[redacted]

UNCLASSIFIED

PUBLICLY RELEASABLE
Per J. Brown, FSS-16 Date: 8-26-93
By Marlene Lujan CIC-14 Date: 8-7-96

LA 671

[redacted]

[redacted]

~~Classification changed to Official Use Only~~
~~by authority of the U. S. A... Adcs-4373 dtd 3/31/54~~
~~Per L. M. Ced... (Date)~~
~~(Person authorizing change in classification)~~
~~By ... Campas~~
~~(Signature of person making the change, and date)~~

C.3

VERIFIED UNCLASSIFIED
Per J. Brown 8-26-93
By Marlene Lujan CIC-14 8-7-96

[redacted]

[redacted]

UNCLASSIFIED

[redacted]

[redacted]

[redacted]

UNCLASSIFIED [redacted]

LA-671

Series A

[redacted]

9 March 1948

This document contains 18 pages.

PROPOSED NUMERICAL METHOD FOR CALCULATION OF SHOCKS

Work Done By:

R. D. Richtmyer

Report Written By:

R. D. Richtmyer

[redacted]

UNCLASSIFIED

[redacted]

[redacted]

[redacted]

ABSTRACT

This report gives a method of handling the finite difference equations of hydrodynamics of compressible fluids. Shocks are automatically taken care of by the expedient of introducing a (real or fictitious) dissipation term which serves to make the solution continuous across the shock front. The effect of this term is to "smear" the shock somewhat, but the Hugoniot-Rankine conditions are satisfied and the entropy increase is correct; thus the equations describe the motion correctly except for the fine structure of the shock. Stability conditions are derived and are not unduly severe.

[redacted]

PROPOSED [redacted] CALCULATION OF SHOCKS

INTRODUCTION.

As is known from the investigations of vonNeumann and Peierls, and from the experience of the IBM group, the finite-difference equations that are used for numerical solution of the differential equations of hydrodynamics of compressible fluids are capable of giving an approximate representation of shocks. That is, if the calculations are allowed to proceed with the difference equations as though no shock discontinuity were present, the results are found to show an approximate discontinuity of conditions at approximately the place where the shock should occur and moving at approximately the right speed. Behind the shock the average density, pressure and material velocity are approximately correct, and there is superposed on these an irregular motion that represents in a sense the thermal agitation of the particles of the material that should result from the shock heating. In all these respects the calculation gives approximately correct results for weak shocks, but not for strong ones. For strong shocks it has been customary to interrupt the normal calculating routine at the discontinuity and perform a special calculation ("shock-fitting") based on the Rankine-Hugoniot theory.

In this report we consider modifications of the standard difference equations which, it is hoped, will allow them to give automatically an approximately correct representation of all shocks, weak or strong.

THE DIFFERENTIAL EQUATIONS.

We consider a one-dimensional flow and introduce the following notations:

χ = Lagrangean coordinate = initial coördinate of a given mass point

$X(\chi,t)$ = coordinate, at time t, of a particle initially at χ.

$V(\chi,t) = V_0 \dfrac{\partial X}{\partial \chi}$ = specific volume

V_0 = initial spec. vol. (assumed uniform)

$p(\chi,t)$ or $p(V,T)$ = pressure

$T = T(\chi,t)$ = temperature

[redacted]

[redacted]

$\mathcal{E} = \mathcal{E}(x,t)$ or $\mathcal{E}(V,T)$ = internal energy of fluid per unit mass

