutron at a will eventually escape from the medium.

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Before we use this result to find the albedo of such a one-dimensional medium for neutrons incident on the boundary from the outside, let us redo the problem in another way. In addition to the methods of diffusion, there is a more exact, more rigorous way to attack problems of the type being discussed. To find the neutron density in a prticular volume V at a time t, one could investigate the density of neutrons that are moving toward V and are in other volumes at various earlier times t' so that (considering their velocities and distances from V) they would be in V at the time t. The neutron density at V at the time t, could be expressed as some sort of sum or integral of these other neutron densities. We would be led to an integral equation in the neutron density n.

Thus in addition to the differential equation method of solving diffusion problems, there is an integral equation method too. It would be well to stop a moment and compare the relative merits of the two approaches. In setting up the diffusion differential equation it had to be assumed the quantities like n, $\frac{dn}{dx}$ etc. vary slowly with respect to the mean free path of the diffusing particles. Further it was assumed that densities of particles were large enough so that speaking of quantities like $\frac{dn}{dx}$ made sense. In particular one would not expect that the solution of a problem like the following by diffusion methods would give physically true results: - "Find $n(r,\theta,\phi)$, the density of slow neutrons in a sphere of radius $\frac{\lambda}{2}$, if there is a point slow neutron source at the center of the sphere. (λ is the mean free path of slow neutrons in the medium of the sphere.)" There are, however, no such restrictions on the use of integral equation methods. No assumptions about the variation of $\frac{dn}{dx}$ with distance, etc., need be made. Integral equation methods are more general. However, they are usually more difficult. It often becomes expedient to do the diffusion problems by means of the differential equation and proper boundary conditions first in order to get a rough idea about the function in question. Then one gets the more

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exact solution by means of an integral equation. This is the procedure we shall follow here. We have obtained $p(d)$ the probability that a neutron d units from the boundary of a one dimensional medium will escape it, by means of the diffusion differential equation. Let us now apply integral methods.

Consider a neutron at d. As it leaves d, one of two things may occur. It may go to the left or it may go to the right, each with a 50% chance. If it goes to the left, it may escape before it suffers a collision or it may collide. The probability that a neutron at d will escape without a collision is therefore the product

$$\frac{1}{2} \times e^{-\frac{d}{\lambda}}$$

where λ is the total mean free path. However a neutron may escape even it it suffers a collision. Say the neutron suffers its first collision at x, and that $\pi(x)$ is the probability that a neutron from d suffers its first collision at x (either to the right or left of d). $p(x)$ is the probability that a neutron at x will eventually escape. Clearly then, $p(d)$ can be written as some sort of sum or integral

$$p(d) = \frac{1}{2} \cdot e^{-\frac{d}{\lambda}} + \sum_{\text{all } x} \pi(x)\, p(x)$$

where it is assumed that there is isotropic scattering in the lab system, i.e., that $p(x)$ depends only on x and not on the side from which the neutron arrives at x.

In detail the sum should be written:

$$\left.\begin{aligned} &\frac{1}{2}\int_0^d e^{-\frac{d-x}{\lambda}} \frac{dx}{\lambda}\, p(x)\, \frac{N-1}{N} \\ &+ \frac{1}{2}\int_d^\infty e^{-\frac{x-d}{\lambda}} \frac{dx}{\lambda}\, p(x)\, \frac{N-1}{N} \end{aligned}\right\}$$

where the first integral gives the probability that a neutron starting from d will go left, suffer a collision at x, but will eventually escape.

The second integral gives the probability for the same thing with initial motion to the right.

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The factor $\frac{N-1}{N}$ appearing in these integrals gives theprobability that the collision at x is a scattering collision. There are N scattering collisions per absorbing collision. Thus the integral equation is

$$p(d) = \frac{1}{2} e^{-\frac{d}{\lambda}} + \frac{1}{2}\int_0^\infty e^{-\frac{|x-d|}{\lambda}} \frac{dx}{\lambda}\, p(x)\, \frac{N-1}{N}$$

The solution of the differential equation gave $p(d) = e^{-d/\ell}$. Let us therefore try $p(d) = A e^{-\alpha d}$ as the solution of this integral equation. So doing, we find

$$p(d) = \frac{\sqrt{N}}{\sqrt{N}+1}\, e^{-\frac{d}{\lambda\sqrt{N}}}$$

($\lambda\sqrt{N}$ is $\sqrt{3}\,\ell$; see p.55). Thus this solution goes as $e^{-\frac{d}{\sqrt{3}\,\ell}}$ rather than as the differential equation solution $e^{-\frac{d}{\ell}}$.

We are now in a position to do what we originally set out to do -- to find the albedo. Say a beam of slow neutrons moving along the x axis from the left hits the plane $x=0$. That probability that a neutron of the beam will make its first collision in dx at x is $e^{-\frac{x}{\lambda}}\frac{dx}{\lambda}$ The probability of not being absorbed and escaping from here is $p(x)\cdot\frac{N-1}{N}$ Hence the albedo is,

$$\beta = \int_0^\infty e^{-\frac{x}{\lambda}} \frac{dx}{\lambda}\, p(x)\, \frac{N-1}{N} = \frac{\sqrt{N}-1}{\sqrt{N}+1}$$

A non-absorbing medium would eventually return all neutrons and have an albedo of 1.

If we wish to know β for an angle of incidence θ, we are forced to drop the one-dimensional attack and the problem becomes more difficult. The result is that

$$\beta(\theta) = \frac{\sqrt{N}-1}{\sqrt{N}+\sqrt{3}\cos\theta}$$

This is inconsistent with the first result ($\theta=0$), only because this solution allows for the fact that although $\theta=0$ for the incident neutrons, they are not restricted to move along the x axis in the medium. The effect of allowing motion at angles to the x axis is to allow longer paths and hence more chance for absorption. This makes β slightly smaller.

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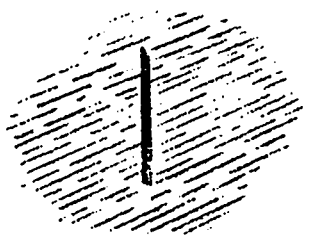
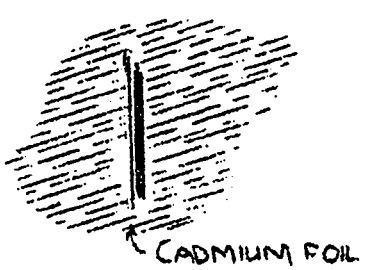
Suppose we wish to measure the albedo for slow neutrons on paraffin. A direct measurement would be difficult, for even if a collimated slow neutron beam can be made to impinge on some paraffin, neutrons would be coming off at all angles from all over the surface of the paraffin and their detection would be no easy matter. A much neater way of finding the albedo is the following. Place a thin foil slow neutron detector somewhere in the middle of a mass of paraffin whose boundaries are far enough away from the foil that the paraffin can be considered infinite in extent. (By a thin foil is meant one where there is little modification of the neutron distribution due to the presence of the foil, i.e., one for which $\sigma n \delta \ll 1$, where σ is the atomic cross-section, n the number of atoms/cc. and δ the thickness of the foil.) By means of some neutron source we induce an activity in the foil. Call this activity A. Next back the foil on one side with some cadmium (enough so that it can be assumed that the cadmium absorbs practically all the slow neutrons hitting it, but not enough to distort the neutron flux field appreciably.) Remeasure the activity in a detector foil. Call this activity B. Now the ratio $\frac{A}{B}$ bears a simple relation to the albedo. To see how this comes about, consider the number of neutrons N hitting the foil each second in situation B. It is clear that for a uniform distribution of slow neutrons, the foil in case A would have N neutrons per second hitting it from each side, and would have at least $2N$ slow neutrons hitting it per second. But that is not all,





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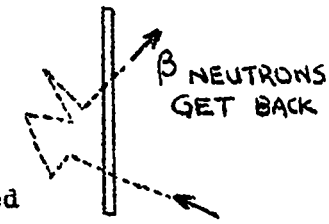

for some of the neutrons passing through the foil can return and pass through it again (for there is no cadmium about to prevent this.) In fact we can calculate the average number of times a neutron about to impinge on the foil will pass through it before it is eventually absorbed in the paraffin. Certainly the probability that this neutron will return through the foil is β the albedo of the paraffin for slow neutrons. The probability that it will make at least two trips is $\beta \times \beta$ or β^2 and so on. Thus the total number of passages through the foil for a neutron about to hit it is on the average

$$1 + \beta + \beta^2 + \beta^3 + \ldots\ldots = \frac{1}{1-\beta}$$

Hence there would be $\frac{2N}{1-\beta}$ slow neutrons hitting the foil each second, rather than simply $2N$ neutrons. From this it follows that $\frac{A}{B} = \frac{2\cancel{N}}{(1-\beta)\cancel{N}}$ Hence a measurement of the two activities A and B suffices to let us calculate the albedo. For paraffin $\frac{A}{B}$ is 11. This makes $\beta = .82$. This type of measurement would not be feasible with poor absorbers of neutrons for it has to be assumed that the diffusion length in the medium is small compared to the foil size, i.e., that most of the "reflection" takes place close to the foil. Further, if the detector is not thin it acts as its own cadmium, so to speak, and a correction must be made for the absorption in the foil.

If ξ is the average fraction (averaged over angles of incidence) of incident neutrons absorbed in the foil, find the relation between $\frac{A}{B}$ and the albedo.

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CHAPTER VIII

Nuclear Fission

(1) The Binding Energies of Nuclei.

Before we proceed to take up the subject of fission, it is necessary to get some understanding of what it is that holds nuclei together. For this purpose we should like to find an expression for the nuclear binding energy [or the nuclear mass -- since mass and binding energy are related through

nuclear mass = (number of neutrons × neutron mass) + (number of protons × proton mass) − binding energy]

in terms of the general nuclear parameters, A and Z. A is the number of particles in the nucleus and Z the number of protons (A − Z) is therefore the number of neutrons.

In the absence of exact knowledge concerning the nuclear forces, the problem of finding the dependence of binding energy on Z and A is a difficult one. We must examine our empirical knowledge about nuclei for implications concerning the nuclear forces or the binding energy.

A. The sizes of nuclei and the binding energy. From scattering and other experiments with heavy nuclei it is found that nuclear radii go as $A^{1/3}$ In fact,

$$R = 1.48 \times 10^{-13} A^{1/3} \text{ cm.}$$

fairly well fits the known data. (It should be remarked that this formula does not mean very much if applied to the very lightest nuclei.) For our purposes the formula implies that the average density of constituent particles is about the same in all nuclei. It is quite likely that the density within a single nucleus does not vary much from one region within the nucleus to another. If a certain binding energy due to nuclear forces is to be associated with two nuclear particles within the nucleus a given distance apart, it is clear from the foregoing that this binding energy per unit volume of nucleus is constant inasmuch as the average distances between constituent particles is everywhere the same. We conclude, therefore, that the binding energy

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of nuclei is essentially proportional to their volume or to A. In terms of the energy of a nucleus (the negative of the binding energy) we have then $E \sim -a_1 A$ where a_1 is some positive coefficient which these considerations have not sufficed to determine.

B. The surface of the nucleus and the binding energy. We know that the above discussion is not the whole story. For example, even if it is assumed that nuclear constituents are everywhere spaced the same distance apart and are subject to equal forces everywhere, it is clear that an exception must be made of the particles at the surface. They are not surrounded by as many particles and are therefore not bound as strongly as particles inside a nucleus. The number of such particles is proportional to the surface, and we must subtract a number proportional to them from our previously discussed binding energy, for we see now that we have somewhat overestimated the binding energy. Since surface area goes as R^2 we must add to E a factor $a_2 A^{2/3}$ Thus $E = -a_1 A + a_2 A^{2/3}$

C. The fact that Z tends to be $\frac{A}{2}$ and its relation to the binding energy. In addition to the foregoing, we have other bits of information about nuclei that have certain implications about how E must depend on Z and A. Let us investigate these relations and incorporate them into our expression for E as correction terms. For example, it is found that the number of protons and neutrons in any nucleus are very nearly the same. (It is true that for heavy nuclei there are less protons than neutrons, but we shall assume that this is due to the electrostatic repulsion between protons which we shall consider next. That is, we assume that if it were not for the Coulomb forces between protons, there would be equal numbers of protons and neutrons in nuclei.) There are at least 3 types of nuclear forces within a nucleus; neutron-proton, proton-proton, neutron-neutron. In view of the equality of the number of protons and neutrons in nuclei, the last two types of forces must be of the same order of magnitude. For were the proton-proton forces stronger, nuclei with more protons than neutrons would

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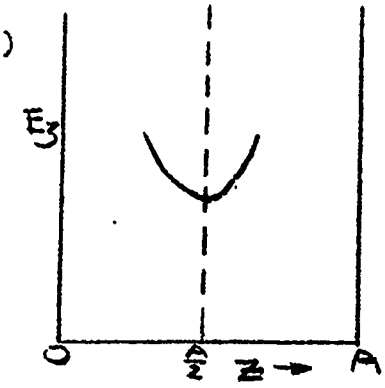
tend to be more strongly bound and hence more stable than those with equal numbers of each. If the energy E of isobars (same A, different Z) were plotted against Z, we should get a curve symmetric about $Z = \frac{A}{2}$ for nuclei with Z protons and $(A-Z)$ neutrons would have the same E as those with Z neutrons and $(A-Z)$ protons since neutron-neutron and proton-proton forces are assumed equal. The curve as drawn shows a minimum at $Z = \frac{A}{2}$ since nuclei for which $Z = \frac{A}{2}$ are the most stable. Thus E_3 the energy associated with the departure from equality in the number of protons and neutrons must go as some even power of $\left(Z - \frac{A}{2}\right)$ For simplicity consider that in the neighborhood of $\frac{A}{2}$, E_3 goes as $\left(Z - \frac{A}{2}\right)^2$ To see what the "dimensions" of the coefficient should be consider two nuclei with the same value for $\frac{Z}{A}$, one having an A twice the other. (Thus both nuclei have the same fractional excess of neutrons over protons, but one has twice as many particles.) The larger nucleus will have twice the E_3 if we associate with each extra or unpaired particle a certain fixed energy. It appears, therefore, that E_3 should be proportional to A. We write



$$E_3 = +a_3 A\left(\frac{Z}{A} - \frac{1}{2}\right)^2 \text{ or } a_3 \frac{\left(\frac{A}{2} - Z\right)^2}{A}$$

D. The Coulomb forces between protons and the binding energy. The problem of finding the energy, E_4, due to the electrostatic forces between the protons can be approximated by the solution of the following straightforward electrostatic problem. "What is the energy of a charge Z uniformly distributed throughout a sphere of radius R?"

The result is $\frac{3}{5}\frac{Z^2}{R}$

written with a proper coefficient so that the energy is in mass units

$$E_4 = +.000627\frac{Z^2}{A^{1/3}}$$

E. Even or odd numbers of protons and neutrons and the binding energy. It is found empirically that there are very few stable nuclei with even atomic weight A

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and odd atomic number (Z). In fact it can be said that the most stable nuclei tend to have both Z and $(A-Z)$ even. Slightly less stability occurs in the cases Z-odd, $(A-Z)$ even and Z-even, $(A-Z)$ odd. And, as we have said, Z - odd, $(A-Z)_{odd}$ is the least stable arrangement for a nucleus. Clearly forces between nuclear constituents must therefore show a dependence on whether an even or odd number of neutrons and protons are about, and so must the binding energy. Some sort of explanation has been advanced based on the idea that constituents tend to fill the nucleus' lowest energy levels and that strong forces exist between the pairs of neutrons or protons that can fill the same level. It has been empirically determined that $E_5 = \delta$ can be assigned as a correction term to our expression for E on the following basis:

$$\delta = \begin{cases} 0 & A \text{ odd} \\ -\dfrac{.036}{A^{3/4}} & A \text{ even} \begin{cases} Z \text{ even} \\ Z \text{ odd} \end{cases} \end{cases}$$

We must now evaluate the coefficients a_1, a_2, a_3 in our complete expression:

$$M(A,Z) = 1.00893A - .00081Z - a_1A + a_2A^{2/3} + a_3\frac{\left(\frac{A}{2}-Z\right)^2}{A} + .000627\frac{Z^2}{A^{1/3}} + \delta$$

To find a_3 get $\frac{dM}{dZ}$ and set this equal to zero. The resulting equation between Z and A

$$Z_A = \frac{.00081 + a_3}{2a_3 + .001254A^{2/3}} \cdot A$$

is one for which M is a minimum and therefore gives the stablest values of Z for any A. Fitting this equation to the known stable isotopes gives one a best value for a_3. It is .083.

