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AMPLIFIER RESPONSE

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ABSTRACT:

The response of amplifiers to sharp pulses and step pulses is studied for amplifiers with various kinds of attenuation functions at high and at low frequencies.

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AMPLIFIER RESPONSE

To design an amplifier to amplify pulses from a counter or ionization chamber one must have some idea of the distortion which the pulse will undergo in passing through the amplifier. Distortion occurs because the amplification is not the same at all the frequencies out of which the pulse can be considered to be made up. In addition there may be a relative shift in phase for these different frequencies.[1] A good amplifier has a large region of frequencies over which it is "flat", that is amplification is nearly independent of frequency. Since we are only interested in distortion and not absolute amplification, we shall call the amplification in this region unity. For very high frequencies, and usually for very low frequency, the amplification falls off. For high frequencies the amplification usually varies inversely as some power of the frequency[2] namely as $(\omega_o/\omega)^k$. The logarithm of the amplification falls linearly with the logarithm of the frequency. That is, the output falls at the rate of 6k decibels per octave.

The frequency ω_o is the frequency at which the straight line on the attennation vs log ω plot would intercept the amplification, 1, of the flat region (See Fig. 1 for example).

Most amplifiers also have a low-frequency cutoff, where the amplification falls, usually directly as a power of the frequency, $(\omega/\omega_o)^k$. The k and ω_o are analogous but bear no direct relation to the corresponding quantities at the high-frequency end. In practice the high- and low-frequency cutoffs are separated by such a long stretch of flat response that the effects on the pulse shape of these two regions can be clearly distinguished. The high-frequency response effects the shape of a sharp pulse, the rate of rise, the accuracy with which the pulse is

1. A phase shift varing linearly with frequency produces no distortion but only a delay in the amplifier. We shall assume such a linear variation has been subtracted from the phase shift and shall not consider the delay.
2. We shall use throughout the so-called angular frequency, ω, which is $1/2\pi$ times the usual frequency.

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followed, etc. The low-frequency response determines the response over long times. After a short pulse the amplifier does not return immediately to zero, but drifts slowly back and may overshoot somewhat, etc.

To separate these effects for analysis we shall study two types of amplifiers. One which has only a high-frequency cutoff, but which is flat for low frequencies, all the way down to zero (it passes DC). The other kind will be imagined to have only a low-frequency cutoff, and to pass with unit amplification all frequencies no matter how large (it follows faithfully instantaneous changes). The effect on a pulse of a real amplifier with both types of cut off can be studied by imagining the pulse to go first through one amplifier with only high cutoff, and the resultant of this then to go through a second amplifier with low cutoff.

Since the amplifier is linear we do not have to find the response to all different kinds of pulses individually. Suppose we know the responce $O(t)$ to an infinitely sharp (and high) pulse, $\delta(t)$, describable by Dirac's delta function. Then, since a pulse of any shape $f(t)$ can be looked at as the superposition of a very large number of infinitely sharp pulses occuring at different times, the response can be built up from an infinite number of responses, $O(t)$. That is since

$$f(t) = \int_{-\infty}^{\infty} f(t')\,\delta\,(t - t')\,dt' \qquad (1)$$

the response of the amplifier to $f(t)$ is

$$R(t) = \int_{-\infty}^{\infty} f(t')\quad O(t - t')\,dt' \qquad (2)$$

Since $O(t)$ is the response to $\delta(t)$.

Of particular interest also is the response $I(t)$ to a step function $1(t)$ (which is 0 if $t < 0$ and 1 for $t > 0$). This is often called the indical admittance. Since the delta function is the derivative of the step function, $O(t)$ is the derivative of $I\,(t)$:

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$$O(t) = d/dt \; I(t) \qquad \text{or,} \quad I(t) = \int_{0}^{t} O(t') \, dt' \quad I(t) = 0 \text{ for } t < 0. \qquad (3)$$

We have calculated O(t) and I(t) for amplifiers with different kinds of high- and low-frequency cutoff.

What is the relation between O(t) and the amplification and phase shift of the amplifier as a function of frequency? If a sine wave of frequency ω is put into the amplifier, we can imagine that the output has amplitude $a(\omega)$ times as much and is shifted by $b(\omega)$ in phase. Using the usual complex notation then we would say that for an input wave $e^{i\omega t}$ the output wave is $a(\omega) e^{ib(\omega)} e^{i\omega t}$. We can combine $a(\omega)\, e^{ib(\omega)}$ into the single complex function $A(\omega)$.