Then the differential equations governing the continuous part of the motion are:

$$(1) \quad \left(\frac{\partial \mathcal{E}}{\partial t}\right)_x + p\left(\frac{\partial V}{\partial t}\right)_x = 0$$

$$(2) \quad \frac{1}{V_o} \frac{\partial^2 X}{\partial t^2} = -\frac{\partial p}{\partial x}$$

$$(3) \quad V = V_o \frac{\partial X}{\partial x}$$

and are to be taken in conjunction with the relations (assumed known) between $p, \mathcal{E}, V, T$:

$$(4) \quad p = p(V,T) \qquad \mathcal{E} = \mathcal{E}(V,T)$$

Heat conduction and production terms, if present, would be on the right-hand side of (1), but for the present discussion we assume that they vanish. Equation (1) then merely states that the specific entropy is constant (independent of t) at each x. For this reason equation (1) is not usually considered at all, in dealing with hydrodynamics without shocks. The entropy, which is another known function of V and T, is simply put equal to its initial value, and from this relation and relation (4) for the pressure the temperature T is eliminated, leaving a single relation ("adiabatic") between p and V which is then used directly in connection with equations (2) and (3). We retain equation (1), however, because entropy changes cannot be neglected when shocks (except weak ones) are present.

At a shock $\mathcal{E}$, p, and V are discontinuous, and X has discontinuous derivatives. Equations (1) and (2) then no longer apply, and must be replaced by suitable equations from the Rankine-Hugoniot theory. For example, if $x = x_o(t)$ is the position of the shock, so that $S = \frac{dx_o}{dt}$ is the shock speed, and subscripts i and f denote conditions immediately preceding and immediately following the shock, respectively, then

$$(5) \quad \mathcal{E}_i - \mathcal{E}_f + \frac{p_i + p_f}{2} (V_i - V_f) = 0,$$

and

$$(6) \quad S^2 = V_o^2 \frac{p_f - p_i}{V_i - V_f}$$ [redacted]

take the place of equations (1) and (2).

The first proposal [redacted] for modifying the usual *calculational* technique was based on the observation that equation (1) in integrated form and equation (5) can be combined in the Stieltjes-integral formula

$$(7) \qquad \mathcal{E}(t) + \int \frac{p(t+0)+p(t-0)}{2}\, dV(t) = \text{constant},$$

where t is the variable of integration, and x is held constant. The proposal was to replace equation (1) by

$$(8) \qquad \left(\frac{\partial \mathcal{E}}{\partial t}\right)_x = -\frac{p(x+x_1,t)+p(x-x_1,t)}{2}\left(\frac{\partial V}{\partial t}\right)_x$$

where x_1 is a small quantity, of the dimension of length, of the same order of magnitude as the interval Δx used in the numerical integration. After a few test calculations, this proposal was abandoned, for two reasons. The first is that the irregular, oscillatory motion behind the shock, familiar in the "vonNeumann method", described in the first paragraph, was present with about the same intensity as in previous calculations with the unmodified equations. These oscillations are undesirable in themselves, because they are quite violent for strong shocks, and furthermore they make it impossible to interpret the results to determine whether the relations (5) and (6) are approximately satisfied across the shock. Secondly, equation (8), being invariant for change of t into -t, can make no distinction between positive and negative shocks. This reversibility probably explains why the oscillations behind the shock were not damped: the entropy increase during the compression part of the oscillation is balanced by an entropy decrease (sic) during the expansion part. This first proposal will not be discussed further.

ENERGY DISSIPATION.

It was pointed out by vonNeumann that what is clearly needed is to take into account in the equations the dissipative mechanisms that operate in a physical shock to convert mechanical energy irreversibly into heat energy. This is the basis of the second proposal. Examples of such mechanisms are conduction of heat from a region, that has been heated by compression, to a cooler region, and viscosity.

[redacted]

These effects are usually [illegible] continuous part of the motion. Their magnitudes depend on the rates of change of physical quantities, so that in a shock they are no longer negligible, because there the rates of change are very rapid. In fact, in the usual simplified picture of a shock as a discontinuity, the rates of change become infinite. Clearly a real shock has a finite (though small) thickness, so that quantities do not change discontinuously, but nevertheless at a very rapid rate. In fact, as shown by various investigators (e.g. Rayleigh, Theory of Sound, and R. Becker, Zs.f. Phys. 8, 321, 1922), the thickness of the shock is controlled by the dissipative mechanisms, and adjusts itself automatically so as to permit the dissipative mechanisms to produce precisely the entropy increase demanded by the Hugoniot relations (5) and (6). This is not surprising, because the Hugoniot relations are based on the fundamental conservation laws of mass, energy and momentum. The details of the dissipative mechanism influence the thickness and "fine structure" of the shock, but not the net result of it.