This makes $Z_A = \frac{A}{1.981 + .015A^{1/3}}$ Compare this with the known stable isotopes by picking ~ 10 of them and plotting a graph.

One determines a_1 and a_2 by fitting the equation for $M(A,Z)$ to the known data for nuclear masses. The result is that $a_2 = .014$ and $a_1 = .00504$

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Hence

$$M(A,Z) = .99389\,A - .00081\,Z + .014\,A^{2/3} + .083\frac{\left(\frac{A}{2}-Z\right)^2}{A} + .000627\frac{Z^2}{A^{1/3}} + \delta$$

Dempster in the Physical Review (53 p. 870 '38) gives a curve of packing fraction, $\frac{M(A,Z)-A}{A}$, vs. A. Using the above equation, plot packing fractions expected "theoretically" along with Dempster's experimental curve and note the degree of agreement.

Using the formula for M one can calculate the binding energies of neutrons to isotopes of uranium. This information will be very closely connected with the ability of slow neutrons to fission these various isotopes as we shall see. Let us calculate the binding energy of a neutron to U^{235}

	$M(U^{235})$	=	235.11240 (given by our formula)
+	$M(n)$	=	1.00893
	Sum	=	236.12133
−	$M(U^{236})$	=	236.11401
	BINDING ENERGY	=	.00732 mass units = 6.81 Mev.

In such a way one finds that the binding energies of neutrons to U^{236}, U^{237}, U^{238}, would be 5.51, 6.56, 5.31 Mev, respectively. This alternation of the magnitudes of the binding energies comes from the factor δ. This alternation is superposed on the regular variation of $M(A,Z)$ with A and Z given by the other 5 terms.

Calculate the binding energies of neutrons to

Th^{232} Au^{197} Sm^{149} In^{115} Mn^{55}

Use the formula for $M(A,Z)$ as we have done to obtain the masses. Sufficiently accurate <u>experimental</u> data for nuclear masses (for such an application as we have here) exists only for the lightest nuclei.

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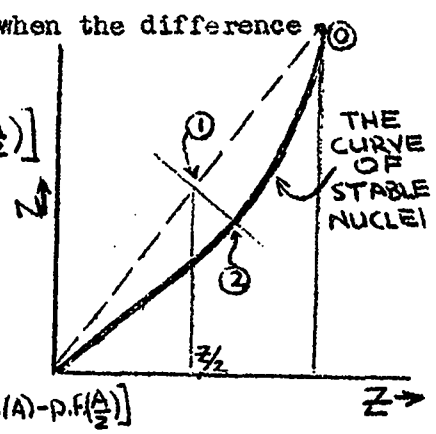
(c) The Fission Process.

In what follows, we shall try to account in a rough way for the existence of fission in the heavy isotopes.

An examination of the packing fraction curve shows that in the region of uranium the packing fraction is of the order + .0006 whereas for middle weight nuclei the packing fractions run of the order — .0007. This implies that the heavy nuclei are not energetically stable against breaking into 2 middle sized nuclei. Let us look into this more closely. Clearly the energy that would be released in such a splitting is of the order $M(A,Z) - 2M(\frac{A}{2},\frac{Z}{2})$ and if this is positive, the splitting is energetically possible. This difference can be written

$$A\left[\frac{M(A,Z)-A}{A} - \frac{M(\frac{A}{2},\frac{Z}{2})-\frac{A}{2}}{A/2}\right]$$

or A times the difference in the packing fractions. So that when the difference of the packing fractions is positive fission is energetically possible. It is to be noted, however, that $A\left[p.f.(A) - p.f.(\frac{A}{2})\right]$ does not give the energy released in a fission process. It gives the energy for the transition from ⓪ to ① (see curve →). Actually the end state ② is of lower energy than ① Hence the total energy released in a fission is $> A\left[p.f.(A) - p.f.(\frac{A}{2})\right]$



If A were 240, $\frac{A}{2}$ is 120 and Z_A comes out 93.74 for the nucleus A (p. 65). $\frac{Z_A}{2}$ is then 46.87, but using $\frac{A}{2}$ and the formula for $Z_{\frac{A}{2}}$ the stable Z_{120} is 51.15 or about 4 units from $\frac{Z_A}{2}$ This means that about 4 β particles will be emitted per fragment. From the packing fraction curves it appears fission is exoenergetic for all nuclei with A greater than about 100. Why then is fission such a rare process? Consider a nucleus that breaks into 2 fission

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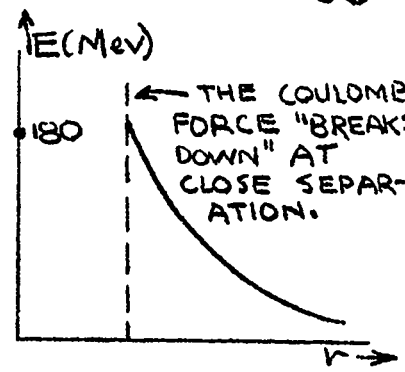
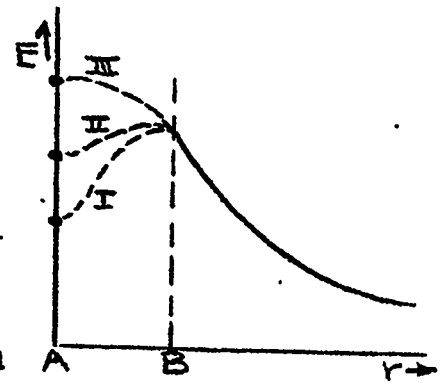
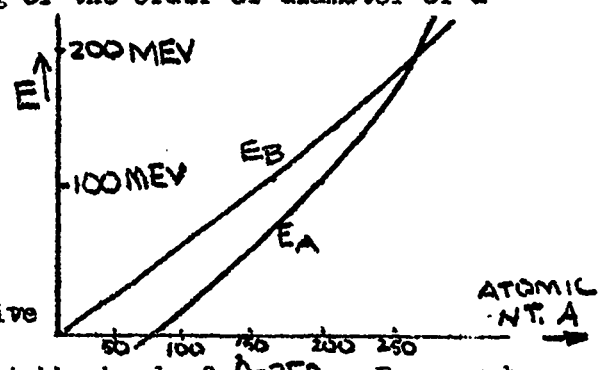


fragments. Plot the energy of the nucleus (i.e., the fragments) as a function of the distance between the 2 parts. At infinite separation we take the energy at zero. When the fragments are combined ($r=0$) we know from measurements that E is about 180 or 200 Mev greater. This lets us plot a point for $r=0$. What about points between $r=0$ and infinity? Up to distance of the order of the diameter of the fragments, it is the Coulomb energy between the particles that alone contributes to the energy between particles since that is the only force acting between them. This energy is $\frac{(\frac{Z}{2}e)^2}{r}$. The value of this term at distances the order of the nuclear diameter is either smaller, equal to, or greater than the 180 or 200 Mev at $r=0$. We would be able to draw three different transition curves from $r=0$ to such distances. (See diagram) Presumably stable nuclei with $A>100$ are represented by curves of the type I, with barrier heights of the order of 50 Mev. Presumably, too, uranium would be represented by a curve like II where the barrier is about 6 Mev. Substances whose curve would be given by III would naturally not exist for long in nature. This curve presumably represents non-existing transuranics. For a somewhat more detailed discussion of the transition distances ($r=0$ to $r=B$), see the Bohr-Wheeler paper in the Physical Review of Sept. 1939. Let us think of $r=B$ as being of the order of diameter of a fission fragment. Then $B=2\times1.48\times10^{-13}\left(\frac{A}{2}\right)^{1/3}$



We can draw a curve for E_B as a function of atomic weight. The curve for E_A gives the excess of mass or energy of a mother nucleus of weight A over that of its two fragments. This curve becomes negative below $A=85$ It crosses the curve for E_B in the neighborhood of $A=250$. From such a graph one can get (E_B-E_A) for any A. (E_B-E_A) is a measure of the height of the energy barrier against fission.



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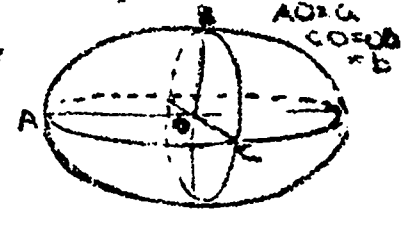
One can of course investigate more precisely the shape of the energy vs separation curve in the neighborhood of $r=0$ if some specific model is assumed. Consider the Bohr liquid drop model. Here we assume that the original nucleus is a sphere and we calculate the change in energy for a small deformation. We assume that the first deformation of a sphere that is beginning to split is a very simple one, namely that the sphere stretches a bit in one direction and flattens out perpendicular to this direction. The sphere becomes an ellipsoid.

If we assume that the sphere does not change its volume on becoming an ellipsoid (and this is reasonable in view of the fact that all nuclei tend to maintain the same density of nuclear particles, as we have seen), the change in the energy of the nucleus upon deformation will be due to only two of the five factors discussed in the last section. First there will be the surface energy which will tend to increase with deformation because more surface will be exposed. And second we have the electrostatic energy which will decrease upon deformation because the repelling charges are effectively separated a little.

Thus we have at least two energies changing in opposite ways with deformation of a spherical nucleus. Since the surface or capillary energy goes as the surface area or $A^{2/3}$ and the electrostatic energy $\sim \frac{Z^2}{A^{1/3}}$ or $\sim A^{5/3}$ the latter energy becomes more important for heavy nuclei. Thus, for heavy nuclei it is likely that the energy of a nucleus tends to decrease with deformation and a spherical nucleus is unstable. The opposite is true for light nuclei. From this picture it is in heavy nuclei that we would expect fission. Let us investigate the change of energy of a spherical nucleus upon distortion, in some detail. The electrostatic energy of a charge Z distributed throughout the volume of an ellipsoid



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can be shown to be:

$$\frac{3}{10}\frac{Z^2e^2}{\sqrt{a^2-b^2}}\log\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}$$

And the surface energy is proportional to the surface area which can be shown to be:

$$2\pi b^2+2\pi\frac{a^2b}{\sqrt{a^2-b^2}}\arccos\frac{b}{a}$$

Now consider a sphere of original radius R. We stretch it in one direction $a=R(1+\epsilon)$ What should b be in order that the volume of the ellipsoid is that of the original sphere? $\frac{4\pi}{3}ab^2=\frac{4\pi}{3}R^3$ whence $b=\frac{R}{\sqrt{1+\epsilon}}$ Substituting for a and b in the two energy expressions above and developing the results in powers of ϵ we find:

the electrostatic energy is

$$\frac{3}{5}\frac{Z^2e^2}{R}\left(1-\frac{1}{5}\epsilon^2+\ldots\ldots\right)$$

(where it is to be noted that the first term is simply the electrostatic energy of the sphere and the second, a correction term that gives a decrease of energy with deformation as predicted.)

The surface energy is proportional to:

$$4\pi R^2\left(1+\frac{2}{5}\epsilon^2+\ldots\ldots\right)$$

Using the proper coefficients for these energies from the formula for $M(A,Z)$ (p.66), the excess in energy of the ellipsoid over the sphere is:

$$\epsilon^2\left[\frac{2}{5}\times .014\,A^{2/3}-\frac{1}{5}\times .000627\frac{Z^2}{A^{1/3}}\right]$$

The condition for stability against deformation is that the bracket be positive. That is, that

$$\frac{Z^2}{A}<44.7$$

For uranium $\frac{Z^2}{A}=36$ and for lower elements its value is even smaller.

At what atomic number is instability reached according to this inequality? Use the expression for Z_A the proper value of Z for a nucleus of weight A that has been developed (p.65)

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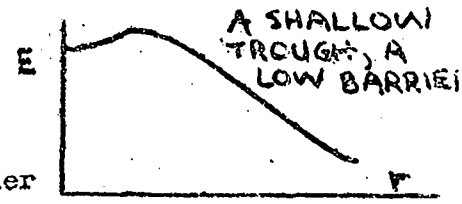

Actually instability will come quite a bit before that point given by the solution of the preceding problem. For curves of the shape shown, the barrier is already rather transparent and one could expect appreciable spontaneous fission. Even for U^{238} there are $\sim$ 20 fissions per gm per hour spontaneously, so that for heavier atoms this quickly becomes a prominent phenomenon. For example, for plutonium 238, the spontaneous fission rate is already 10^7 per gm hour.

We shall be more interested, in what follows, in fissions brought about by neutrons than in spontaneous fissions. Neutrons can cause fission by contributing their kinetic energy and their binding energy to the nucleus. This energy is at least $\sim$ 5 or 6 Mev (the binding energy of the neutron) and may raise the energy of the nucleus high enough within the barrier for a fission to take place before the excess energy is lost by a γ radiation. Because of the fact that the binding energy of neutrons to nuclei with an odd number of neutrons is larger than it is to those with an even number of neutrons (p.66), it is reasonable to expect fission for thermal neuts to be more prevalent for those nuclei with an odd number of neutrons. This is so. (238 won't fission with thermals but 235 will.)

From facts like these and photofission thresholds, one can estimate that for uranium the height of the fission barrier is of the order 5 Mev.

It is to be kept in mind that in the consideration of the competition of fission with other processes, it is not sufficient to consider energies alone as we have done. For fission, one must have the energy rise to the top of the barrier, but in addition it is necessary that this energy be concentrated in the proper modes of motion for fission. This may take some time and competing processes may therefore occur at the expense of fission. Since the number of modes (and hence of useless non-fission modes) increases with excitation energy, it may therefore be very likely

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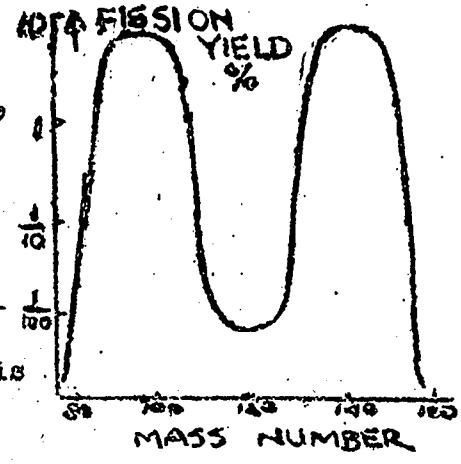
(3) <u>The Particles of Fission.</u>

In addition to the appearance of the fission fragments, one can observe neutrons, β particles, γ rays and very often fast α particles following or accompanying fission.

As to the fission fragments themselves, their atomic weights present an interesting problem. The distribution of fission fragments as a function of atomic weight appears at the right. At the half point (if U^{235} is fissioned by a neutron and 2 are emitted in the course of fission, one would get A = 117 if it is assumed that a nucleus splits into 2 equal fragments.), the observed yield is very low. Instead of even splitting, the splitting is very uneven as this experimental curve shows. This phenomenon has not as yet been explained theoretically.



As for the neutrons of fission, they are classed as either prompt or delayed, "prompt" meaning that they leave the fission fragment after its formation in times shorter than we can measure. One can estimate that they leave within 10^{-15} seconds under circumstances such as the following. Consider the fission process once again as the splitting of a drop.