If any other wave form f(t) is put into the amplifier it can be decomposed into its component sine waves by Fouriers theoreom:

$$f(t) = \int_{-\infty}^{\infty} \phi(\omega) \, e^{i\omega t} \, d\omega \quad \text{where } \phi(\omega) = (1/2\pi) \int_{-\infty}^{\infty} f(t) \, e^{-i\omega t} \, dt \qquad (4)$$

Each component wave on going through the amplifier, then, gets multiplied by its corresponding amplitude and phase factor $A(\omega)$. These waves are then be combined to give the response:

$$R(t) = \int_{-\infty}^{\infty} A(\omega) \, \phi(\omega) \, e^{i\omega t} \, d\omega \qquad (5)$$

The delta function $\delta(t)$ has as its fourier transform simply $\phi(\omega) = 1/2\pi$. All frequencies are equally represented, with no relative phase shift. Hence, the function O(t) is given by (5) with $\phi(\omega)$ replaced by $1/2\pi$:

$$O(t) = \int_{-\infty}^{\infty} A(\omega) \, e^{i\omega t} \, d\omega/2\pi \qquad (6)$$

The function I(t) can then be got from (3) or by what amounts to the same thing, putting $\phi(\omega) = 1/2\pi\omega$ into (5). These results can be summarized in the following table I.

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TABLE I

INPUT	OUTPUT
$e^{i\omega t}$	$A(\omega)\ e^{i\omega t}$
$f(t)=\int_{-\infty}^{\infty}\phi(\omega)\ e^{i\omega t}\,d\omega$	$R(t)=\int_{-\infty}^{\infty}A(\omega)\ \phi(\omega)\ e^{i\omega t}\,d\omega$
$\delta(t)$	$O(t)=\int_{-\infty}^{\infty}A(\omega)\ e^{i\omega t}\,d\omega/2\pi$
$1(t)$	$I(t)=\int_{-\infty}^{t}O(t')\,dt'$
$f(t)$	$R(t)=\int_{-\infty}^{\infty}O(t-t')\ f(t')\,dt'$

We can now find $O(t)$ for various choices of $A(\omega)$, in equation (6), representing high- and low-frequency cutoffs. Not any arbitrary choice for the function $A(\omega)$ can be realized in a real amplifier however. There are certain necessary relations between the phase shift and frequency.[3] If these relations are not satisfied by $A(\omega)$ the corresponding amplifier would have signals at the output starting before the input terminals are excited, which is, of course, impossible for any real amplifier. The condition that this does not happen is that all of the singularities branch points, etc. of the function $A(\omega)$ lie on the positive imaginary half of the ω complex plane.

One possible function[4] which satisfies these conditions and which represents a high-frequency cutoff is:

$$A(\omega)=\left[\sqrt{1-\omega^2/4\omega_0^2}+i\omega/2\omega_0\right]^{-k} \tag{7}$$

When the frequency ω is less than $2\omega_0$ it is clear that the absolute magnitude of A is unity. Therefore, for all $\omega < 2\omega_0$ the amplification is constant (but the phase shift varies). Above this frequency the function is i^{-k} times a real function. In this region the phase shift is constant, therefore. For very large frequency $A(\omega)$ behaves as $i^{-k}\,(\omega_0/\omega)^k$ so that the amplification falls off

3 See "Network Analysis and Fred Bach Amplifier Design." H.W. Bode 1945
4 Footnote 3, page 333.

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(but the phase shift is constant) at the rate of 6k decibels per octave. Different rates of cut off are obtained by using different values of k. The amplification and phase shift for this function are plotted in Figs. 1 and 2.

If this form for $A(\omega)$ is put into (6), the integrals can be performed by contour integrations and the result is ($J_k(x)$ is the k^{th} order Bessel Function)

$$O(t) = (k/t)\ J_k(2\omega_o t) \tag{7'}$$

This function is plotted for k = 1.0, 1.5, 2.) (corresponding to cutoffs of 6, 9 and 12 db/ octave respectively) as heavy lines in Fig. 3. The integrated functions I(t) giving the responses to the step function are given in Fig. 4.

Another type of response can be represented by

$$A(\omega) = (1 + i\omega/\omega_o)^{-k} \tag{8}$$

Suppose we have an R, C Circuit consisting of a simple resistance R and capacitance C in series. The voltage across the capacitance then bears the ratio $1/(1 + i\omega Rc)$ to the voltage across the circuit. Hence $A(\omega)$ with $k = 1$ and $\omega_o = 1/Rc$ is represented by such a circuit. Its response to a delta function pulse is a sudden rise followed by a simple exponential fall. If two such RC circuits are put together, one following the other the response would be given by $A(\infty)$ for $k = 2$. Although a fractional value of k cannot be represented in this way by RC circuits, the resulting $A(\omega)$ could nevertheless be roughly the response of some amplifier. The amplification and phase shift for this function is given by the dotted curves of Figs. 1 and 2. If this expression for $A(\omega)$ is put into (6) the function O(t) is found to be

Equivalent Circuits, Eq. (8)

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$$O(t) = \frac{(\omega_0 t)^k}{t\,\Gamma(k)}\; e^{-\omega_0 t} \qquad (8')$$

Here $\Gamma(k)$ is the gamma function of k, which is $(k-1)!$ for integral k. This response curve for $k = 1.0$, 1.5 and 2.0 is given by the dotted curves of Fig. 3. The response $I(t)$ to a step function is given in Fig. 4, dotted curves.