This suggests that we should introduce a (real or fictitious) dissipative mechanism into our equations. The only requirements on it would seem to be:

1. It should be strong enough to damp the oscillations that occur immediately behind the shock in the calculation .
2. It should not be strong enough to have an appreciable influence on the continuous part of the motion.
3. The natural thickness that it gives to the shock should not be much greater than the interval Δx used in the numerical calculation.
4. It should not interfere with the stability of the difference equations.

Requirements 1, 2 and 3 would seem to imply that the natural thickness given to the shock should be of the order of Δx or slightly larger, because the wavelengths of the oscillations observed is of this order, and the continuous part of the motion is by definition those regions in which quantities change by only a small fractional amount in one interval of the calculation.

[redacted]

THE PROPOSED EQUATIONS. [redacted]

The form of dissipation proposed is dilatational viscosity. In equations (1) and (2) we replace p by $p - \mu \frac{\partial V}{\partial t}$, where p represents the equilibrium part of the pressure, to be calculated as before from the temperature T and the specific volume V, and the second term represents a correction to the pressure that tends to resist changes of volume while these changes are taking place. For the time being we regard μ as a constant, though later this will be modified. The equations then become:

$$(9) \quad \frac{\partial \mathcal{E}}{\partial t} + p\frac{\partial V}{\partial t} = \mu\left(\frac{\partial V}{\partial t}\right)^2$$

and

$$(10) \quad \frac{1}{V_0}\frac{\partial^2 x}{\partial t^2} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 V}{\partial x \partial t} \quad .$$

We next show that the Hugoniot relations (5) and (6) follow from these equations. For this purpose we consider a steady-state shock. That is, we consider a motion in which all quantities depend on x and t through the combination $\xi = x - St$ only, S being the speed, in Lagrangean coordinates, of the shock. We assume furthermore that the variation of p, V, and $\mathcal{E}$ is continuous and mostly occurs in a small interval of the variable ξ : to the right of this interval the said variables have very nearly constant values (the initial values); and to the left of this interval they also have very nearly constant values (the final values). It will be seen later that our equations actually lead to a motion of this kind in the steady state case. Equations (9) and (10) now take the form:

$$(11) \quad \frac{d\mathcal{E}}{d\xi} + p\frac{dV}{d\xi} = -\mu S\left(\frac{dV}{d\xi}\right)^2 ,$$

and

$$(12) \quad \frac{S^2}{V_0}\frac{d^2x}{d\xi^2} = \frac{S^2}{V_0^2}\frac{dV}{d\xi} = -\frac{dp}{d\xi} - \mu S\frac{d^2V}{d\xi^2} \quad .$$

By integrating (12) from $\xi = -\infty$ to $\xi = +\infty$ and setting $\frac{dV}{d\xi} = 0$ at both limits, we find

[redacted]

[redacted]

$$\frac{S^2}{V_o^2}(V_i-V_f) = p_f-p_i$$

which agrees with (6). Integration of (12) also gives

$$p(\xi) = \text{constant} - \frac{S^2}{V_o^2}V(\xi) - \mu S\frac{dV}{d\xi}$$

or

$$(13) \quad p(\xi) = p_i + \frac{S^2}{V_o^2}\left[V_i-V(\xi)\right] - \mu S\frac{dV}{d\xi}$$

and substitution of this expression for $p(\xi)$ into (11) gives

$$(14) \quad \frac{d\mathcal{E}}{d\xi} + \left(p_i + \frac{S^2V_i}{V_o^2}\right)\frac{dV}{d\xi} - \frac{S^2}{V_o^2}\,V\frac{dV}{d\xi} = 0$$