The final fragments are not of spherical shape. Hence there will be a considerable vibrational energy associated with oscillations about the equilibrium (spherical) shape of the fragment. This excitation energy may be sufficient to evaporate a neutron, especially since neutron binding energies in fission fragments are small because of the excess of neutrons. For example, assume that U^{235} is made to fission by a neutron and 2 fragments with A = 118 and Z = 46 appear. Using the formula for $M(A,Z)$ (p. 65) one can calculate binding energies for various nuclei of weight A = 118.

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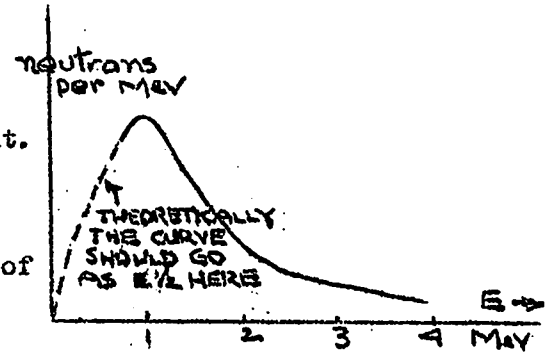
Z	44	45	46	47	48	49	50
BINDING ENERGY	2.5	6.8	3.6	7.8	4.7	9.0	5.8

(where $Z = 50$ is the stable value for Z if $A = 118$). Thus neutrons may be lightly bound to fission fragments. Whenever neutron emission is energetically possible, neutron emission is likely, because of the absence of a barrier for neutrons. As a matter of fact, one would conclude from observations that considerable excitation energy must be present in the fragments because ν, the number of neutrons emitted per fission, is for U^{235}, to take an example, equal to 2.45 or slightly more than one neutron per fission fragment.



The energies of the neutrons that come off at fission are given in the distribution curve at the right. In the center of gravity system of neutron and fission fragment the neutron energy distribution would be sort of Maxwellian with a "temperature" corresponding to the excitation of the fragment. To get the theoretical curve for the distribution in the lab system one would have to take account of the motion of the fission fragment and the dependence of emission probability on neutron energy.

Derive an energy distribution curve for fission neutrons assuming a velocity V for the fission fragment and a Maxwell distribution of energies in the c. of g. system. Also assume that the probability of neutron escape is proportional to their velocity.

The observed results agree roughly with those to be expected on these considerations except below 1 Mev where no sufficiently accurate measurements have yet been made.

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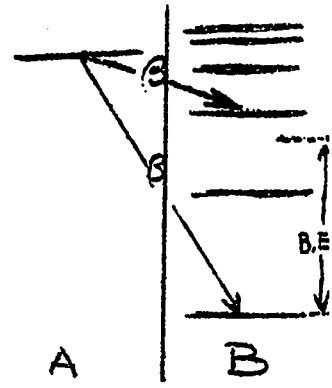


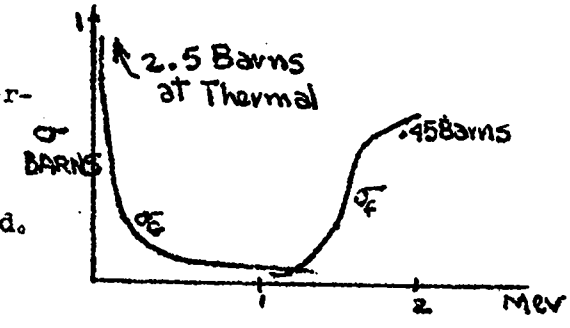
In addition to these prompt neutrons some (~ 1%) come off delayed. To explain the emission of delayed neutrons, consider a fragment A which undergoes a β-disintegration to a nucleus B. Usually this disintegration will go to the ground state of B, but occasionally the nucleus B may end up excited with excitation energy greater than B.E., the binding energy of a neutron. In such a case neutron emission becomes quite likely. Such neutrons would come off very quickly after the β decay and would therefore show decay periods that correspond to the periods of the β disintegration of A to the excited state of B.

The delayed neutron periods that have been observed and their yields are at the right. The first 2 periods are rather well verified. The short time ones are as yet not definitely confirmed.

1/2 life	delayed neuts per prompt
55.6 sec	.00021
22.0	.00139
4.51	.00178
1.53	.00201
.42	.00071
Total	.006

(3) <u>The Cross-Sections for Fission and Competing Processes in Heavy Isotopes.</u>

In U^{238}, as in other heavy elements, the only likely processes caused by neutrons outside of scattering are (n,γ) capture and (n,f) processes. The cross-sections for these processes in U^{238} are plotted. Note that capture is more probable than fission for neutron energies up to ~ 1 Mev.



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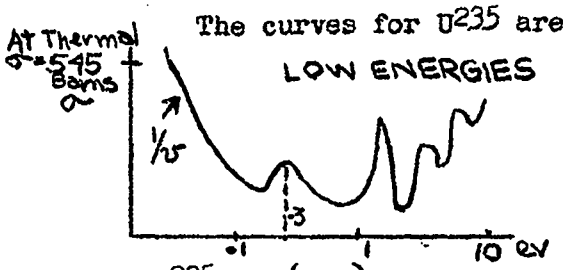
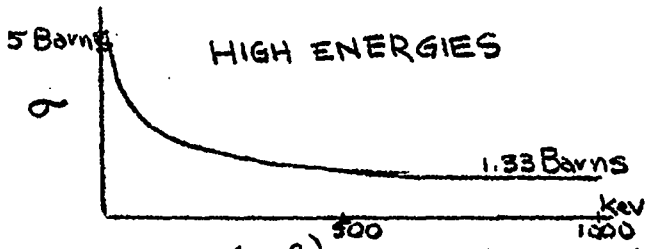
The curves for U^{235} are drawn below





For U^{235} the (n,γ) cross-section follows the shape of the (n,f) curve at low energies (being about 20% as probable at thermal energies). The (n,γ) process does however have a few of its own resonances (~ 2 volts, ~ 5 volts) also. At high energies the (n,γ) process is quite improbable compared to the (n,f) process.

For Pu^{239}, the thermal (.025 ev) cross-sections are σ_f = 705 barns, $\sigma_{n,\gamma}$ = 345 barns, and the total cross-section is therefore ~ 1050 barns. The variation of fission cross-section with energy is indicated in the table at the right. The capture cross-section becomes relatively less important at higher energies.

ev	σ_f (Barns)
Thermal	705
.1 to .2	1/v Law
.3	6000 (Resonance)
10^4	3
3×10^5	2
levels off	1.96

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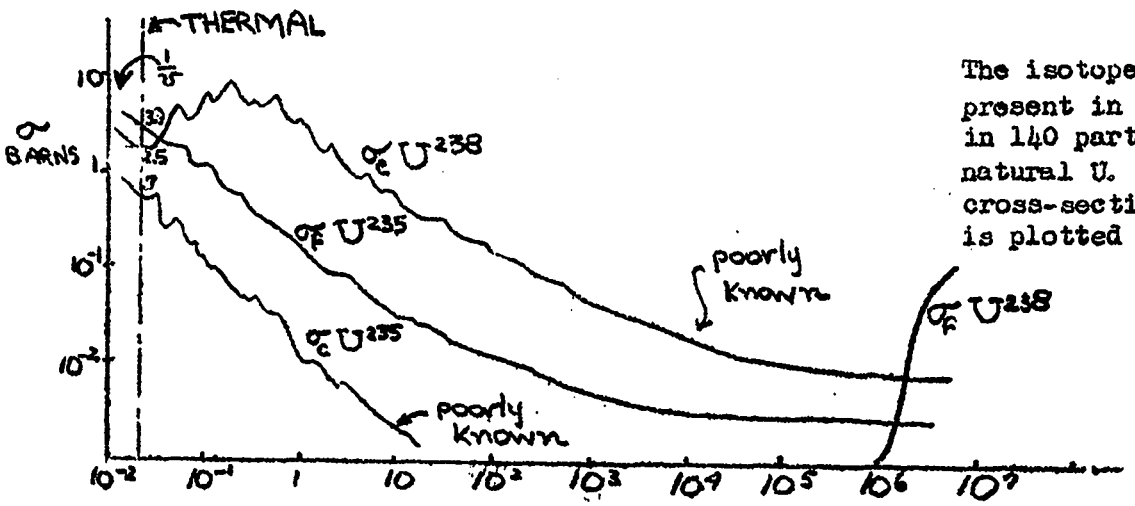
CHAPTER IX

Chain Reactions

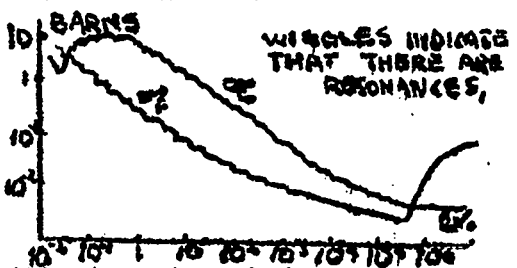
(1) The Properties of Natural Uranium.

It will be shown in this section that a neutron chain reaction cannot maintain itself in uranium of the isotopic composition found in nature. Whether or not a chain can be maintained will depend on the relative probabilities of (n,f) to (n,γ) processes for the neutrons in the material. Although a fission in uranium will release on the average 2.4 neutrons, it will be seen that the (n,γ) cross-sections (for the various energies in the neutron spectrum) are sufficiently high to drain away most of these neutrons, and to make the probability, for the progeny of one fission to cause at least another, less than unity. Let us plot on a single graph the capture & fission cross-sections for the two main isotopes of U.



The isotope U^{235} is present in about 1 part in 140 parts of U^{238} in natural U. Its effective cross-section, $\sigma/140$ is plotted here.

The cross-sections observed with a piece of natural U would be the sums of the above cross-sections and these are plotted at the right. It is to be noted that below ~ 1 ev and above $\sim$ 1.4 Mev $\sigma_f > \sigma_c$ and in between σ_c is greater.



One would conclude, therefore, that if it were possible to get a chain reaction going in natural uranium, it would be sustained by neutrons whose energy

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is greater than ~ 1.4 Mev or by neutrons near thermal, for in intermediate regions the neutrons would more likely be absorbed with the emission of a γ, than lead to new fissions.

Consider first the high energy region. It will be shown that the existence of a large inelastic scattering cross-section prevents the realization of a chain reaction. Given a big mass of natural U (big enough so that surface leakage of neutrons is no consideration) and a fission occurring somewhere in the middle, let us see what is likely to happen. About 2.4 neutrons will be produced. The fission neutron spectrum is such that we can assume that 1 neutron is produced with energy below the effective fission threshold of U^{238} (~1.5 Mev) and 1.5 neutrons with energy > 1.5 Mev. The low energy neutron will probably be absorbed because of the high capture cross-section at its energy. (In fact, the probability that it causes a fission is only ~ .05.) Were there no inelastic scattering, the remaining 1.5 neutrons could start a chain reaction for they have ~ four times the probability of leading to fission rather than to capture (i.e. ~ 1.2 of our 1.5 fast neutrons would lead to new fissions, and since the necessary condition for a chain is that the number of neutrons per fission that lead to new fissions be greater than one, this would certainly start a chain reaction). But there is inelastic scattering of these fast neutrons and the cross-section for scattering from above to below the U^{238} threshold is of the order 2.3 barns. Since σ_f is only ~ .4 and σ_c ~.1 only $\frac{4}{28}$ of the 1.5 fast neutrons (rather than $\frac{4}{5}$ of them) will produce fissions. This is about .2 neutrons and there can be no self-sustaining chain due to the high energy neutrons of the neutron fission spectrum. We say the "reproduction factor" of the structure is 0.2.

Assuming the following data,

	Below Threshold	Above Threshold
$\sigma_{inelastic}$ (above to below)		2.3 barns
σ_f	.0096	.47
σ_c	.17	.12

and a fission spectrum where 40% of the neutrons are emitted below threshold and 60% above, what is the reproduction factor for an infinite mass of metal?

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Now, let us consider the thermal region. For .025 volt neutrons $\sigma_f = 3.9$ and $\sigma_c = 3.2$ barns, or $\sigma_{total} = 7.1$ barns. All of these cross-sections behave as $\frac{1}{v}$ and so their relative values are roughly the same throughout the region. If the 2.4 neutrons produced in a fission manage to get slowed down to thermal energies without absorption, one would expect them to make $\frac{3.9}{7.1} \times 2.4 = 1.32$ new fissions. One would have a "going" chain.

(2) <u>The Use of a Moderator</u>.

The problem, then is to reduce the fission neutron energies to thermal before the neutrons are absorbed. One would be led to a scheme such as the following. Mix the uranium with a substance that doesn't absorb neutrons. Spread the uranium thin enough so that a fission neutron will make many collisions with this other substance, called a moderator, before it is likely to meet some U again. In fact it should make enough collisions so that it is probably thermal when it hits its first U nucleus.

One might consider using He^4 as the moderator for it is believed He^5 does not exist, i.e., He^4 will not capture neutrons. However such a moderator would not be practical, the mean free paths would be long, and a structure of about one cubic mile might be required. One might think of using liquid helium in such a structure, called a "pile". This is also obviously not very practical, but a remark might be in order about dimensions of piles in general. If one can succeed in increasing the density of a pile by a factor f, as in the case of liquifying helium, then the linear dimensions of the pile can each be reduced by the factor f and the operation of the pile will be exactly as it was before changes were made, because the ratio of mean free path to a distance in the pile stays fixed. The overall result is to multiply the volume by f^{-3} and the mass by f^{-2}.

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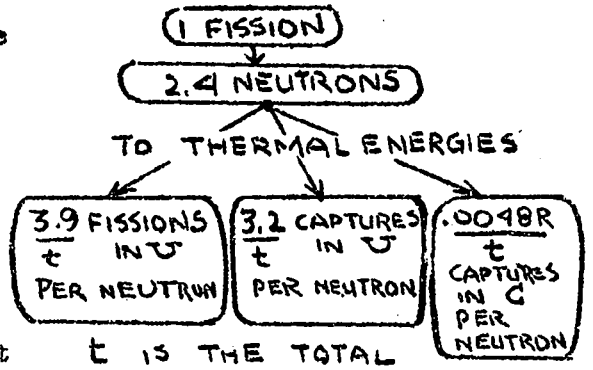
One is forced to use substances that do absorb neutrons, but that do not absorb them very much. Chain reactions have been obtained with graphite and heavy water. Possibly Be or even H_2O may be successful moderators.

(3) Homogeneous vs. "Lumped" Graphite Piles

Consider a graphite pile and assume that it is so large that leakage from the surface can be neglected. Assume that the U is uniformly spread through the pile. For graphite at thermal energies, $\sigma_{scat} = 4.8$ barns, $\sigma_c = .0048$ barns. What is the maximum ratio, R, of the number of carbon to uranium atoms required to keep the pile going? The maximum is set by the absorption of the carbon at thermal energies. Assume $\nu = 2.4$, i.e., there are 2.4 neutrons emitted per fission. Assume further (in order to get the upper limit on R) that no absorption takes place during the slowing down to thermal. From the diagram at the right, it is seen that for the pile to go $\frac{3.9}{t} \times 2.4$ must be > 1

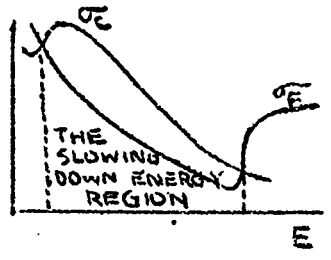


t IS THE TOTAL CROSS-SECTION FOR ABSORBING PROCESSES AT THERMAL ENERGIES.
$t = 3.9 + 3.2 + .0048R$ Barns

This gives $.00051R < .24$ or $R < 470$

or in terms of weight, the ratio of weight of carbon to U must be < 25. Actually a consideration of losses during slowing down, and the consideration that U is not spread uniformly through actual piles (resulting in higher neutron densities in the C, away from the U neutron "sinks") both tend to make even lower this upper limit. There is clearly a lower limit for R (for, as we have seen, a pile will not go without moderator) and this lower limit must be less than our upper limit if our whole scheme is to be feasible with C.