It is seen that the amplifiers with the function (8) are slower to respond but less liable to overshoot than are the corresponding amplifiers with the function (7). Many real amplifiers will probably have curves of amplification lying between the two cases (7) and (8). It often is the case, however, that the amplification rises a little just before the cutoff sets in. Other forms of $A(\omega)$ might be analyzed. The best procecure would of course be to work out $A(\omega)$ for each proposed amplifier, and then get $O(t)$. The curves given here should be a fair guide as to what to expect of most amplifiers.

For the low-frequency-cutoff amplifiers, reasonable curves for $A(\omega)$ can be gotten by replacing ω_0/ω by ω/ω_0 (and i by $-i$) in the expressions for $A(\omega)$ for high frequency. The behavior of amplification and phase shift are exactly similar when plotted against $\ln \omega$ (but are reversed high for low frequency) and will not be plotted again. The low frequency attenuation function corresponding to (7) is:

$$A(\omega) = \left[\sqrt{1 - \omega_0^2/4\omega^2} - i\omega_0/2\omega\right]^{-k} \qquad (9)$$

It gives constant amplification above $\omega_0/2$ and constant phase shift below. The integral (6) for $O(t)$ is more difficult to calculate in this case:

$$\text{For } k = 1, \quad O(t) = \delta(t) - \omega_0/2 + (\omega_0/2 \int_0^t (1/t)\, J_1(\omega_0 t/2)\; dt$$

$$\text{For } k = 2, \quad O(t) = \delta(t) - \omega_0 J_0(\omega_0 t/2) + (\omega_0^2 t/2)\left[1 - \int_0^t t J_1(\omega_0 t/2)\; dt\right] \qquad (9')$$

These are plotted as heavy lines in Fig. 5. Their integrals $I(t)$, the response to a step in plotted in 6. The appearance of the $\delta(t)$ in $O(t)$ is due to the fact that,

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there being no high-frequency cutoff, the sharpest features of the input are reproduced in the output.

The expression for A (ω) corresponding to (8), but for low-frequency cutoff is:

$$A(\omega) = \left(1 - i\omega_0/\omega\right)^{-k} \tag{10}$$

This can be obtained from k (where k is integral) RC circuits similar to those for (8) but with the role of capacity and resistance interchanged. The response O(t) for this case are given for k = 1 and 2 by

For k = 1 $\quad O(t) = \delta(t) - \omega_0 e^{-\omega_0 t}$

For k = 2 $\quad O(t) = \delta(t) - \omega_0(2 - \omega_0 t)\, e^{-\omega_0 t}$

(10') Equivalent Circuits, Eq.(10)

They are plotted as dotted curves in Fig. 5. Their integrals are plotted in Fig. 6.