Integrating (14) through the shock region gives

$$\mathcal{E}_i - \mathcal{E}_f + \left(p_i + \frac{S^2V_i}{V_o^2}\right)(V_i-V_f) - \frac{S^2}{2V_o^2}(V_i^2-V_f^2) = 0$$

which can be written as

$$\mathcal{E}_i - \mathcal{E}_f + \frac{p_i+p_f}{2}(V_i-V_f) = 0$$

by use of (6), thus verifying also (5). We have shown, in other words, that the added terms in (9) and (10) do not interfere with the conservation laws of energy and momentum.

For calculating the shock thickness that results from these equations, we specialize the discussion to a fluid obeying a relation

$$\mathcal{E} = \frac{pV}{\gamma-1}$$

where γ is a constant. By integrating (14) and dividing through by $V(\xi)$, we find:

$$(15) \quad \frac{p(\xi)}{\gamma-1} + p_i + \frac{S^2V_i}{V_o^2} - \frac{S^2}{2V_o^2}\,V(\xi) = \frac{\text{const}}{V(\xi)}$$

where the value of the constant is

[redacted]

(16) $$\text{const.} = p_i V_i \frac{\gamma}{\gamma - 1} + \text{[redacted]}$$

Eliminating $p(\xi)$ from (13) and (15) gives for $V(\xi)$ the differential equation

(17) $$\frac{dV}{d\xi} = A - BV - \frac{C}{V}.$$

where:

(18) $$A = \left(\frac{p_i}{\mu S} + \frac{S V_i}{\mu V_o^2} \right) \gamma$$

(19) $$B = \frac{S(\gamma + 1)}{2 \mu V_o^2}$$

(20) $$C = \frac{\gamma p_i V_i}{\mu S} + (\gamma - 1) \frac{S V_i^2}{2 \mu V_o^2}$$

It is readily verified, by means of (5), (6), (18), (19), (20), that the right hand side of (17) vanishes for $V = V_i$ or $V = V_f$ and is positive for $V_f < V < V_i$. Therefore a solution of (17) has the general appearance of Fig. 1

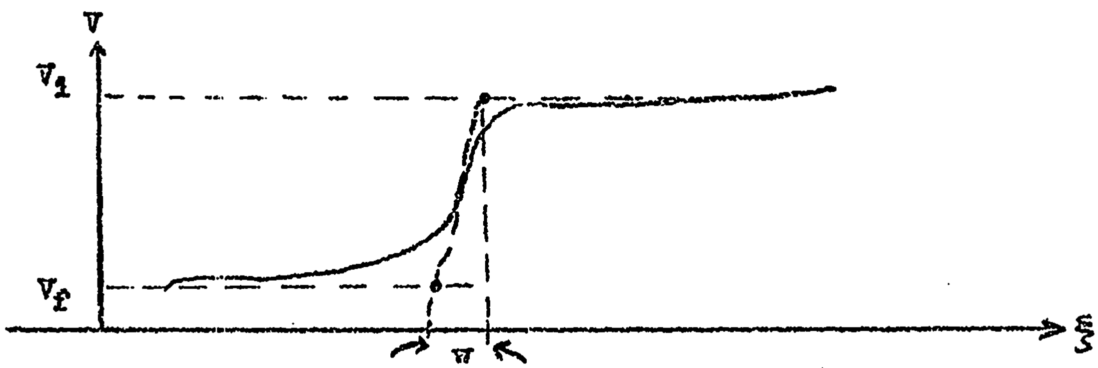

Fig. 1

As indicated in the figure, we take the effective thickness of the shock to be

(21) $$w = \frac{V_i - V_f}{\left(\frac{dV}{d\xi} \right)_{max.}}$$

From (17) the maximum slope occurs at

$V = \sqrt{C/B}$ and has the value $A - 2\sqrt{BC}$.