These considerations lead us to the problem of capture during the slowing down, which we have hitherto neglected. Assume that q, the slowing down density, is constant. That is that the number of neutrons/c.c. that change from energy above E to below E in unit time, (1) does not depend on position, (2) does not depend on E. The physical conditions that would give such a q would be a uniform medium with a uniformly spread steady source.



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We have seen (p.45) that the velocity with which a neutron drifts down the ϵ ($=\log E$) axis is $\xi \frac{v(\epsilon)}{\lambda}$ where $\xi = \overline{\log \frac{E_i}{E_f}}$ Hence $q = n(\epsilon)\,\xi\,\frac{v(\epsilon)}{\lambda(\epsilon)}$

We can write the number of neutrons per c.c. in the logarithmic energy interval $d\epsilon$ at ϵ,

$n(\epsilon)\,d\epsilon = q\frac{\lambda}{\xi}\frac{1}{v}\frac{dE}{E}$ Now consider an atom of U in the pile. How many neutrons in this energy interval will it absorb in unit time? Clearly this will be

$$n(\epsilon)\,d\epsilon \cdot v \cdot \sigma(\epsilon)$$

and in terms of E, the total number of neutrons absorbed per atom per unit time is then

$$q\frac{\lambda}{\xi}\int \sigma(E)\frac{dE}{E}$$

The upper limit of the integral need not be picked too carefully for the integrand decreases rapidly as E increases. The lower limit is usually taken as the cadmium absorption edge $\sim .3$ ev. Under these circumstances the integral (without the $\frac{q\lambda}{\xi}$) is, for U, 240 barns. (This is determined experimentally.)

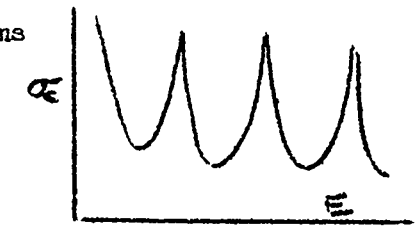
Since q neuts are produced per unit time per unit volume, and each atom absorbs $q \times \frac{\lambda}{\xi}\int_{.3}\sigma\,\frac{dE}{E}$ neutrons in unit time, one can associate with each atom an "absorption volume" $\frac{\lambda}{\xi}\int_{.3}\sigma\,\frac{dE}{E}$ (note that this truly has the dimensions of volume) where we can say that each atom will absorb all those neutrons made in the "absorption volume" surrounding it, and only those. For U, this volume is 4×10^{-21} c.c. which is much greater than the size of a U nucleus (10^{-35} c.c.) as is to be expected.

The total number of neutrons absorbed per c.c. per sec. in a pile would be

$$q \times 4\times 10^{-21} \times \frac{\text{Number of Atoms of U}}{\text{c.c.}}$$

and the number of atoms of U per c.c. would be 1.6×10^{20} and the number of carbon atoms per c.c. would be 8×10^{22} if we take $\frac{\text{Number of C Atoms}}{\text{Number of U Atoms}} = 500$. Thus $.64\,q$ neutrons are absorbed per c.c. per sec. or 64% of the neutrons are absorbed.

However this is a considerable overestimate. This stems from the fact that the absorption cross-section has sharp resonances. The absorption of n atoms is <u>not</u> the same as n times the absorption of one atom. The atoms absorb each other's neutrons, so to speak, and so "shield" each other. This can be clarified



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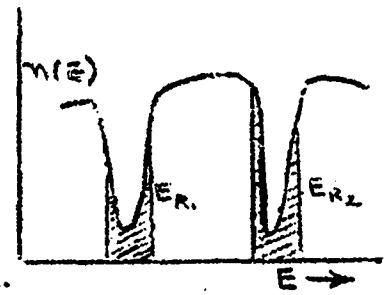
somewhat as follows. The spectrum of neutrons in the pile is not what it would be if σ_c were smooth. There are depressions in the spectrum at resonance energies simply because neuts with such energies are quickly removed. Nuclei of U are therefore presented with relatively few (cross-hatched areas) absorbable nuclei.



To cut down the slowing down absorption, one would lump the U in the pile rather than spread it, for then within a lump there would be no moderator which would tend to even out the neutron spectrum by continually feeding neutrons into resonance regions. Those nuclei on the surface of a lump would quickly remove all incident resonance neutrons and interior nuclei would see very few.

But how would lumping effect the absorption of <u>thermal</u> neuts by the U and the C ? The carbon absorption would be

$$n_c \times \sigma_c \times v \times N_c$$

where n_c is the density of neutrons of thermal energies at the carbon, and σ_c and N_c are the neutron capture cross-section of carbon and the number of atoms of carbon/c.c. respectively. For uranium, the absorption per c.c. per second is similarly

$$n_U \times \sigma_U \times v \times N_U$$

and the ratio of absorptions is $\frac{\sigma_c N_c n_c}{\sigma_U N_U n_U}$. Since the uranium acts as a neutron sink if it is lumped $n_c > n_U$ whereas n_c would be equal to n_U for a uniform distribution. Hence from the point of view of absorption of thermals to get fissions, lumping is undesirable.

This result and the previous one that lumping <u>is</u> desirable from the point of view of cutting down absorption during the slowing down of neutrons suggest that a complete consideration of the problem may lead to values for optimum sizes of the lumps. In the next section we shall calculate the reproduction factor for a pile with spherical lumps and shall see how the various factors that go to make up the reproduction factor depend on the size of the lumps. For the present we shall record the empirical results

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for the self-absorption effects in homogeneous and lumped piles. It is found empirically that for a homogeneous mixture of U & C

$$\left[\int \sigma \frac{dE}{E}\right]_{effective} = 2.95\,(\sigma_{scat})^{.47}$$

(holds for $\sigma_{scat} < 1000$)

where σ_{scat} is the scattering cross-section of 1 atom of U plus that of the number of moderator atoms per U atom. We have seen that if we were simply to add atomic absorptions, $\int \sigma \frac{dE}{E}$ is 240 barns. Substituting the known scattering cross-sections for U & C in the empirical formula, we get ~ 124. Even for a uniform distribution the "self-absorption effect" is of the order of a factor 2, i.e., the "absorption volume" around each U nucleus is but half as big as it would be were there no resonance absorption effect.

If, however, one lumps the U, the discrepancy should be even greater. It is found empirically that the "absorption volume" of a lump of U in some carbon is

$$\text{Vol.} = (.385 \times \text{Mass of U in gms}) + (1.03 \times \text{Surface of U in cm}^2)$$

Apply this to a U sphere of 3 cm radius.

Actual volume = 112 cc

Empirical absorption volume = 990 cc

Absorption volume calculated neglecting

self screening or

self absorption = 23,000 cc (or 20 times the observed one)

Can one construct a pile by uniformly disbursing U in C ? To find out, let us calculate the reproduction factor. Take $\frac{C}{U}$ 200 (i.e. 200 C atoms / U atom) to start with. Take ν (no. of neuts/fiss.) = 2.4. Empirically (we use the formula of above), $\left[\int \sigma \frac{dE}{E}\right]_{eff}$ = 75, using the fact that σ_{sc} for carbon is ~ 4.8 barns and σ_{sc} is ~ 10 for U. This gives as the total absorption per c.c. during slowing down

$$q \times \frac{2.6}{.158} \times 75 \times 10^{-24} \times 4 \times 10^{20} \cong .5\,q \qquad \text{(in each sec.)}$$

or ~ 50% of the neutrons are absorbed during slowing down. Thus 1.2 neutrons per fission reach thermal. Now σ_c for carbon is .0048 barns and σ_c = 7.1 for U

Since there are 200 C atoms per U atom, the number of neuts absorbed by the U is

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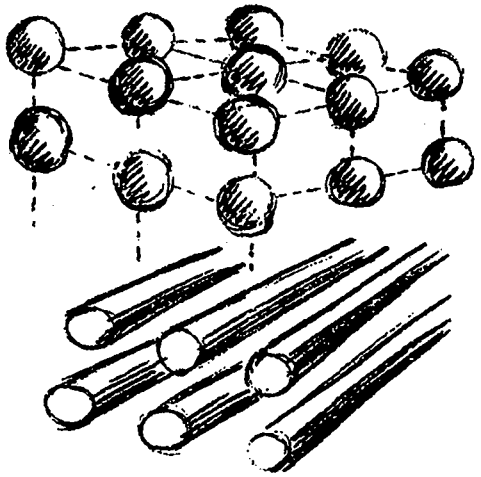
$$\frac{7.1}{(200 \times .0048)+7.1} \times 1.2$$

or 1.06 neuts are absorbed by the U. Since $\frac{3.9}{7.1}$ absorptions in U lead to fission, we get $1.06 \times \frac{3.9}{7.1} = .58$ fissions. This pile will not "go".

Show that for no <u>uniform</u> distribution (i.e. for no value of $\frac{C}{U}$) is a chain reacting pile possible, by finding the maximum reproduction factor as a function of $\frac{C}{U}$ and <u>seeing that it is less than 1</u>.

One could design an inhomogeneous pile by using one of the geometries at the right. The cubic lattice of spherical lumps of uranium is slightly more efficient from the point of view of neutron utilization, but the cylindrical geometry has the advantage that energy can be removed (the pile cooled) by blowing air, say, through the cylinders.



(4) <u>The Reproduction Factor in a Lumped Pile.</u>

The reproduction factor for a pile with spherical lumps. Start with one neutron in a sphere. ϵ neutrons will escape the sphere, where ϵ is somewhat greater than one, for there is a small probability that the original fission neut will (while still fast) give rise to a "fast fission". Let f_R be the probability that a neutron is either resonance or graphite absorbed during slowing down. Then $\epsilon(1-f_R)$ neutrons will reach thermal energy. Let f_T be the probability that a <u>thermal</u> neut is absorbed by U rather than C. Then $f_T \epsilon (1-f_R)$ thermal neutrons will be absorbed by the uranium. In order to find out how many new neutrons, on the average, are made by such an absorption, multiply the above by η which is

$$\left[\begin{array}{l}\text{probability that absorption of} \\ \text{thermal neut will give a} \\ \text{fission} = \frac{3.9}{7.1} \quad \text{for } U\end{array}\right] \times \left[\begin{array}{l}\nu = \text{no. of neuts} \\ \text{per fission} = 2.4 \\ \text{for } U\end{array}\right]$$

Thus $\eta =$ 1.32 and the reproduction factor is $K = \eta f_T (1-f_R)\epsilon$

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Let us see how the 3 factors $f_T, (1-f_R)$ and ϵ can be calculated. Let us first get an order of magnitude calculation for ϵ. The average length of path of a neutron escaping from a spherical lump is the radius. Assume that there is only a 60% chance of the original fission neutron to lie at the high end of the fission spectrum (in particular that it has energy > 1.4 Mev, taken as the U^{238} fission threshold). Now $\sigma_f \sim .45$ barns and $\lambda_{fiss} \sim 50$ cm. Then our original neutron would make this many new ones:

$$.60 \times \frac{R}{\lambda} \times (\nu - 1)$$

(where $(\nu-1)$ is the net neutron production per fission)

due to fast neutron fission. Thus in order of magnitude

$$\epsilon \sim .6 \times \frac{R}{50} \times 1.4$$

(where we have taken $\lambda \sim 50$, $\nu = 2.4$).

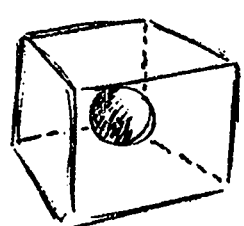
Actually ϵ can be calculated by somewhat more refined methods, giving more accurate results.

Next we calculate $(1-f_R)$. We have seen that empirically the absorption volume of a lump of U for absorption during slowing down is (p.)

$$Vol = .385 \times Mass + 1.03 \times Surface\ Area$$

$q \times Vol.$ neutrons are absorbed per second by the lump. But $q\, V_c$ (where V_c is the volume of the graphite cell) neutrons are produced per second per lump. Thus

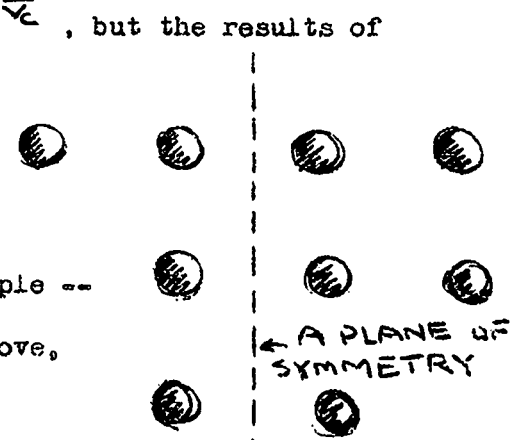
$$f_R = \frac{Vol.}{V_c}$$



A "CELL" IN A CUBIC LATTICE WOULD BE THAT VOLUME OF GRAPHITE, CONTAINING A U SPHERE AT ITS CENTER, AND WHICH COULD BE CONSIDERED THE BUILDING UNIT OF THE LATTICE.

Actually a somewhat better calculation gives $f_R = 1 - e^{-\frac{Vol}{V_c}}$, but the results of this and the simpler formula are not too different.

Now it remains to see how f_T is calculated. If the lattice is infinite, then at any symmetry plane, the gradient of neutron density -- for thermal neuts for example -- must vanish. Now the boundaries of a cell, as defined above, are all symmetry planes for an infinite medium and the problem of finding the neutron density as a function of



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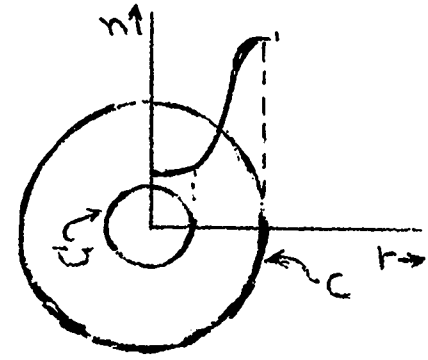
position becomes a typical boundary value problem. Actually, for purposes of calculation the cubic cell is replaced by a spherical one and one finds n, the thermal neutron density, as a function of position in the cell under the condition $\nabla n = 0$ at the boundaries. Having found this, one can calculate the rate of absorption of thermal neutrons by the U



In this way one obtains

$$f_T = \frac{3\alpha^2(1-\alpha)(1+\beta)e^{-(\beta-\alpha)} - (1+\alpha)(1-\beta)e^{\beta-\alpha}}{\beta^3 - \alpha^3(\alpha+s-s\alpha)(1+\beta)e^{-(\beta-\alpha)} - (\alpha+s+\alpha s)(1-\beta)e^{-\beta-\alpha}}$$

where $\alpha = \dfrac{\text{Radius of Lump}}{\text{Diffus. Length in C}}$ $\qquad \beta = \dfrac{\text{Radius of Cell}}{\text{Diffus. Length in C}}$

and $s = \dfrac{1+\gamma}{1-\gamma}\,\dfrac{1}{\sqrt{N}}$ where $N = \dfrac{\sigma_{total}}{\sigma_{abs}}$ (for graphite $N \sim 1100$)

and γ is the albedo of the sphere (looking from the outside)

$$\frac{1-\gamma}{1+\gamma} = \frac{\sqrt{3}\,l_i}{\lambda_i N_i}\left\{\frac{e^{\frac{2R}{l_i}}+1}{e^{\frac{2R}{l_i}}-1} - \frac{l_i}{R}\right\}$$

where l_i & N_i are the diffusion length and $\sigma_{tot}/\sigma_{absorp.}$ (for U) respectively.

For U, $l_i = \dfrac{24.0}{\rho}$; $\lambda_i = \dfrac{21.1}{\rho}$; $N_i = 2.79$

(ρ ← density)

> Take a 3000 gram lump of U ($\rho = 18$) which is spherical in cubic cells of side 20 cm. Calculate f_T. [One must use the above formula with high accuracy, because numbers tend to cancel.]