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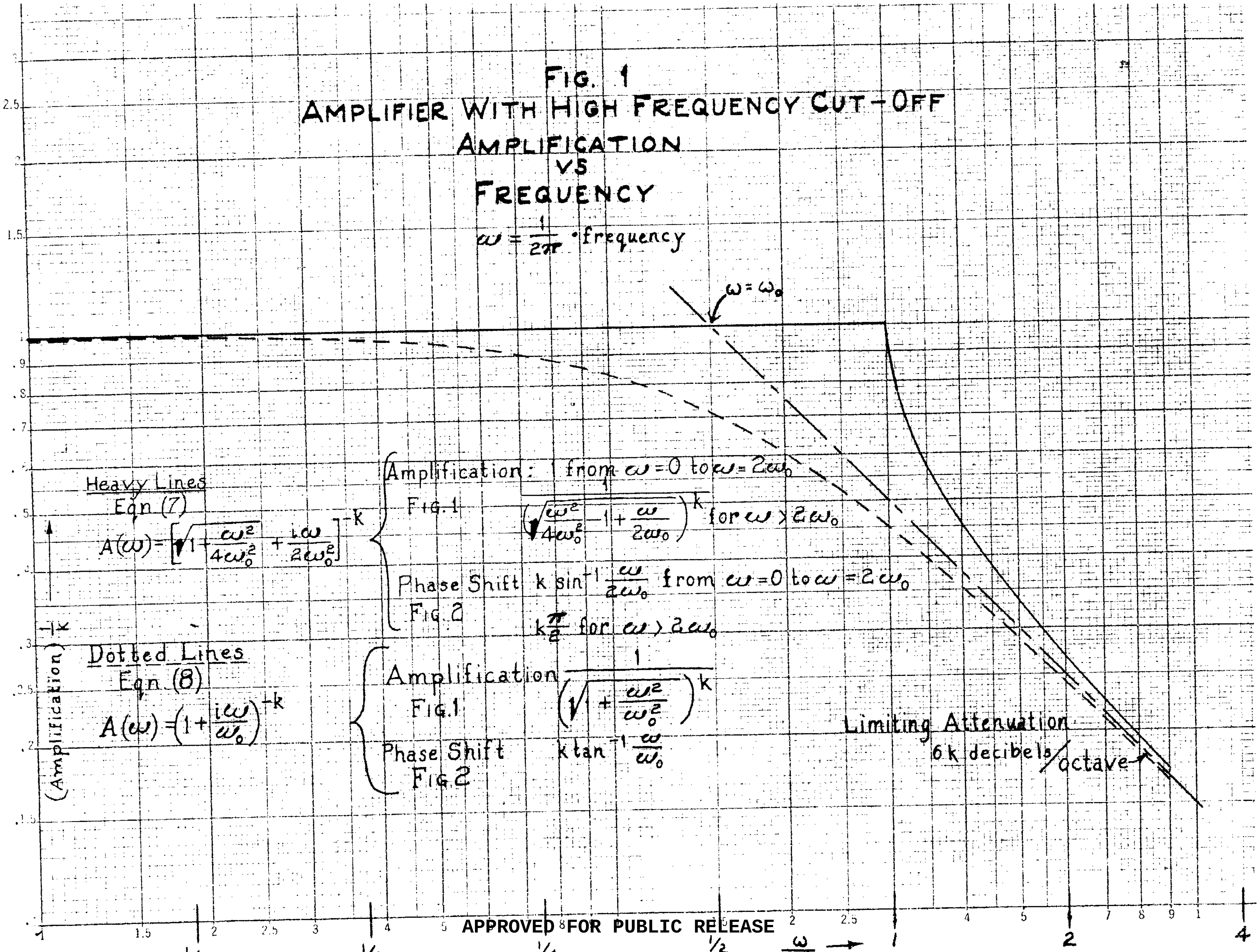


FIG. 1
AMPLIFIER WITH HIGH FREQUENCY CUT-OFF
AMPLIFICATION
VS
FREQUENCY
$\omega = \frac{1}{2\pi} \cdot$ frequency
$\omega = \omega_0$
Heavy Lines
Eqn (7)
$A(\omega) = \left[\sqrt{1 + \frac{\omega^2}{4\omega_0^2}} + \frac{i\omega}{2\omega_0}\right]^{-k}$
Amplification: 1 from $\omega = 0$ to $\omega = 2\omega_0$
Fig. 1
$\frac{1}{\left(\sqrt{\frac{\omega^2}{4\omega_0^2} - 1} + \frac{\omega}{2\omega_0}\right)^k}$ for $\omega > 2\omega_0$
Phase Shift $k \sin^{-1} \frac{\omega}{2\omega_0}$ from $\omega = 0$ to $\omega = 2\omega_0$
Fig. 2
$k\frac{\pi}{2}$ for $\omega > 2\omega_0$
Dotted Lines
Eqn (8)
$A(\omega) = \left(1 + \frac{i\omega}{\omega_0}\right)^{-k}$
Amplification $\frac{1}{\left(\sqrt{1 + \frac{\omega^2}{\omega_0^2}}\right)^k}$
Fig. 1
Phase Shift $k \tan^{-1} \frac{\omega}{\omega_0}$
Fig. 2
Limiting Attenuation
6 k decibels/octave
(Amplification)$^{-\frac{1}{k}}$
$\omega \rightarrow$
1/16
1/8
1/4
1/2
1
2
4