[redacted]

Furthermore, $V_i - V_f = \frac{\sqrt{A^2 - 4BC}}{B}$, so [redacted]

(22) $$w = \frac{1}{B} \sqrt{\frac{A + 2\sqrt{BC}}{A - 2\sqrt{BC}}} .$$

By virtue of (18), (19), and (20), this gives the thickness in terms of the initial conditions, the value of μ and the shock strength.

For strong shocks the expression (22) for the thickness w reduces approximately to $1/B$, which varies inversely as the shock speed, and hence goes to zero as the shock strength increases indefinitely. For weak shocks, on the other hand, (22) reduces to $\frac{2A}{B^2(V_i - V_f)}$ so that the thickness goes to infinity as the shock strength decreases to zero.

Our aim is to obtain a natural thickness of order Δx regardless of the strength of the shock, and consequently the above formulation has to be modified. As a guide in the modification we solve (22) for μ with the help of (18), (19), and (20) to see how μ would have to vary with shock strength to give approximately the same thickness for all shocks. The result in two limiting cases is:

strong: $(V_f \ll V_i)$ $$\mu \approx \frac{\gamma+1}{2} \frac{Sw}{V_o^2} \approx \frac{\gamma+1}{2} \frac{Sw}{V_o^2} \frac{V_i - V_f}{V_i}$$

weak: $(V_i - V_f \ll V_i)$ $$\mu \approx \frac{\gamma+1}{2} \frac{\pi}{4} \frac{Sw}{V_o^2} \frac{V_i - V_f}{V_i}$$

The last factor in the third member of the equation for strong shocks is approximately unity, by definition of a strong shock, and is introduced merely to make the expressions more similar. The dependence of these expressions on the shock strength is contained in the factors

$$S(V_i - V_f) \approx w \left| \frac{\partial V}{\partial t} \right|_{max}$$

which suggests that in our formulation we should replace the constant μ by something proportional to $\left| \frac{\partial V}{\partial t} \right|_{max}$ [redacted] The expressions above for strong and weak shocks fail to agree, by a factor $\gamma/4$, on [redacted] size, and to resolve the disagreement we give priority to the strong shocks (this is a purely nominal decision, because the optimum value of w, on which the coefficient depends, will have to be found in any case by test calculations), and the final formulas are:

$$\mu \longrightarrow \frac{\gamma+1}{2}\left(\frac{w}{V_0}\right)^2 \frac{1}{V_1}\left|\frac{\partial V}{\partial t}\right|$$

so that

(23) $$\frac{\partial \varepsilon}{\partial t} + p\frac{\partial V}{\partial t} = \epsilon\left|\frac{\partial V}{\partial t}\right|^3$$

(24) $$\frac{1}{V_0}\frac{\partial^2 x}{\partial t^2} = -\frac{\partial p}{\partial x} + 2\epsilon\left|\frac{\partial V}{\partial t}\right|\frac{\partial^2 V}{\partial x \partial t}$$

(25) where $\epsilon = \frac{\gamma+1}{2}\frac{w^2}{V_0^2 V_1}$. It is expected that the effective shock thickness obtained from (23) and (24) will not vary more than about by a factor 3 for the entire range of shock strength.

The Hugoniot relations (5) and (6) follow from (23) and (24) in about the same way that they did from (9) and (10).

V_1 will not generally be known in advance of the calculation, except in problems where it is known that all shocks run into previously undisturbed material, but in many problems it will nevertheless be a fair approximation to set V_1 equal to V_0 in the expression for ϵ, so that finally,

(26) $$\epsilon \approx \frac{\gamma+1}{2}\frac{w^2}{V_0^3}$$

It is believed that the added terms in (23) and (24) should be at least as easy to incorporate in a routinized calculation as the conventional shock fitting, and furthermore they automatically take care of all shock, whenever and wherever they arise. These terms should have no appreciable influence on the continuous part of the motion because of their strong dependence on the rate of change of V.

[redacted]

The optimum value of v will have to be determined by trial calculations. It is expected that it will tu [redacted] tiple of Δx.