And so one calculates the four factors in K. For a typical lattice, these numbers may be of order $\eta = 1.32$ $\quad f_T = .88$ $\quad f_R = .12$ $\quad \epsilon = 1.05$

For such a lattice $K = 1.07$ (the best K'S for a U-C system are $\sim$ 1.10)

In industrial piles -- or even in lab piles to give strong neutron fluxes -- one runs the pile at high power. The Argonne pile ($\sim$ 20 ft on a side) can't be run

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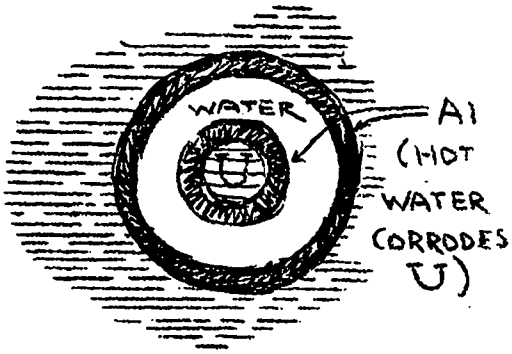
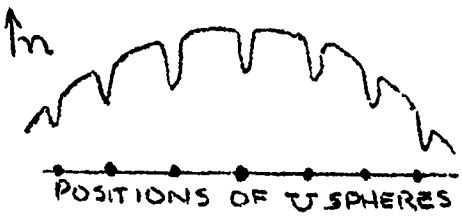
higher than 2 KW because no special provision was made for cooling. For higher power piles, one must introduce a cooling system. In Clinton, for example, there is a rod lattice with air cooling through the channels in which the rods lie. The power of the Clinton pile is ~5000 KW and the air coming out is at 100°. Other piles are water cooled and Al jackets must be used. The point of mentioning the cooling problem here, is to indicate that because cooling material must be introduced high power piles cannot maintain K at 1.1 but will show a lower value of K, the cooling material being a non-productive absorber.



(5) <u>The Relation between the Reproduction Factor and the Critical Size of the Pile.</u>

Assume we have calculated K, the reproduction factor, for a particular lattice and it turns out to be slightly greater than one, say, 1.05. Were we to have an infinite lattice one would have a supercritical pile. How big should we make the pile so that it can still be under control, i.e., just a little over critical? If n is the thermal neutron density in the pile, it will have a maximum at the middle of the pile and show a general dependence on distance within the pile as shown. Superposed on this will be the fluctuations due to the uranium lumps in each lattice cell. Let us assume that there are many lattice cells and that we can, in what follows, consider only the general shape of n as shown above, i.e., assume that the bumps in n are ironed out. We have derived once before (p. 49) the differential equation for n, the density of thermal neutrons, $\frac{\lambda v}{3}\Delta n - \frac{v}{\Lambda} n - q_T = 0$



It is to be noted that Λ, the absorption mean free path, is to be considered as some sort of average of Λ for pure uranium and of Λ for graphite. For pure

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graphite, Λ is $\sim$ 25m. and for the proportions of U and graphite usually used the "average" Λ is about a tenth of this.

In addition to the slow neutrons in the pile, there are fast neutrons from fission. For these we also assume that the density is a smooth function of position within the pile. The equation obeyed at equilibrium is the slowing down equation

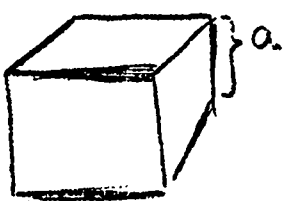
$$\Delta q = \frac{\partial q}{\partial \tau}$$

This equation and the preceding one are simultaneous descriptions of a pile at equilibrium. The q of this equation and the n of the thermal neutron equation are related. Let us investigate the relation. $\frac{v}{\Lambda} n$ thermal neutrons are absorbed per cc. per sec. Of these $f_T \frac{v}{\Lambda} n$ are absorbed by the U, and as a result $\eta f_T \frac{v}{\Lambda} n$ new (fast) neutrons appear. One would be inclined to write for $q(\tau)$ at $\tau = 0$, which is the number of fast neutrons made per second per c.c., simply $q(0) = \eta f_T \frac{v}{\Lambda} n$ However one would have the problem of reconciling the continual absorption of neutrons during the slowing down with the use of the slowing down (or age) equation. For in the derivation of the age equation no absorption term was assumed. It was assumed instead that all the fast neutrons eventually become thermal (p.14). We can however use the age equation if we decide not to be interested in the actual amount of fast neutrons in the pile, but only in those that will become thermal. These neutrons not absorbed will obey the age equation and they will be $\epsilon(1-f_R)$ of all the fast neutrons due to fission. Thus $\epsilon(1-f_R)\eta f_T \frac{v}{\Lambda} n$ fast neutrons are made per c.c. per second that eventually become thermal. This is the effective $q(0)$ and we shall write

$$q(0) = \epsilon(1-f_R)\eta f_T \frac{v}{\Lambda} n = k \frac{v}{\Lambda} n$$

This is the desired connection between q and n, and we are ready to apply our equations to an actual pile to determine its size for equilibrium.

For simplicity assume that the pile is shaped as a cube. Because n vanishes at the boundary (actually, as we have seen, the true boundary condition is $n=0$ at $.7\lambda$ outside the surface, but in this case this is but a few cm. which is small compared to "a" for piles) we expand $n(x,y,z)$ as $n(x,y,z) = \sum_{jk} n_{ijk} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{a} \sin\frac{k\pi z}{a}$



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Take only the n_{111} term for our solution. That is, we assume $n = n' \sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \sin\frac{\pi z}{a}$ (calling $n_{111} \equiv n'$) and also write $q = q' \sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \sin\frac{\pi z}{a}$ and $q(0) = q'(0) \sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \sin\frac{\pi z}{a}$ These must solve the pair of equations

$$\frac{\lambda v}{3} \Delta n - \frac{v}{\Lambda} n + q = 0 \qquad \text{(for slow neutrons)}$$

$$\Delta q = \frac{\partial q}{\partial \tau} \qquad \text{(slowing down for the effective number of fast neutrons.)}$$

subject to $q(0) = K \frac{v}{\Lambda} n$ Substituting the assumed solutions

$$-\frac{\lambda v}{3} \frac{3\pi^2}{a^2} n' - \frac{v}{\Lambda} n' + q' = 0 \qquad (\mathrm{I})$$

$$-\frac{3\pi^2}{a^2} q' = \frac{\partial q'}{\partial \tau} \qquad (\mathrm{II})$$

$$K \frac{v}{\Lambda} n' = q'(0) \qquad (\mathrm{III})$$

The second of these readily lets us evaluate the constant q'

$$q' = q'(0)\, e^{-\frac{3\pi^2}{a^2}\tau} \qquad \text{or} \qquad q' = K \frac{v}{\Lambda} n'\, e^{-\frac{3\pi^2}{a^2}\tau}$$

If this is substituted into the first equation, we get a condition on the constants of the pile that must be satisfied for a solution.

$$n'\left(-\frac{\lambda v}{3} \frac{3\pi^2}{a^2} - \frac{v}{\Lambda} + K \frac{v}{\Lambda} e^{-\frac{3\pi^2}{a^2}\tau}\right) = 0$$

This can be written $K = e^{\frac{3\pi^2}{a^2}\tau}\left(1 + \frac{3\pi^2}{a^2} \frac{\lambda\Lambda}{3}\right)$ where K is the K for the infinite lattice. Since our equations have been solved for an <u>equilibrium</u> pile, this equation gives the length of side "a" of a pile that would be in equilibrium if K were the reproduction factor for the infinite lattice. This can be written in somewhat simpler form if the exponential is expanded. $K \cong 1 + \frac{3\pi^2}{a^2}\left(\tau + \frac{\lambda\Lambda}{3}\right)$. Let us put in some numbers. τ is usually $\sim 300\ cm^2$ and $\frac{\lambda\Lambda}{3} \sim 350\ cm^2$. Thus $K \cong 1 + \frac{20000}{a^2}$ or $K \cong 1 + \frac{2}{a_m^2}$ where a_m is in meters. Since the best K for a graphite pile is ~ 1.1 , then $a_m \sim 4.5\ m$. For $K \sim 1.03$ we would need an 8 meter pile. In general $a_m \cong \frac{\sqrt{2}}{\sqrt{K-1}}$ or the volume is $a^3 = \left(\frac{2}{K-1}\right)^{3/2}$ As K approaches one, "a" becomes infinite, as is reasonable.

If the pile were a rectangular parallelpiped, it is easy to see that the condition on the dimensions would be $K = 1 + \pi^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\left(\tau + \frac{\lambda\Lambda}{3}\right)$ instead of those

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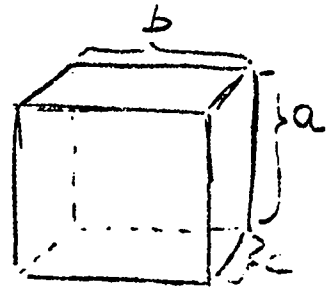

we found. It can now be shown that our neglect of the solutions n_{ijk} $(i,j,k > 1)$ in the Fourier expansion is justified for an actual pile. Had we assumed the i,j,k harmonic to exist in the pile we would find that the dimensions of the pile, when the i,j,k harmonic is in equilibrium would be given by $K = 1 + \pi^2\left(\frac{i^2}{a^2} + \frac{j^2}{b^2} + \frac{k^2}{c^2}\right)\left(\tau + \frac{\lambda\Lambda}{3}\right)$. If the pile is of such a size that the 1,1,1 harmonic is slightly over critical, say 1.021 it can be seen that for a pile of the dimensions to make the 1,1,1 harmonic critical, the higher harmonics will <u>not</u> be critical and will die out. For any shape of pile it can be shown that the main harmonic present in a pile is the fundamental.

For $a = 6m$, $b = 7m$, $c = 8m$ what values of K are needed for criticality in n_{111}, n_{211}, n_{121}, n_{112} ?

Calculate the relation between the height and the radius of a cylindrical pile given K for the infinite lattice if the pile is to be at equilibrium.

(6) <u>The Time Dependent Equations of the Pile.</u>

We have derived 5 equations to describe a pile at equilibrium. We shall here discuss the time dependence of the behavior of a pile. We shall assume for simplicity at first that all the neutrons produced are prompt. Actually the delayed neutrons are very important in running a pile. They often make the difference between running and not running. However, for simplicity let us at first make the assumption of no delayed neutrons. We also assume, and this not such a bad assumption, that the slowing down takes a time that is negligible compared to the time spent by a neutron as a thermal (actually the ratio of time is of the order 1 : 100). The slow neutron diffusion equation becomes in the time dependent case,

$$\frac{\lambda v}{3}\Delta n - \frac{v}{\Lambda} n + q = \frac{\partial n}{\partial t}$$

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and the other equations of the equilibrium description

$$\Delta q = \frac{\partial q}{\partial \tau} \qquad\qquad q(0) = k\frac{v}{\Lambda} n$$

remain unchanged because of the above assumptions, i.e., it is assumed that $q(0)$ is still proportional to n at all times, there being no delayed neutrons and no delay in the slowing down of prompt ones.

For a general shape of pile, the problem is to solve a differential equation of the form $\Delta\phi + \omega\phi = 0$ where the ω are eigenvalues, and the boundary condition is that $\phi = 0$ at (really near) the boundary. We assume that the time dependent solutions are expressible as products of a space dependent ϕ and a function depending only on the time.

$$n = f(t)\phi(x,y,z) \qquad q = g(t,\tau)\,\phi(x,y,z) \qquad q(0) = g(0,t)\,\phi(x,y,z)$$

The age equation becomes

$$\Delta q = g(t,\tau)\Delta\phi = -\omega\, g(t,\tau)\phi$$
$$= \frac{\partial q}{\partial \tau} = \phi\frac{\partial g}{\partial \tau} \qquad \text{or} \qquad \frac{\partial g}{\partial \tau} = -\omega g$$

whence $g = g(0)e^{-\omega\tau}$

From the 2nd equation at the top of the page, $q(0) = k\frac{v}{\Lambda} f$ and $g = \frac{kv}{\Lambda} f(t)\, e^{-\omega\tau}$

we can therefore write for q,

$$q = k\frac{v}{\Lambda} f(t)\,\phi\, e^{-\omega\tau}$$

Now let us find the form of solution for n. From the thermal neutron diffusion equation and $\Delta n = f(t)\Delta\phi = -\omega f(t)\,\phi = -\omega n$ we get

$$-\frac{\lambda v}{3}\omega n - \frac{v}{\Lambda} n + \frac{kv}{\Lambda} e^{-\omega\tau} n = \frac{\partial n}{\partial t}$$

where τ is the age of thermal neuts. The solution is $n = n(0)\, e^{t\left[k\frac{v}{\Lambda}e^{-\omega\tau} - \frac{v}{\Lambda} - \frac{\lambda v}{3}\omega\right]}$ an exponential.

Let us consider the exponent, $k\frac{v}{\Lambda}e^{-\omega\tau} - \frac{v}{\Lambda} - \frac{\lambda v}{3}\omega$. It is a decreasing function of ω. Recall that ω is a constant denoting some harmonic of the stationary solution. Because the set of equations is linear, the time dependence of the various harmonics is $e^{t\,F(\omega_1)}$, $e^{t\,F(\omega_2)}$ ..... etc.

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and since $F(\omega)$ decreases as ω increases, it means that after a time the fundamental harmonic will outgrow the others even if we started at $\tau = 0$ with a complicated distribution of neutrons in which some higher harmonics were important.

When the exponent is zero, we have a time-independent or equilibrium solution

$$K\frac{v}{\Lambda}e^{-\omega_1\tau} - \frac{v}{\Lambda} - \frac{\lambda v}{3}\omega_1 = 0$$

This is precisely the equation we developed giving pile sizes for equilibrium, i.e., for piles just critical, just "going".

$$K = e^{\omega_1\tau}\left(1 + \frac{\lambda v}{3}\omega_1\right)$$

where, of course, the proper ω is substituted as determined by the shape of the pile. (For a cube $\omega_1 = \frac{3\pi^2}{a^2}$; for a sphere $\omega_1 = \frac{\pi^2}{R^2}$ etc.) Taking ω to mean ω_1, the ω for the fundamental, let us rewrite the exponent

$$K\frac{v}{\Lambda}e^{-\omega\tau}\left[1 - \frac{e^{\omega\tau}}{K}\left(1 + \frac{\lambda\Lambda}{3}\omega\right)\right]$$

If the reproduction factor were $K = \left(1 + \frac{\lambda\Lambda}{3}\omega\right)e^{\omega\tau}$ then the pile would be just going, as we have seen. Calling this value of K simply K_0, the exponent can be written

$$K\frac{v}{\Lambda}e^{-\omega\tau}\left(1 - \frac{K_0}{K}\right)$$

or the time dependence is $e^{\frac{v}{\Lambda}e^{-\omega\tau}(K-K_0)t}$.

Write $\frac{n}{n(0)} \cong e^{\frac{v}{\Lambda}(K-K_0)t} = e^{t/T}$ where T is the period of the pile. (We have neglected $e^{-\omega\tau}$ because it is of order unity). T can be written

$$\frac{\Lambda/v}{K-K_0} \qquad \text{or } T \text{ is } \frac{\text{LIFE TIME OF A NEUTRON}}{K-K_0}$$

Let us reconsider our 3 equations in order to find a better expression for T, taking into account this time the delayed neutrons. The equations

$$\frac{\lambda v}{3}\Delta n - \frac{v}{\Lambda}n + q = \frac{\partial n}{\partial t} \qquad ; \quad \Delta q = \frac{\partial q}{\partial \tau}$$

remain unchanged. The last equation must be changed in the following way. If a neutron is absorbed at $\tau = 0$, 99% of the fission neutrons are delivered immediately and 1% are delivered later. At any instant therefore ~99% of the neutrons delivered are prompts and ~1% are delayed neutrons from previous fissions. If p is the fraction of neutrons

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that are delayed, the number of neuts made per unit time in an equilibrium pile is

$$q(0) = k\frac{\nu}{\Lambda}(1-p)n + pk\frac{\nu}{\Lambda}n = k\frac{\nu}{\Lambda}n$$

PROMPTS DELAYED NEUTRONS FROM PREVIOUS FISSIONS.

or the same number as if all neuts are prompt (as is expected).