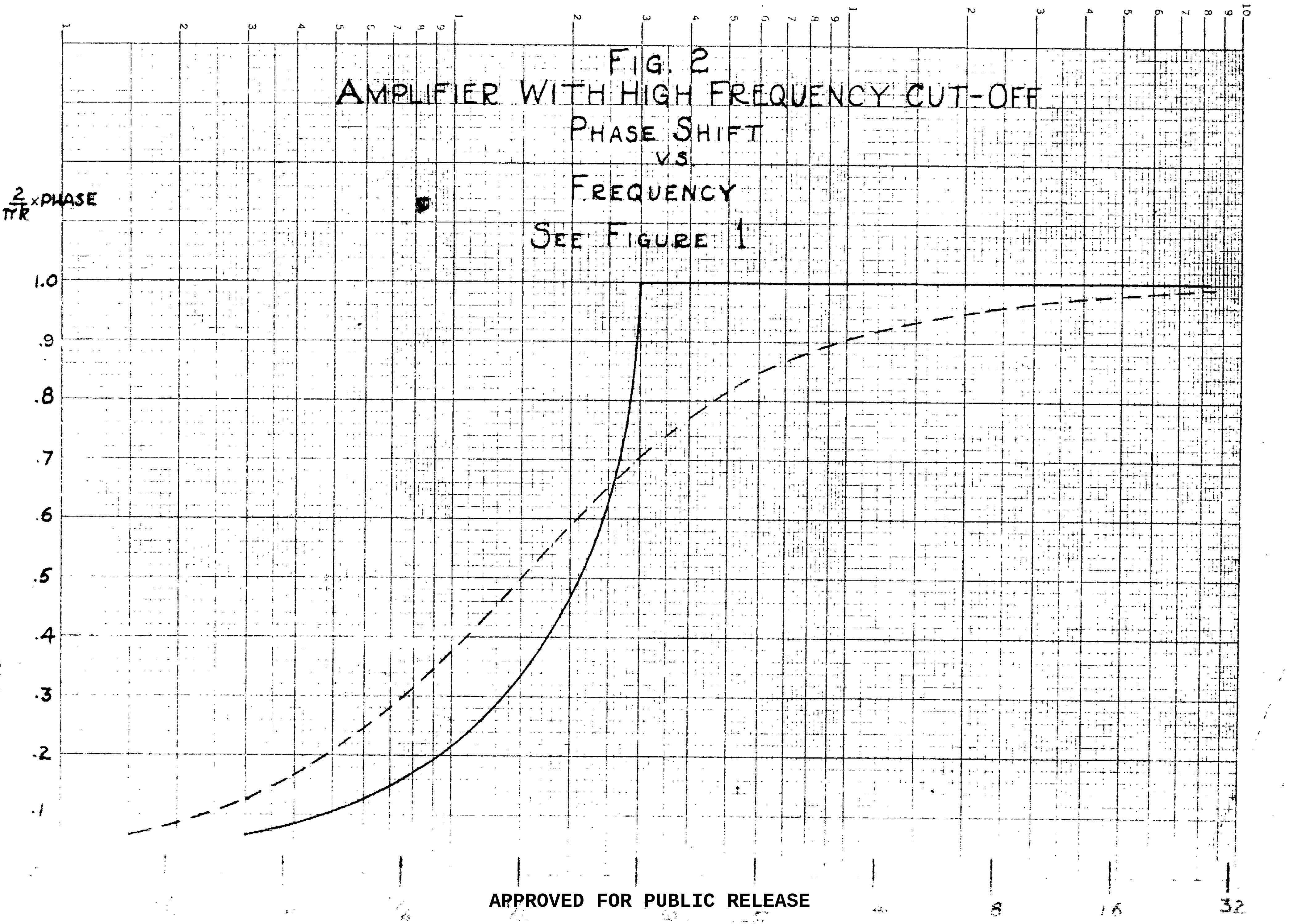
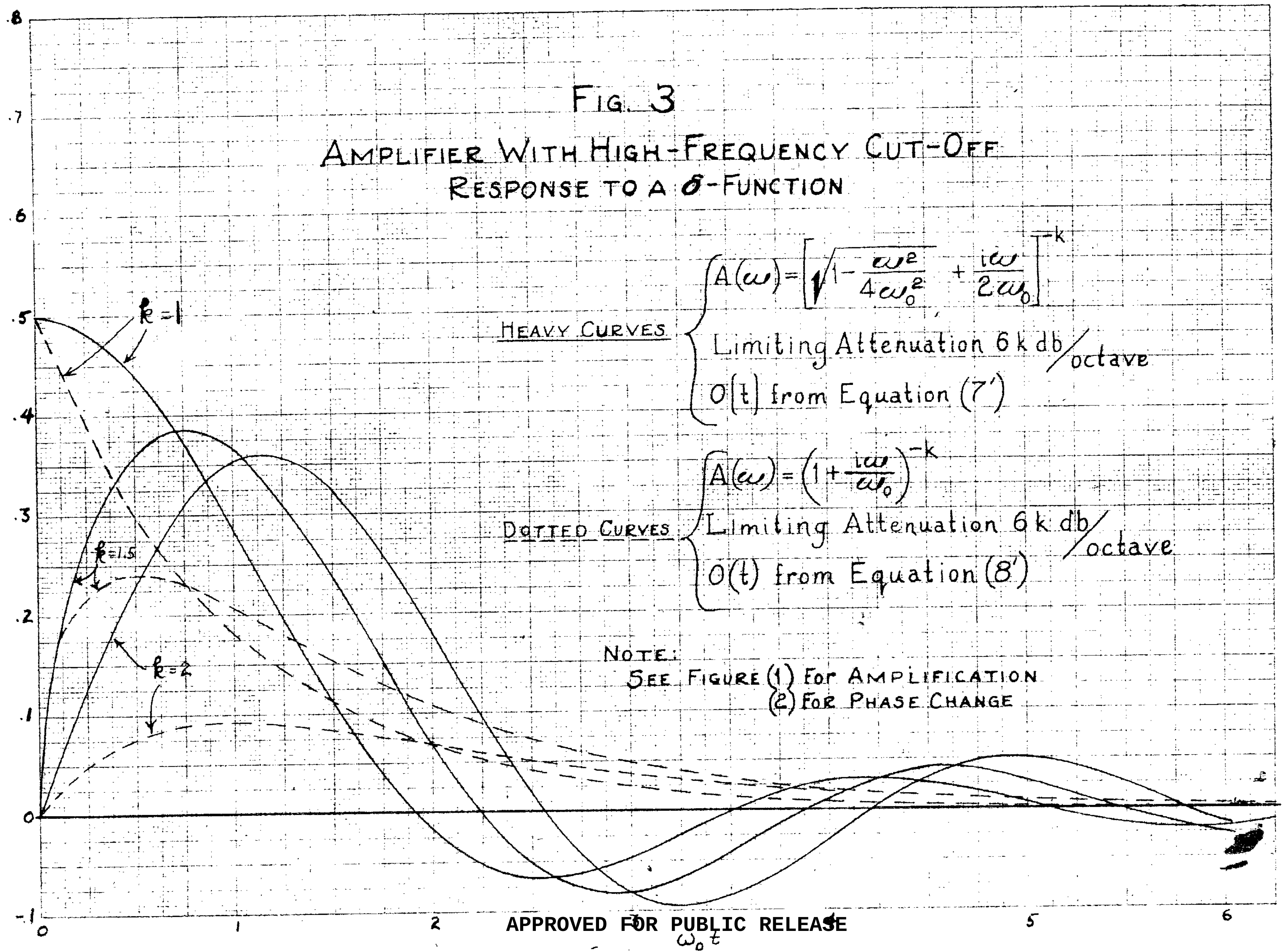