THE DIFFERENCE EQUATIONS.

For numerical calculations, the system (23) and (24), together with (3), must be formulated as difference equations. Consider a uniform rectangular mesh of points in the x-t plane, with coordinates x_l, t^n, spacings Δx and Δt, and let the value of any function $F(x,t)$ at a mesh point be denoted by $F^n_l = F(x_l, t^n)$. Suppose that at a certain stage of the calculation all quantities are known for time t^n, and that in particular, the values X^n_l of the coordinate X are known at the mesh points. Then clearly the values of the specific volume V will have been determined by straightforward application of (3) at midway points, $(x_{\ell+\frac{1}{2}}, t^n)$, (cf. equ. (29) below). Since p, V, T, $\mathcal{E}$ are all connected by functional relations (4), it is natural to suppose that T and $\mathcal{E}$ will also be known at midway points for time t^n. Then the difference equations can be written as:

(27) $$p^n_{\ell+\frac{1}{2}} = p\left(V^n_{\ell+\frac{1}{2}}, T^n_{\ell+\frac{1}{2}}\right)$$

(28) $$\frac{1}{V_0}\,\frac{X^{n+1}_{\ell}-2X^{n}_{\ell}+X^{n-1}_{\ell}}{(\Delta t)^2} = \frac{p^n_{\ell-\frac{1}{2}}-p^n_{\ell+\frac{1}{2}}}{\Delta x} + 2\epsilon\,\frac{\left|V^n_{\ell+\frac{1}{2}}+V^n_{\ell-\frac{1}{2}}-V^{n-1}_{\ell+\frac{1}{2}}-V^{n-1}_{\ell-\frac{1}{2}}\right|}{2\Delta t}$$

$$\cdot\,\frac{V^n_{\ell+\frac{1}{2}}-V^{n-1}_{\ell+\frac{1}{2}}-V^n_{\ell-\frac{1}{2}}+V^{n-1}_{\ell-\frac{1}{2}}}{\Delta x\,\Delta t}$$

(29) $$V^{n+1}_{\ell+\frac{1}{2}} = V_0\,\frac{X^{n+1}_{\ell+1}-X^{n+1}_{\ell}}{\Delta x}$$

(30) $$\frac{\mathcal{E}^{n+1}_{\ell+\frac{1}{2}}-\mathcal{E}^{n}_{\ell+\frac{1}{2}}}{\Delta t}\; p^n_{\ell+\frac{1}{2}}\,\frac{V^{n+1}_{\ell+\frac{1}{2}}-V^{n}_{\ell+\frac{1}{2}}}{\Delta t} = \frac{\left|V^{n+1}_{\ell+\frac{1}{2}}-V^{n}_{\ell+\frac{1}{2}}\right|^3}{(\Delta t)^3}$$

(31) $$\mathcal{E}^{n+1}_{\ell+\frac{1}{2}} = \mathcal{E}\left(V^{n+1}_{\ell+\frac{1}{2}},\ T^{n+1}_{\ell+\frac{1}{2}}\right)$$

and are to be solved, in order, for $p^n_{\ell+\frac{1}{2}}$ ($\ell = 0,1,2,\ldots$), X^{n+1}_{ℓ} ($\ell = 0,1,2,\ldots$),

[redacted]

condition is, of course, needed for the first and for the last value of ℓ.) As written, equations (28) and (30) are not correct to second order in Δt, and cannot be so made without rendering their solution much more difficult. In some problems the system can be so rewritten as to make the most important terms correct to second order by using specific entropy S, instead of temperature T, as a dependent variable. $S = S(V,T)$. Then (30) and (31) are replaced by

$$(32) \quad T^n_{\ell+\frac{1}{2}} \frac{S^{n+1}_{\ell+\frac{1}{2}} - S^n_{\ell+\frac{1}{2}}}{\Delta t} = \epsilon \frac{\left|V^{n+1}_{\ell+\frac{1}{2}} - V^n_{\ell+\frac{1}{2}}\right|^3}{(\Delta t)^3}$$

and

$$(33) \quad S^{n+1}_{\ell+\frac{1}{2}} = S(V^{n+1}_{\ell+\frac{1}{2}}, T^{n+1}_{\ell+\frac{1}{2}})$$

and are solved for $S^{n+1}_{\ell+\frac{1}{2}}$ ($\ell = 0,1,2,\ldots$) and $T^{n+1}_{\ell+\frac{1}{2}}$ ($\ell = 0,1,\ldots$) respectively.