But if the n's in the 2 terms are different, i.e., if we are not dealing with a pile in equilibrium, it is necessary to know the time distribution of delays. The fraction of delayed neuts emitted in interval dt at the time t after fission is

$$dt\sum_{i=1}^{m\sim 10}\frac{p_i}{\theta_i}e^{-\frac{t}{\theta_i}}$$

where the θ_i are the various delayed neutron periods. If, however, we are at a time τ, and we look back to an interval dt' at a time $t-t'$, we see that there were $k\frac{\nu}{\Lambda}n(t-t')\,dt'$ absorptions then. The number of neutrons appearing as delays from these absorptions in unit time at t is $k\frac{\nu}{\Lambda}n(t-t')\,dt'\sum_i\frac{p_i}{\theta_i}e^{-\frac{t'}{\theta_i}}$ Thus

$$q(0) = \underbrace{k\frac{\nu}{\Lambda}n(t)(1-p)}_{\text{PROMPTS}} + \underbrace{k\frac{\nu}{\Lambda}\int_0^\infty n(t-t')\,dt'\left(\sum_i\frac{p_i}{\theta_i}e^{-t'/\theta_i}\right)}_{\text{DELAYS}}$$

This, then, is the equation we must use instead of simply $q(0)=k\frac{\nu}{\Lambda}n$ when we wish to take the delayed neutrons into account.

We seek a solution of the same form as before

$$n = \text{constant} \times \phi(x,y,z) \times e^{t/T}$$

Substituting this into the $q(0)$ equation,

$$q(0) = k\frac{\nu}{\Lambda}(1-p)n + \frac{k\nu}{\Lambda}\cdot\text{const}\;\phi e^{t/T}\int_0^\infty e^{-t'/T}dt'\left(\sum_i\frac{p_i}{\theta_i}e^{-\frac{t'}{\theta_i}}\right)$$

$$\text{whence } q(0) = k\frac{\nu}{\Lambda}n(t)\left[(1-p)+\sum_i\left(\frac{p_i}{\frac{\theta_i}{T}+1}\right)\right] \equiv An$$

Just as before we find

$$q = q(0)e^{-\omega\tau} = Ane^{-\omega\tau}$$

for the q equation, but for the n equation,

$$-\frac{\Lambda\nu}{3}\omega n - \frac{\nu}{\Lambda}n + Ane^{-\omega\tau} = \frac{n}{T}$$

so that the life time is

$$T = \left[-\frac{\Lambda\nu}{3} - \frac{\nu}{\Lambda} + k\frac{\nu}{\Lambda}\left(1-p+\sum_i\left(\frac{p_i}{\frac{\theta_i}{T}+1}\right)\right)e^{-\omega\tau}\right]^{-1}$$

This expression for T can be rewritten $T=\left[-\frac{\Lambda\nu}{3}-\frac{\nu}{\Lambda}+\frac{k\nu}{\Lambda}\left(1-\sum_i\left(\frac{p_i\theta_i}{\theta_i+T}\right)\right)e^{-\omega\tau}\right]^{-1}$

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and one can solve for K as a function of the period T

$$K = \frac{e^{\omega\tau}\left[\frac{\Lambda}{v}\frac{1}{T} + 1 + \frac{\lambda\Lambda}{3}\omega\right]}{1 - \sum_i \frac{p_i\theta_i}{T+\theta_i}}$$

where $\left(1 + \frac{\lambda\Lambda}{3}\omega\right)e^{\omega\tau}$ is recognized as K_0, the K needed to keep the pile in equilibrium, i.e.,

$$K = \frac{K_0 + \frac{\Lambda}{v}\frac{e^{\omega\tau}}{T}}{1 - \sum_i \frac{p_i\theta_i}{T+\theta_i}}$$

note that as $T \to \infty$ (pile at equilibrium) $K \longrightarrow K_0$ as it should.

Since the 2nd term of the denominator is small, we can write without appreciable error,

$$K = \left(k_0 + \frac{\Lambda}{v}\frac{e^{\omega\tau}}{T}\right)\left(1 + \sum_i \frac{p_i\theta_i}{T+\theta_i}\right)$$

The orders of magnitude are $\Lambda \sim 300$ cm, $v \sim 2\times10^5 \frac{cm}{sec}$ $e^{\omega\tau} \sim 1$ $T \sim 1$ sec. so that the 2nd term in the first factor is small compared to K_0 which is ~ 1. Therefore, multiplying out the expression for K and neglecting the product $\frac{\Lambda}{v}\frac{e^{\omega\tau}}{T}$ times $\sum_i \frac{p_i\theta_i}{T+\theta_i}$ since both factors are small,

$$K = k_0\left[1 + \frac{\Lambda e^{\omega\tau}}{v k_0 T} + \sum \frac{p_i\theta_i}{T+\theta_i}\right]$$

The relative excess of reactivity $\frac{K-k_0}{k_0}$ can be written

$$\frac{K-k_0}{k_0} = \frac{\Theta_0}{T} + \sum_i \frac{p_i\theta_i}{\theta_i + T} \qquad \text{where } \Theta_0 \equiv \frac{\Lambda e^{\omega\tau}}{v k_0}$$

Θ_0 is approximately $\frac{\Lambda}{v}$ (since $e^{\omega\tau}$ & k_0 are of order 1) which is the lifetime of a neutron.

The expression for $\frac{K-k_0}{k_0}$ can be interpreted physically quite easily if T is either very big or very small. For T large, $\frac{K-k_0}{k_0}$ becomes $\frac{\Theta_0 + \sum p_i\theta_i}{T}$ What is the physical meaning of the numerator in this expression? The average length of time of a generation is simply $\Theta_0(1-p) + \sum_i p_i(\theta_i + \Theta_0)$ where Θ_0 is the generation time for prompts and $\Theta_0 + \theta_i$ is the generation time for a neutron delayed θ_i seconds. This expression is exactly equal to the numerator in $\frac{K-k_0}{k_0}$ for T large.

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Some of the known delayed neutron periods for U^{235} are given at the right. Using this data let us get an idea of the magnitude of the average length of time of a generation. $\Theta_0 \sim .0015$ sec. $\sum_i p_i \Theta_i \sim .083$

$p_i \times 100$	Θ_i,
.077	.6 sec
.220	2.2
.194	6.51
.152	31.7
.023	80.2
.666	

Thus the contribution to the generation time of the delays is ~ 60 times that of the prompts. Were there no delays, it is seen that T would be very short even for small excess K.

Writing the formula for $\frac{k-k_0}{k_0}$ for short periods, i.e., $T \ll \Theta_i$ Then $\frac{k-k_0}{k_0} = \frac{\Theta_0}{T} + p$ and the excess k depends mainly on the prompts for p is small. In practice one operates a pile with $T \gg \Theta_i$ and there, we have seen, that the delayed neutrons do play a considerable role.

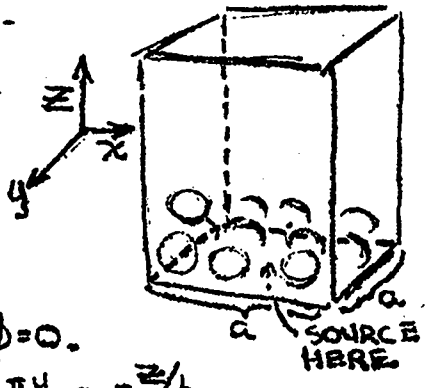
Given $\Theta_0 = .0015$, plot $\frac{k-k_0}{k_0}$ vs $\frac{1}{T}$ using the complete formula $\frac{k-k_0}{k_0} = \frac{\Theta_0}{T} + \sum_i \frac{p_i \Theta_i}{T+\Theta_i}$

(7) <u>Finding K and Proper Pile Dimensions Experimentally</u>.

Since the critical dimensions of a pile depend on $K-1$ where K is close to one, small errors in K give rise to large uncertainties in the pile dimensions required.

To determine K experimentally for a particular lattice one can proceed as follows. Construct a small (say $\frac{1}{10}$ pile size) lattice-work of the type to be used and place a constant neutron source at the middle of one end. An equilibrium condition will be set up which will still be described by the equations $\frac{\Lambda v}{3}\Delta n - \frac{v}{\Lambda} n + q = 0$ $\quad \Delta q = \frac{\partial q}{\partial t}$

$$q(0) = K \frac{v}{\Lambda} n$$



We assume again the solutions for q and n have the space dependence of a function $\phi(x,y,z)$ obeying $\Delta\phi + \omega\phi = 0$. Because of the boundary conditions, assume $\phi = \sin\frac{\pi x}{a}\sin\frac{\pi y}{a} e^{-z/b}$ (where "a" is the side of the column and b is the decay factor in the exponential). For such a ϕ it

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can easily be shown that $\omega = \frac{2\pi^2}{a^2} - \frac{1}{b^2}$. We assert that the equation relating the reproduction factor for the infinite lattice to ω (of page 91) is still true. That is

$$K = \left(1 + \frac{\lambda\Lambda}{3}\omega\right) e^{\omega\tau}$$ where τ is $\tau_{thermal}$ again.

This is true for essentially the same reasons as for the full size pile. Given a K, the ω in this equation will give the dimensions of the pile at equilibrium. In this case "a" is a true dimension and "b" behaves as one. Were we to keep "a" fixed and enrich the lattice (K larger) certainly the decay would be slower -- i.e., b longer, and this is borne out by the formula in the same way as the dependence of the true dimensions on K. For a cubical pile in equilibrium $K = \left(1 + \frac{\lambda\Lambda}{3}\omega\right) e^{\omega\tau}$ where ω is $\frac{3\pi^2}{A^2}$ this time, A being a side of the cube. Hence $\frac{3\pi^2}{A^2} = \frac{2\pi^2}{a^2} - \frac{1}{b^2}$ or a measurement of "b" in a model lattice permits one to find the size of the A of the equilibrium pile.

One can find the properties of graphite or uranium that determine K for the pile in another way. This method has to do with the effect on equilibrium piles by absorbers. Let us consider for a moment the problem of the control of a pile. A pile is always constructed enough over critical size so that it can easily be brought below critical by means of absorbers. It should be added, K is temperature dependent and one must be able to compensate for this. One must also be able to compensate for the absorption due to foreign materials, if one wants to use it for, say, making radioactive substances. Thus a pile is planned with excess activity and control rods are relied upon to vary K from below to above 1. One might use a cadmium rod which one inserts to various depths in a channel of the pile. To see the effect, suppose one placed an absorber in the pile where the neutron density was n. It is not true (as might be expected) that the absorbtivity at any point in the pile is proportional to n. Consider an absorbing atom at the center of a pile and one near the periphery. The absorbing effect of these atoms is proportional approximately to the square of n at the 2 positions. The reason for

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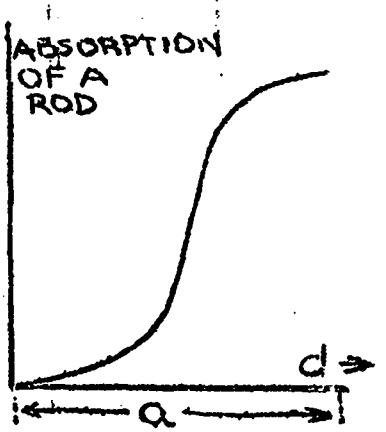
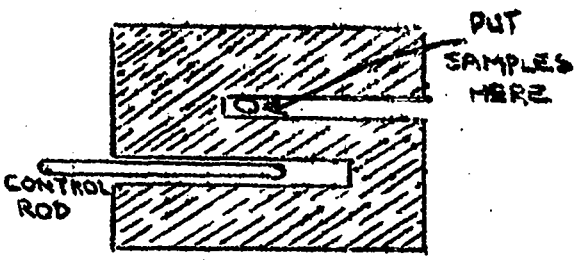
this is along the following lines. Neutrons absorbed by the centrally located atom are more likely to be of service in the pile than peripheral neutrons. The effective absorption is really (neutron density $\times$ effectiveness of neuts). It can be shown that the second factor is proportional to n and so the product goes as n^2. Thus if a control rod is inserted into a pile to depth d, the dependence of the absorption of the rod on d is proportional to $\int_0^d \left(\sin \frac{\pi x}{a}\right)^2 dx$ in a cubical pile of side a.



One can use control rods to test samples of graphite or absorbing materials as follows. One builds the pile with a channel for the insertion of samples. The critical (equilibrium) position for the control rod with no sample is determined. Then new critical positions for the various samples are determined and the differences in critical position are a measure of absorptivity of the sample. One can use such a scheme to determine reactivities for pieces of uranium correct to 5 places.



(8) <u>The Energy and Radiation Production of a Pile</u>.

The number of fissions produced per c.c. per second in a pile is $f_T n \frac{v}{\Lambda} \frac{3.9}{7.2}$ where the factors are: the fraction of thermals absorbed by the V, the neutron density, the reciprocal lifetime of a thermal, and the fraction of neuts absorbed by V that lead to fissions. Assuming 200 Mev per fission, we must multiply the number of fissions/sec. cc. by $200 \frac{\text{Mev}}{\text{Fission}} = 3.2 \times 10^{-4} \frac{\text{ergs}}{\text{fission}}$ in order to get $\frac{\text{ergs}}{\text{sec/c.c.}}$. In the resulting expression the product nv, the flux, appears. We should use $\overline{nv}$, the average flux and it can be shown that for a cube, this is related to $n_0 v$, the flux at the middle, thru $\overline{nv} = n_0 v \frac{8}{\pi^3}$ Finally, killowatts $= n_0 v \frac{8}{\pi^3} V \frac{f_T}{\Lambda} \frac{3.9 \times 3.2}{7.2} \times 10^{-14}$*

Putting in proper numbers this is $\sim 4 \times 10^{-9}\, n_0 v$. One can obtain powers of the order 5×10^3 KW (Clinton) which means that the flux at the center is $\sim 10^{12}$. At Hanford neutron fluxes at the center of the piles are $\sim 10^{13} \frac{\text{neutrons}}{\text{sec} \times \text{cm}^2}$.

*V is the volume of the pile.

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In a 1 killowatt pile there are $\sim 3 \times 10^{13}$ fissions/sec. Assuming 20 Mev of γ radiation per fission, and that 5% of this is not absorbed, then the total radiation from the pile is 3×10^{13} γ's of $\sim$ 1 Mev or the equivalent per second. For a pile of K killowatts, we must multiply by K. The surface area of a pile is $\sim 4\pi(500)^2 cm^2$ so that someone near the pile would get in one day (10^5 sec)

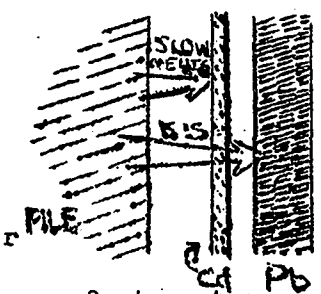
$$\frac{K \cdot 3 \times 10^{13}}{4\pi\ (500)^2} \cdot 10^5 \sim 10^{12} K \quad \frac{\text{One Mev Photons}}{cm^2 \cdot \text{Day}}$$

Now a daily dose is $\sim 10^8$ photons/cm^2 · day. (This corresponds to about 1 R unit). Thus a pile gives of the order

$$10^4 \cdot K \quad \frac{\text{DAILY DOSES}}{\text{DAY}}$$

This gives an idea of the difficult shielding problem, since K is usually $\sim 5 \times 10^3$. For large piles we must shield by a factor of a billion or so.