FIG. 2
AMPLIFIER WITH HIGH FREQUENCY CUT-OFF
PHASE SHIFT
VS.
FREQUENCY
SEE FIGURE 1
$\frac{2}{\pi k}$ × PHASE
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
8
16
32



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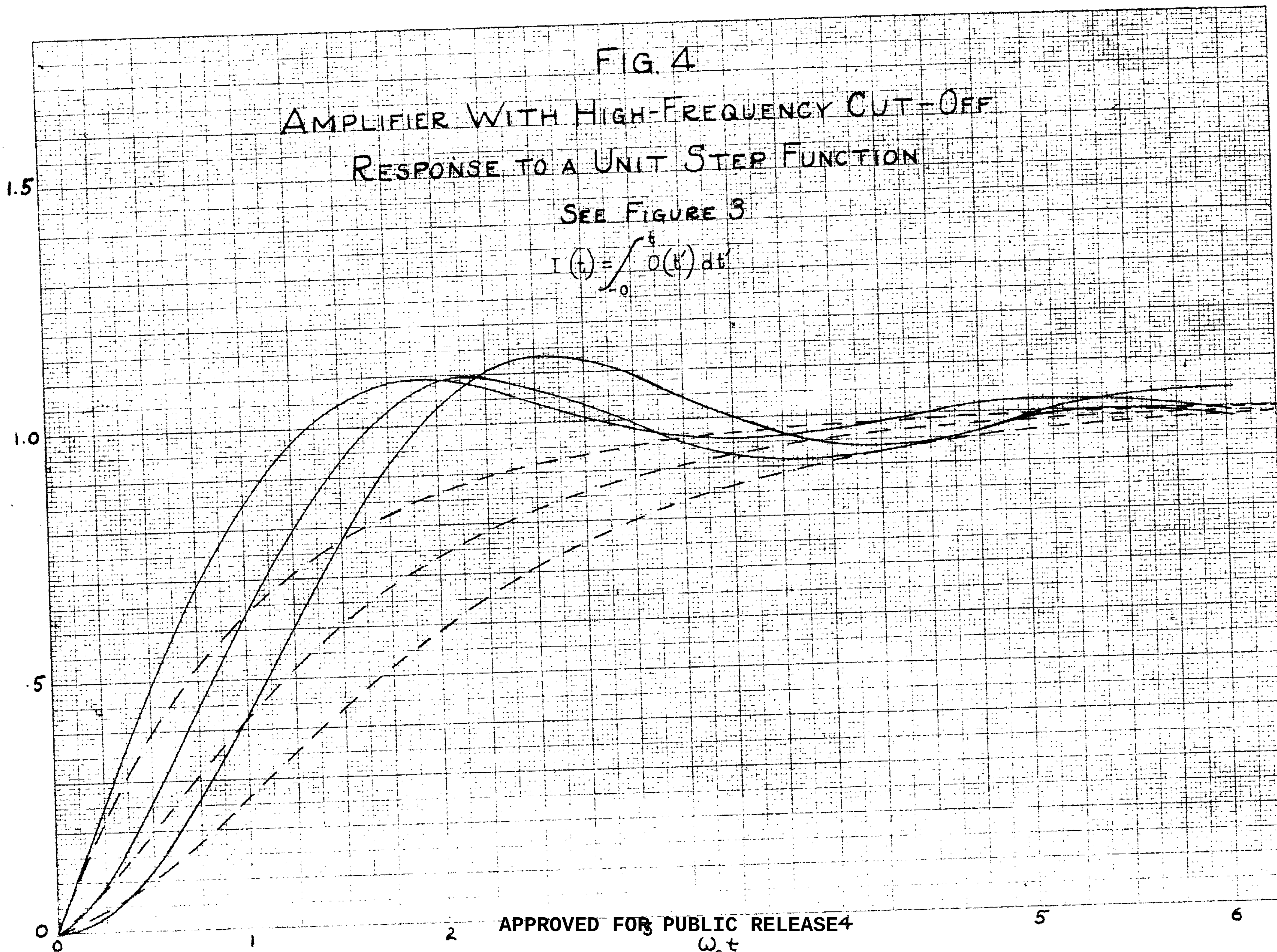
FIG. 4
AMPLIFIER WITH HIGH-FREQUENCY CUT-OFF
RESPONSE TO A UNIT STEP FUNCTION
SEE FIGURE 3
$I(t) = \int_0^t O(t')\,dt'$
1.5
1.0
.5
0
0
1
2
3
4
5
6
$\omega_0 t$