In addition to the problem of shielding from the γ's one must shield from the neutrons. One can use cadmium to shield from thermal neuts, but one must be careful to put the cadmium inside the Pb shield used to cut our the pile's γ's since the (n,γ) reaction in the cadmium makes many γ's. However neither the Cd nor the Pb are very effective against the very fast neutrons. Concrete can be used to help cut out the fast neutrons. It helps slow them down and capture them. However, concrete is bulky. Layers of Fe alternated with layers of paraffin form a less bulky and somewhat more effective shield. About 4 feet of this will absorb by a factor 10^9 for the γ's, and a reasonable amount of the neuts.



Let us estimate the minimum bulk of a radiation shield for a pile. Since γ's will attenuate $\sim \frac{1}{e}$ in 25 $\frac{gm}{cm^2}$ we would need at least 500 $\frac{gm}{cm^2}$ to attenuate only the γ's by 10^8. Since we need neutron protection too, a compromise shield will be at least 1000 $\frac{gm}{cm^2}$. This is the minimum bulk of shield that can do the job. (This, incidentally, is one of the difficulties in making cars diven by piles.) The lightest portable nuclear energy source based on these principles would weigh about 50 tons.

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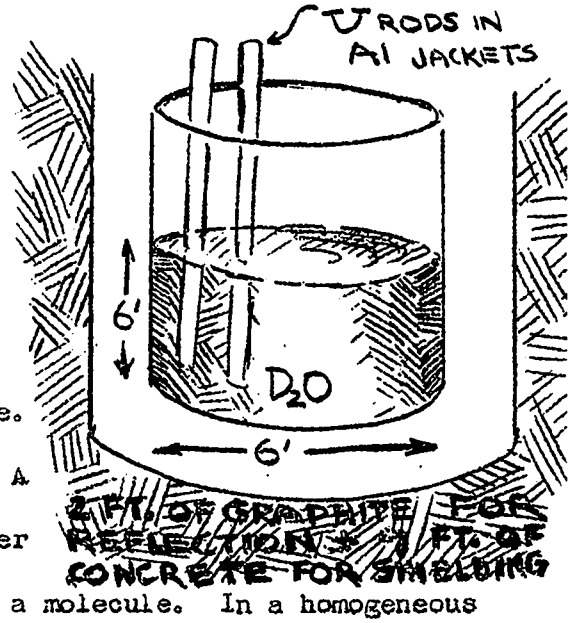
Let us consider the radiation the pile emits via the cooling system. In Clinton the cooling system is an air system, which blows through the channels for the uranium. None of the fission product gases give trouble (the U slugs are in Al cans) but some of the argon in the air becomes radioactive. The air is disposed of through a 100 foot stack and is quite dilute and harmless. At Hanford the cooling system is a water system. There the problem of disposing of radioactive water arises. It is diluted to harmlessness. (The salmon of the Columbia River are still healthy.)

(9) Other Types of Piles.

In addition to graphite piles other types are possible. For example, there's a D_2O pile at Argonne that can run at 300 KW. For this pile $\eta_0 \upsilon = 10^{12}$ or the same as for the 5000 KW Clinton pile. The reason the latter has so much more power is simply due to the fact that one can put more uranium in the Clinton pile. Heavy water has some advantages and some disadvantages over carbon. It makes a pile easier to cool (at low power). It is still, however, expensive. The advantage of smaller size piles is very important. A very important disadvantage is the decomposition of water by the energy of the pile. Only 100 ev will decompose a molecule. In a homogeneous pile, a solution, this would be especially serious for most of the 200 Mev per fission would be picked up by the water and ~ 2 mg per KW per sec. would be decomposed. This would necessitate schemes for recovery of the D_2O.



One can use different materials as moderators if enriched rather than natural U is used. Since for every 3.9 neuts absorbed in 235 giving fission, .7 & 2.5 give γ's in 235 and 238 respectively, it is seen that the number of neuts, per thermal absorbed by U would be $\frac{3.9}{4.6} \times 2.4 = 2.05$ rather than $\frac{3.9}{7.1} \times 2.4 = 1.32$

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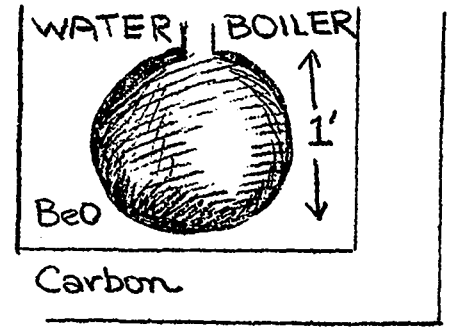
if we used pure 235. Actually even a 14% enrichment of U^{235} makes a considerable difference, and this is the material used with the water boiler at Omega. Water makes a good enough moderator. This particular pile can run at ~5 KW. One problem connected with this pile is the disposal of radioactive fission products. Such piles are small and it is feasible to build them for use as sources of neutrons for experimental purposes.



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NEUTRON PHYSICS

CHAPTER X

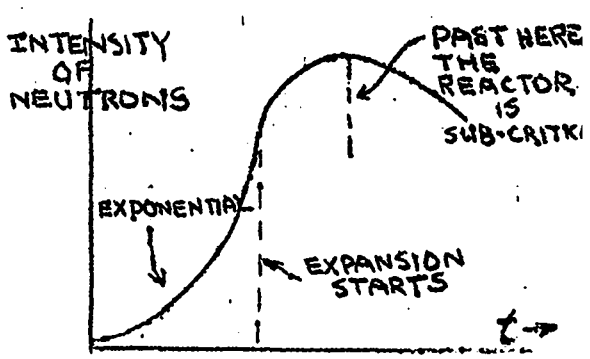
Fast Reactors

(1) Producing a Nuclear Explosion.

We have spoken of chain reactions maintained by thermal neutrons. We shall see that in order to produce nuclear explosions one had best make use of the fast neutrons to maintain the chain. It will be convenient to speak of e-folding times, the time it takes n, the neutron density, to be multiplied by a factor e in a supercritical assembly. We have seen (p.91) that this time is the lifetime of a neutron $\frac{\Lambda}{v}$ divided by $K-K_0$ For a thermal pile, $\frac{\Lambda}{v} \sim 10^{-3}$ sec and $K-K_0 < .1$ or the e-folding time is $\sim 10^{-2}$ seconds or longer. In a water boiler $\Lambda \sim 50$ cm, $v \sim 10^5$ and $K \sim 2$ so that the e-folding time is $\frac{\Lambda}{v}/K-1 \sim 10^{-4}$ seconds. In a fast reactor (say U^{235} or Pu) $\Lambda \sim 10$ cm and $v \sim 10^9$, $K \sim 2$ so that for a fast reactor, we get $\frac{10}{10^9} \cdot 1$ or the e-folding time is $\sim 10^{-8}$ seconds. All the times given are minima -- i.e, we have assumed the most favorable conditions in each type of reactor.

For the industrial problem, the advantage is in long e-folding times, for it is such reactors that are most controllable. For explosives, it is short e-folding time reactors that become important.

In a kg. of fissionable material there are $\sim 2 \times 10^{24}$ atoms. If α is the reciprocal of the e-folding time then a reaction could continue for at most t seconds where $e^{\alpha t} \sim 10^{24}$. This would give times: $t = .5,\ .005,\ .5\mu$, sec for the graphite pile, water boiler and fast reactor respectively. Actually mechanical effects begin to take place 5 or so e-folding times before the using up of all material. What happens here is that after many generations the energy release is great enough so that pressures expand the reactor. Its surface increases, leakage increases and eventually the reactor falls below critical and from there on the reaction is just dying out.



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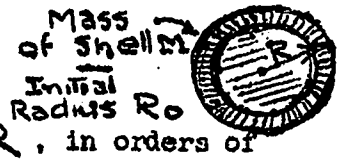
Let us consider this more quantitatively. Say the pressure goes as $p = p_0 e^{\alpha t}$ and we are dealing with a spherical reactor. The surface will feel an acceleration that is of order

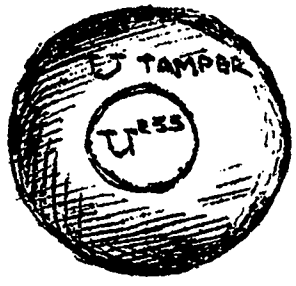
$$\ddot{R} \sim \frac{pR^2}{M} = \frac{p_0 R^2}{M} e^{\alpha t}$$



Since the exponential varies much more rapidly with time than R, in orders of magnitude the displacement of the surface as a function of time is

$$\delta R \sim \frac{p_0 R^2}{M} \frac{e^{\alpha t}}{\alpha^2} = \frac{pR^2}{M\alpha^2}$$

where p is the pressure at the time t. What is the total amount of energy released? The final δR will be of order of the difference of R_0 and the radius of the "just critical" sphere. The energy released up to the point that the criticality goes below 1 is of order $P_{MAX} \times \frac{4\pi R^3}{3}$ Thus, the energy $W \sim M\alpha^2 R(R_{CRIT} - R_0)$ and it is seen that α enters as the square. Clearly, to produce an explosion, we must use a reactor with the shortest possible e-folding time.



(2) <u>The Integral Equations for Neutron Distribution.</u>

Let us calculate the critical properties of a fast reactor of the type shown. In such a reactor the size of active material is ~10 cm and the mean free path is ~10 cm. Consequently we cannot use the approach we used in pile theory, that is we may not use differential diffusion theory, for it is not true that mean free paths are << dimensions of the system. In the following approach we shall have to introduce among other things the transport cross-section (i.e., the cross-section modified to take account of the non-isotropic elastic scattering). Let us call σ the sum of the transport cross-sections(p.44) of all the atoms per c.c. Then (since $\sigma N \lambda = 1$) this cross-section is simply the reciprocal of the mean free path. The processes that can occur upon collison other than transport (elastic scattering) are:

	No. of neutrons hitting nucleus	No. of neutrons leaving
Capture	1	0
Inelastic. Scat.	1	1
Fission	1	ν

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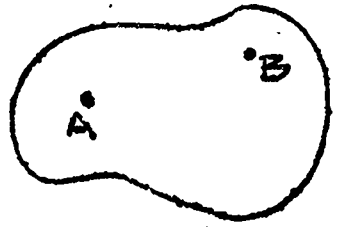
On the average, for every neutron hitting a nucleus, $1+f$ come out, where f is either positive or negative depending on the type of material used. $1+f$ can be written in general:

$$1+f = \frac{0\cdot\sigma_{abs} + 1\cdot\sigma_{sc} + \nu\cdot\sigma_f}{\sigma_{total}} \quad \text{or} \quad f = \frac{-\sigma_{abs} + (\nu-1)\sigma_f}{\sigma_{total}}$$

In a non-fissionable material $f = -\frac{\sigma_a}{\sigma_{sc}+\sigma_a}$ and f is negative & $|f|<1$ both of which are reasonable. For a chain reaction to take place, it is necessary that at least in some part of the system $f>0$.

In a proper calculation of the operation of a fast reactor one would consider what happens to neutrons in various energy ranges, but for our purposes we shall assume that all the neutrons have some average energy (one group theory).

We shall establish an integral equation for the neutron density in terms of σ and f. If at a certain point, one neutron is produced each second, what is the neutron density at a distance r from this point if the emission is isotropic? The time spent in a shell of volume $4\pi r^2 dr$ by a neutron is $\frac{dr}{v}$. The average number of neutrons found in the shell is $\frac{dr}{v}$ since 1 neut is emitted per second. The neutron density at the distance r is then $\frac{dr}{v}$ divided by $4\pi r^2 dr$ or $\frac{1}{4\pi r^2 v}$. It is to be remarked that this result is based on the assumption that there is no absorption in the medium. Suppose we now wish to find n_A, the neutron density at a point A in a critical reactor. To find this, find the contributions to this density of a volume element at B, i.e., we try to find how many collisions at B result in scatterings in the direction of A. We then integrate the volume element dB over all space. In the volume element dB there are $n(B)v\sigma(B)dB$ collisions per second. And $n(B)dB\sigma(B)v(1+f(B))$ neutrons come out of the volume dB each second. Thus dB can be thought of as a neutron source of this strength. But we have seen that such a source will give a density



$$\frac{n(B)\,dB\,v\,\sigma(B)\,[1+f(B)]}{4\pi r^2 v}$$

at a point r away if there is no absorption and the emission is isotropic. We can

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account for the absorption by multiplying by a factor $e^{-\int_A^B \sigma\, dr}$

Thus $$n(A) = \int_{\substack{\text{all}\\ \text{space}}} \frac{n(B)\,\sigma(B)\,[1+f(B)]\,e^{-\int_A^B \sigma\, dr}}{4\pi\, r_{AB}^2}\, dB$$

and this is the integral equation we sought. For a system in equilibrium another equation must hold.

$$v \int_{\substack{\text{all}\\ \text{space}}} n(A)\, dA\, \sigma(A)\, f(A) = 0$$

for there will be no net production of neutrons in the system.

(3) <u>The Critical Size of a Fast Reactor.</u>

These two equations can be considered the equilibrium equations of a fast reactor (assuming a one group theory) just as the differential equations of p. 88 were the equilibrium equations for a pile. We shall use these equations also to determine the critical size.

But before we do let us return to differential theory. We had $l = \lambda\sqrt{\frac{N}{3}}$ (p. 55) where $N = \frac{\sigma_{total}}{\sigma_{abs.}}$ and we assumed that N was a rather large number. This assumption crept in somewhat as follows. We assumed that the flux of neutrons is everywhere proportional to the gradient neglecting higher derivatives. This is the same as assuming that $n(x)$ is a function that is fairly linear for distances of order λ. In a one dimensional problem $n(x)$ goes as $e^{x/l}$ and for this function to show little curvature in a distance λ, it is necessary that $\frac{\lambda}{l}$ be a small number, or that N be large. Thus our original diffusion equation implicitly assumed that N was large. Try to modify differential diffusion theory to be valid for N small. Say we have a gradient of neutron density along the x axis. Because of this gradient, the density of neutrons travelling at any angle θ to the x axis, is not the same for all θ at a given point, but is a maximum for neutrons moving in the direction of the gradient. Write as the neutron density moving at the angle θ at the point x

$n(x,\theta)\cdot d\tau \cdot 2\pi \sin\theta\, d\theta$ or writing $\xi = \cos\theta$, this is

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$n(x,\xi)\,d\tau \cdot 2\pi d\xi$ where $2\pi d\xi$ is the element of solid angle. We wish to establish an equation for $n(x,\xi)$. We follow the procedure of examining all the factors for the variation of $n(x,\xi)$ with time.

Since we are assuming that all the neutrons have the same velocity, neutrons everywhere moving in the direction ξ_0 will have an x component of velocity $v \cdot \xi_0$. If n varies with x, then the change in $n(x,\xi_0)$ at x in time dt will be $n(x - v\xi_0 dt) - n(x)$ which is simply $-v\xi_0 dt \frac{\partial n}{\partial x}$, where we have considered neutrons moving in the specific direction ξ_0

Another source of neutron loss from our element $dx\,d\xi$ is the fact that in each unit time, a number of neutrons, $\frac{v}{\lambda} n$, will suffer collisions. If the neutrons in dx were initially moving in direction ξ, once they have collided they will be moving in some other direction and will no longer belong to the same $dx\,d\xi$.