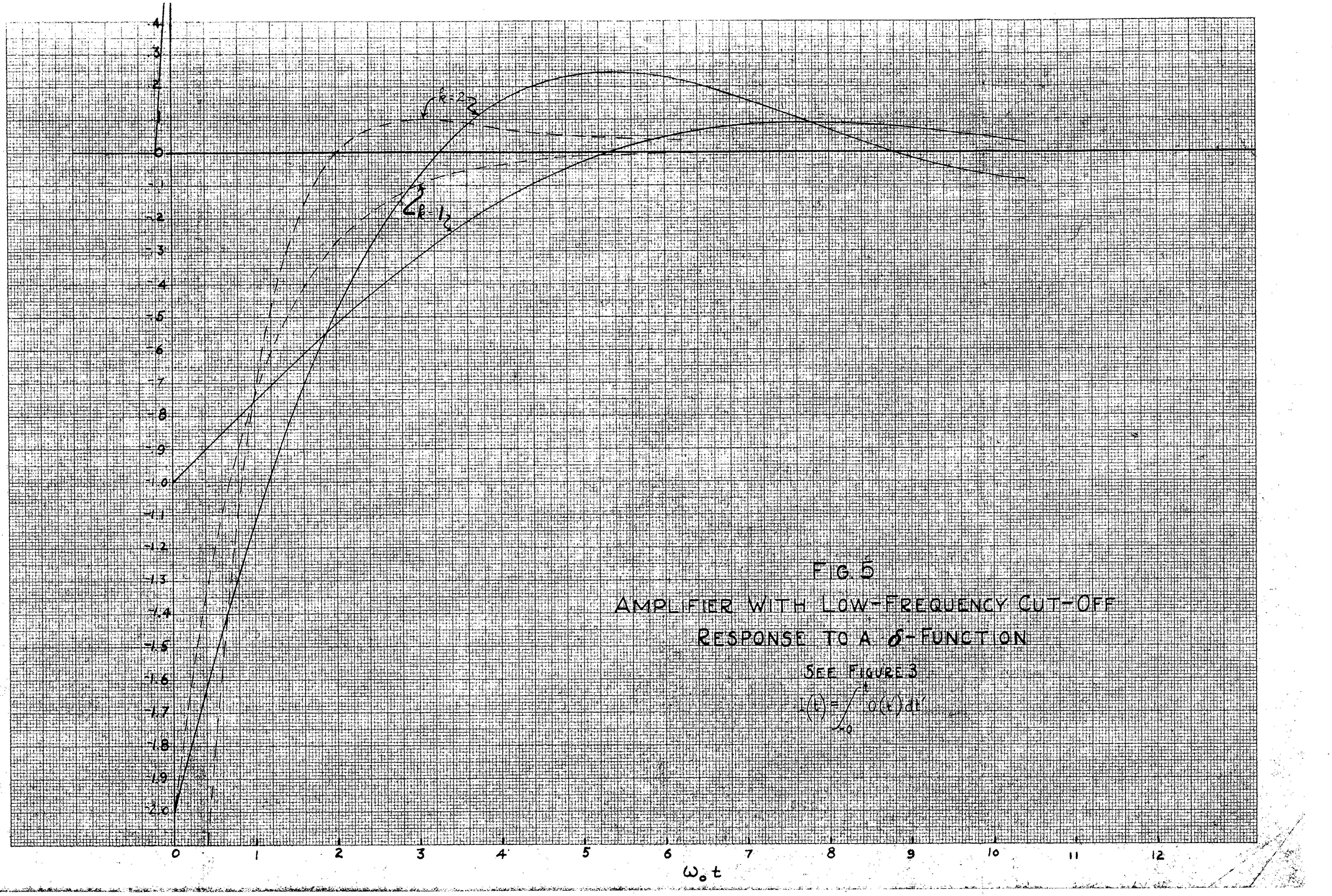

FIG. 5
AMPLIFIER WITH LOW-FREQUENCY CUT-OFF
RESPONSE TO A δ-FUNCTION
SEE FIGURE 3
$$I(t) = \int_0^t O(t')\,dt'$$