We must now find how many neutrons are scattered into the solid angle dw by collisions where dw is $2\pi d\xi_0$. For the various directions ξ' at x there will be $n(x,\xi')dw' \frac{v}{\lambda}$ collisions out of dw' in unit time. $\frac{N-1}{N}$ of these neutrons will be moving in other directions after collision. $\frac{1}{N}$ will be absorbed. Therefore the number of neutrons initially moving in direction ξ' that are scattered into the direction ξ in unit time is

$$n(x,\xi')\,dw' \frac{v}{\lambda} \frac{N-1}{N} \frac{dw}{4\pi}$$

We must integrate this over dw' to get the total number of neutrons at x scattered into the direction ξ_0. Per unit solid angle, the term to be added to our equation, is

$$\frac{N-1}{N} \cdot \frac{v}{2\lambda} \int_{-1}^{1} n(x,\xi')\,d\xi'$$

(where we have put $2\pi d\xi'$ for dw')

At equilibrium then,

$$-\xi \frac{\partial n}{\partial x} - \frac{n}{\lambda} + \frac{N-1}{2N\lambda} \int_{-1}^{1} n(x,\xi')\,d\xi' = 0$$

To solve this we assume that the solution is separable, i.e., $n(x,\xi) = n(x) f(\xi)$ Further, we assume, because of our past experiences that $n(x)$ will be of the form

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$e^{\frac{x}{\ell}}$ where ℓ must be determined from the constants of the problem. Substituting into the equation, we find that $f(\xi)$ must obey the following equation.

$$\frac{1}{\lambda} f(\xi) + \frac{1}{\ell}\xi f(\xi) = \frac{N-1}{2N\lambda}\int_{-1}^{1} f(\xi')\,d\xi'$$

or $\left(\frac{\lambda\xi}{\ell}+1\right) f(\xi) = \frac{N-1}{2N}\cdot\int_{-1}^{1} f(\xi')\,d\xi'$

but the right side of the equation is some constant, so that f must be of the form $f(\xi) = \frac{C}{1+\frac{\lambda}{\ell}\xi}$. We must now make sure, however, that this has at least some values of ℓ that do give solutions. That is, we have established that if there are solutions, they are of the form $\frac{C}{1+\frac{\lambda}{\ell}\xi}$, but we have yet to see if this is true for all ℓ, for some ℓ or for no ℓ. Substituting into the equation, we find:

$$1 = \frac{N-1}{2N}\int_{-1}^{1}\frac{d\xi}{1+\frac{\lambda}{\ell}\xi}$$

whence $1 = \frac{N-1}{2N}\cdot\frac{\ell}{\lambda}\log\left(\frac{\ell+\lambda}{\ell-\lambda}\right)$ This is an equation that tells us what value of ℓ will solve our problem. To see what the solution is when N is still a fairly large number, expand $\frac{2N}{N-1}$ and $\frac{\ell}{\lambda}\log\left(\frac{\ell+\lambda}{\ell-\lambda}\right)$

$\frac{2N}{N-1} = 2\frac{1}{1-\frac{1}{N}} = 2\left(1+\frac{1}{N}+\frac{1}{N^2}+\quad\right)$ which is developed in powers of $\frac{1}{N}$ because N is >1. Writing $\frac{\lambda}{\ell}=\epsilon$, and assuming that ϵ is small,

$$\frac{1}{\epsilon}\log\frac{1+\epsilon}{1-\epsilon} = \frac{1}{\epsilon}\left[\left(\epsilon-\frac{\epsilon^2}{2}+\frac{\epsilon^3}{3}-\frac{\epsilon^4}{4}+\ldots\right)+\left(\epsilon+\frac{\epsilon^2}{2}+\frac{\epsilon^3}{3}+\frac{\epsilon^4}{4}+\ldots\right)\right]$$

$$= 2\left[1+\frac{\epsilon^2}{3}+\frac{\epsilon^4}{5}+\ldots\right]$$

Hence $\frac{1}{N}+\frac{1}{N^2} = \frac{\epsilon^2}{3}+\frac{\epsilon^4}{5}$

The first approximation gives $\epsilon^2 = \frac{3}{N}$, which gives precisely $\ell = \lambda\sqrt{\frac{N}{3}}$, the result from our rougher differential diffusion theory. In the next approximation we may substitute for the correction term, $\frac{\epsilon^4}{5}$, the first approximation result. Then

$$\frac{1}{N}+\frac{1}{N^2} = \frac{\epsilon^2}{3}+\frac{9}{5}\cdot\frac{1}{N^2}$$

or $\frac{1}{N}\left(1-\frac{4}{5N}\right) = \frac{\epsilon^2}{3}$ which gives $\frac{\lambda}{\ell} = \sqrt{\frac{3}{N}\left(1-\frac{4}{5}\frac{1}{N}\right)}$

giving ℓ correct to the second order in N.

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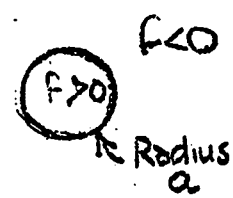
We shall now return to our integral equation making use of what we have here developed. In the integral equation $1+f$ stood for the number of neutrons coming out, on the average, after a collision. For a medium where there is only capture (no fission) $f = -\frac{1}{N}$. σ was $\frac{1}{\lambda}$. One can therefore write for $\frac{1}{\ell}$,

$$\frac{1}{\ell} = \sigma\sqrt{-3f\left(1+\frac{4}{5}f\right)}$$

It can be shown that this formula applies even with f positive (say for U235) in which case the density goes as $e^{\frac{ix}{|\ell|}}$ and $e^{-\frac{ix}{|\ell|}}$ or we have sines and cosines. The method of proof that this applies where f is positive would be essentially a repetition of the calculation, where allowance would be made for fission. It should be remarked that the equation $\Delta n - \frac{n}{\ell^2} = 0$ now holds whether there is production of neutrons throughout the volume or not. The q term in the complete equation (p.48) becomes unnecessary. Neutron production is taken care of by the new more general expression for ℓ. This expression was developed under the assumption that n goes as $e^{\frac{x}{\ell}}$ and is a function of x only. We shall however apply this in our more general 3 dimensional integral equation, (for a more exact and laborious calculation can be shown to give quite similar results). This method was developed by Serber and we shall apply it to the simplest possible geometry, i.e., $f>0$ in a spherical core and $f<0$ outside. We begin by writing $\Delta n - \frac{1}{\ell^2} n = 0$ where ℓ is the corrected ℓ, i.e., we begin with a somewhat improved diffusion theory. On the inside $\ell^2 < 0$ and the solutions are like sines and cosines. Outside $\ell^2 > 0$ and the solutions are exponentials.



We may summarize the essential results to this point.

INSIDE THE SPHERE $(r<a)$

$$\Delta n - \frac{n}{\ell^2} = 0 \qquad \sigma = \frac{1}{\lambda}$$

$$h \equiv \frac{1}{\sigma\ell} \qquad H \equiv -ih$$

whence

$$\frac{1}{1+f} = \frac{1}{2iH}\log\frac{1+iH}{1-iH} = \frac{1}{H}\arctan H$$

OUTSIDE $(r>a)$

$$\Delta n - \frac{n}{\ell'^2} = 0 \qquad \sigma' = \frac{1}{\lambda'}$$

$$h' \equiv \frac{\lambda'}{\ell'} \text{ and from } \frac{N}{N-1} = \frac{1}{2h'}\log\frac{1+h'}{1-h'},$$

$$\frac{1}{1+f'} = \frac{1}{2h'}\log\frac{1+h'}{1-h'}$$

$$\frac{1}{\ell'} = \sigma'\sqrt{-3f'\left(1+\frac{4}{5}f'\right)}$$

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In either case $\Delta n(r) = \frac{d^2 n}{dr^2} + \frac{2}{r}\frac{dn}{dr}$ so that multiplying the equation by r)

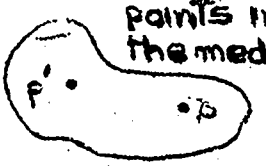
$$r\frac{d^2 n}{dr^2} + 2\frac{dn}{dr} = \frac{1}{\ell^2} nr$$

but the left side is $\frac{d^2}{dr^2}(nr)$, whence we have an equation of the one dimensional form in (nr). The solution is $rn = Ae^{-r/\ell'} + Be^{r/\ell'}$ on the outside. On the inside $\ell = -\frac{1}{\sigma H}$ so that the solution is $rn = C\sin(\sigma Hr) + D\cos(\sigma Hr)$ In terms of σ' rather than ℓ' the outside solution is $Ae^{\sigma' h' r} + Be^{-\sigma' h' r}$ where we may certainly neglect the first term because it represents a diverging solution. Similarly the $D\cos(\sigma Hr)$ term is eliminated on the inside because it gives n infinite at the origin. Let us normalize our solutions so that $n(0) = 1$ Then $n = \frac{C}{r}\sin(\sigma Hr)$ which implies $C = \frac{1}{\sigma H}$. The solution inside is $n = \frac{\sin(\sigma Hr)}{\sigma Hr}$ Outside, the solution is $\frac{Be^{-\sigma' h' r}}{r}$ where B is determined by an appeal to the two integral equations that must be satisfied.

$$n(P') = \int \frac{n(P)\sigma(P)[1+F(P)]e^{-\int_{P'}^{P}\sigma dr}\, dP}{4\pi r_{PP'}^2}$$

$$\int n(P)\sigma(P)F(P)\,dP = 0$$

where P' and P are two points in the medium



We will be unable to satisfy them exactly. We shall satisfy the 2nd equation; but instead of satisfying the first at all points P', we shall do so only at the center. The 2nd equation will suffice to determine B . We then substitute into the first equation. If it checks, we say we are dealing with a critical (equilibrium) assembly. If, however, the first equation does not check, the sphere is either too large or too small for the particular f and f' and we must try a different radius.

Let us therefore substitute into the 2nd equation. The integral must be written in 2 parts (inside and outside)

$$4\pi\int_0^a \sigma f \frac{\sin(\sigma Hr)}{\sigma Hr} r^2 dr + 4\pi\int_a^\infty \sigma' f' \frac{Be^{-\sigma' h' r}}{r} r^2 dr = 0$$

Since f' is negative, we may write

$$\sigma f\int_0^a \frac{\sin(\sigma Hr)}{\sigma Hr} r^2 dr = \sigma'|f'|B\int_a^\infty \frac{e^{-\sigma' h' r}}{r} r^2 dr$$

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These are simple integrals and the result is

$$B = \frac{h'^2\sigma'^2 f}{H^3\sigma^2(-f')} \; \frac{\sin(H\sigma a) - H\sigma a \cos H\sigma a}{1 + h'\sigma' a} \, e^{h'\sigma' a}$$

We now substitute into the first integral equation, letting A refer to the point $r=0$

Since $n(0)=1$, we may write (once again splitting the integral into 2 parts)

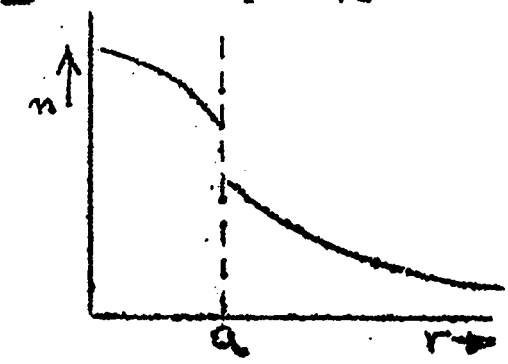
$$1 = \int_0^a \frac{\sin(\sigma H r)}{\sigma H r} \cdot \sigma \cdot (1+f) \cdot e^{-\sigma r} \, \frac{4\pi r^2}{4\pi r^2} \, dr$$
$$+ \int_a^\infty B \cdot \frac{e^{-\sigma' h' r}}{r} \cdot \sigma' \cdot (1+f') \cdot e^{-\sigma a - \sigma'(r-a)} \cdot \frac{4\pi r^2}{4\pi r^2} \, dr$$

These integrals here are slightly more difficult than the previous ones, but they are in the tables. Putting in the value of B, the result is

$$\frac{1+f}{f} H^2 (\sigma a)^2 e^{\sigma a} \frac{\operatorname{ArcTan} H - \int_0^{\sigma a} \frac{\sin(Hx) e^{-x}}{x} dx}{\sin(H\sigma a) - H\sigma a \cos(H\sigma a)} = \frac{1+f'}{-f'} h'^2 (\sigma' a)^2 e^{\sigma' a (1+h')} \frac{\int_{(1+h')\sigma' a}^{\infty} \frac{e^{-x}}{x} dx}{1 + h'\sigma' a}$$

On the left appear quantities related to the core. On the right are the quantities related to the outside reflector or tamper. We substitute in this equation the values for f and H for the particular materials being considered. We can then plot the left and right hand sides as a function of "a". The curves will intersect at some point and this value of "a" is the critical or equilibrium radius.

Having found the critical radius, we may find B and we can plot n as a function of r. It will be found that the curves show a discontinuity. This discontinuity, of course, will not exist -- but it "almost exists", i.e., the gradient of n is fairly sharp at the boundary as can be shown by a more exact calculation.



(4) <u>Supercritical Reactors.</u>

With not very involved modifications, the theory here discussed can give information about supercritical as well as equilibrium systems. We had written

$$n(p') = \int \frac{n(p)\,\sigma(p)\,[1+f(p)]\, \int_{p'}^{p} \sigma\, dr}{4\pi r_{pp'}^2} \, dp$$

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This equation says that all the neutrons in p' at a time t comes from points where they suffered their last collision at some earlier time, t'. In a system at equilibrium the times are of no moment in the final results. But for a supercritical system we should know $t - t' = \frac{r}{v}$. If the intensity goes as $e^{\alpha t}$ all over the system, the intensity at p at the time t' was $e^{-\alpha \frac{r}{v}}$ times what it is at t. Thus we can substitute into the equation for $n(P')$

$$n(P') = \int \frac{\left[n(P) e^{-\alpha \frac{r}{v}} \right] \sigma(P) \left[1 + f(P) \right] e^{-\int_{P'}^{P} \sigma \, dr} dP}{4\pi r_{PP'}^2}$$

The equation is now modified to say: "The neutrons at p' at the time t are the neutrons suffering their last collisions at a distance r away (and coming off in the proper direction) at a time $\frac{r}{v}$ earlier than t." We have simply substituted $n(P) e^{-\alpha \frac{r}{v}}$ for $n(P)$ to take account of the variation of n with time. We can solve for α, the reciprocal of the e-folding time, for a supercritical assembly as follows. Say that we have a core of material with $a = 20$ cm, but that a solution of the equilibrium type indicates that $a = 10$ cm is the critical radius. We solve the problem by finding an α which when substituted into the time dependent equation for $n(P)$ will make that equation hold at the center for $a = 20$, just as it holds there for $\alpha = 0$ and $a = 10$ cm. In this way one can calculate the e-folding time of a fast reactor. This e-folding time will be valid until mechanical effects set in and will be needed in estimates of the efficiency and other properties of such an explosive nuclear reactor.

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Neutron Physics Final Examination

1. Inthe reaction $N^{14} + He^{4} = F^{17} + n^{1}$

the masses of the atomic species involved are:

$N^{14} = 14.00750$

$He^{4} = 4.00390$

$F^{17} = 17.00760$

$n^{1} = 1.00896$

a) Is the reaction endothermic or exothermic?

b) What is the reaction energy in MeV? Q = ...

c) What is the reaction threshold for the α-particle?

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2. Assuming that the cross-section of cadmium for thermal neutrons is 2500×10^{-24}, find the thickness of cadmium needed to reduce the intensity of a thermal neutron beam by a factor of 10^{6}. (density of Cd = 8.65, At wt. = 112.4)

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3. A neutron of 1.5 MeV collides against a proton at rest. Assuming that the neutron is scattered by 30° in the laboratory system, find:

a) The angle formed by the proton trajectory with the original direction of motion of the neutron.

b) The energy acquired by the proton in the collision.

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4. A neutron with initial energy E_0 has two collisions with hydrogen atoms. What is the probability that after the two collisions the energy of the neutron has a value <u>between</u> E and E + dE?

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5. A square graphite column of density 1.6 has sides 150 cm. The intensity of thermal neutrons diffusing up the column decreases by a factor e every 28cm. Find the absorption cross-section for carbon assuming scattering cross-section $= 4.8 \times 10^{-24}$ (Atomic wt. of carbon = 12.)

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6. An atom has Z = 50. What is the minimum value of the atomic weight A for which the atom emits neutrons spontaneously? (For the atomic mass use the formula:

$$M(A,Z) = .99389A - .00081Z + .014A^{2/3} + .083\frac{(\frac{1}{2}A - Z)^2}{A} + .000627Z^2/A^{1/3} + \left(\pm .036/A^{3/4}\right).$$

Mass of the neutron = 1.00893

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