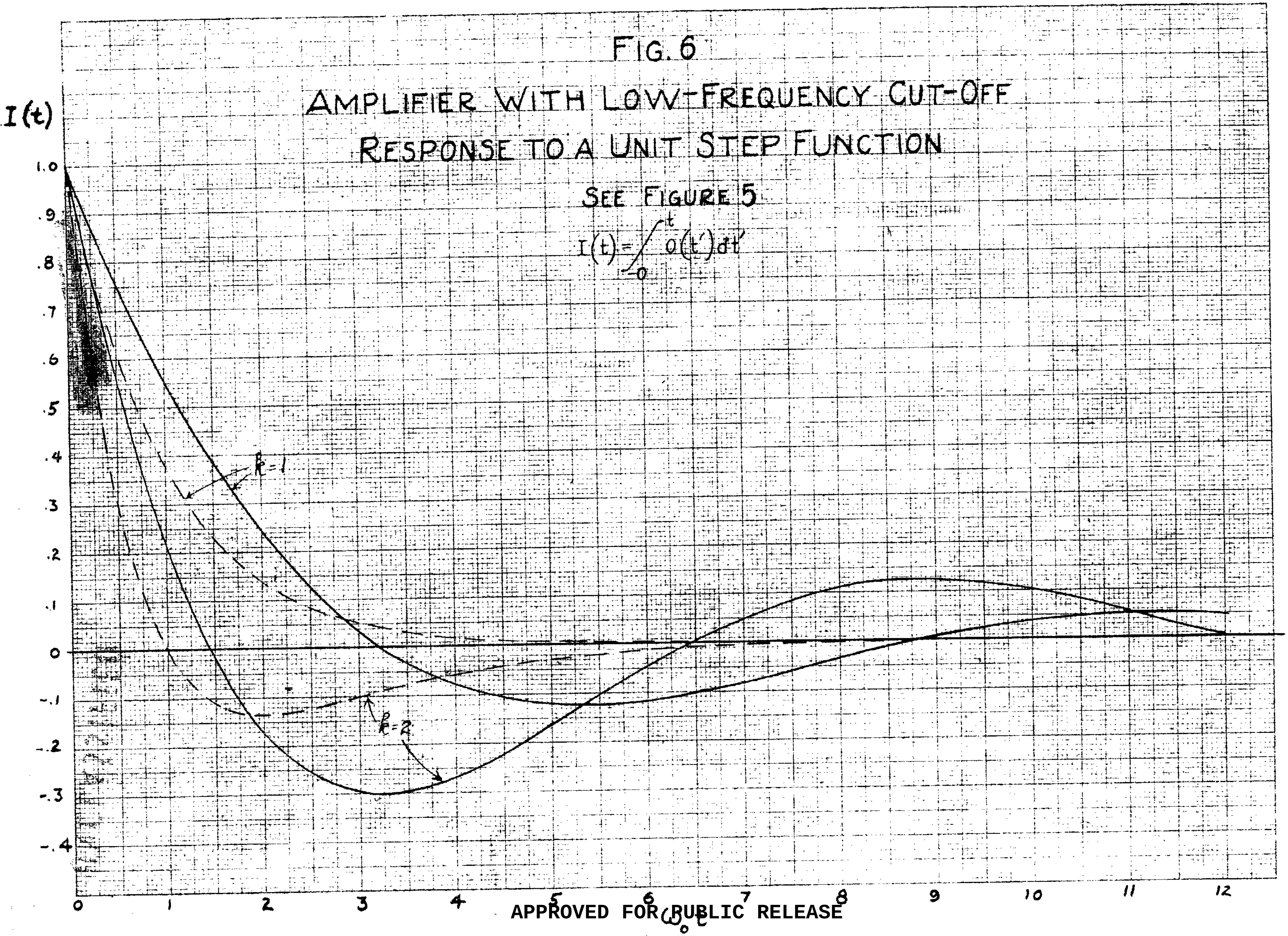

FIG. 6

AMPLIFIER WITH LOW-FREQUENCY CUT-OFF

RESPONSE TO A UNIT STEP FUNCTION

SEE FIGURE 5

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