STABILITY OF DIFFERENCE EQUATIONS.

It is well known that for the continuous part of the motion, where the dissipation is negligible, the difference equations may under some circumstances be unstable, in the sense that small errors introduced at one time, t^n, (for example, errors that are unavoidably introduced by the use of finite differences) appear amplified at each new cycle of the calculation until eventually nothing but gibberish remains. The condition for stability is that the quantity

$$C_1 = \frac{c\Delta t}{\Delta X}$$

should be less than unity throughout the calculation, where c is the local adiabatic sound speed and ΔX is the interval of the Eulerian coordinate X corresponding to the interval Δx of the Lagrangean coordinate x.

It might be hoped that this situation will not be seriously altered by the presence of the dissipative terms, because these terms are important only in shocks, and shock conditions represent only a small portion of the x-t plane. However, the region covered by shock conditions is by no means negligible: The width of this region (i.e., the thickness of the shock) is likely to be at least 2 or 3 times

[redacted]

[redacted]

$\Delta\chi$, the shock speed is less [redacted] the shock, the "Courant number" C_1 is not likely to be more than about $\frac{1}{5}$, so the number of cycles Δt during which a given mass point is under shock conditions is likely to be of the order 5 or more; any appreciable amplification of errors at each of those five or so cycles may well be serious. We shall therefore examine the stability question in a little more detail, taking the dissipative terms into account.

Strictly speaking there is no general theory of stability applicable to problems in which conditions change rapidly from point to point: Intuitively, one can be sure that the overall stability of a problem depends on some sort of average of the Courant number of its analogue, but what sort of average is involved is not clear, and may even depend on what criteria are taken for stability. But to get a qualitative idea, we shall proceed as follows. We find out what conditions - pressures, densities, velocity gradients, etc. - exist in the shock region, and investigate what the stability conditions would be if these conditions persisted in a large region of the x-t plane.

For the remainder of the discussion we shall deal with a fluid obeying a "gamma law"

$$\mathcal{E} = \frac{pV}{\gamma - 1} , \tag{34}$$

and we shall make use of the particle velocity u given by

$$u = \frac{\partial x}{\partial t} ,$$

whereupon the differential equations become:

$$\frac{\partial V}{\partial t} = V_0 \frac{\partial u}{\partial \chi} , \tag{35}$$

$$V \frac{\partial p}{\partial t} + \gamma p V_0 \frac{\partial u}{\partial \chi} = (\gamma - 1) \epsilon V_0^3 \left| \frac{\partial u}{\partial \chi} \right|^3 . \tag{36}$$

and

$$\frac{1}{V_0} \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial \chi} + 2 \epsilon V_0^2 \left| \frac{\partial u}{\partial \chi} \right| \frac{\partial^2 u}{\partial \chi^2} \tag{37}$$

[redacted]

In outline, the calculat[redacted] as follows. We assume that a small perturbation is superposed on the smooth variation of u, V and p, as indicated schematically by the substitutions

$$u \rightarrow u + \delta u,$$
$$V \rightarrow V + \delta V,$$
$$p \rightarrow p + \delta p.$$

It is assumed, however, that the perturbation of u is not sufficiently great to alter the sign of $\left[\frac{\delta u}{\delta \chi}\right]$. We then obtain linear partial differential equations for δu, δV and δp by carrying out the above substitutions in equations (35), (36), and (37). We then look for solutions of the difference equations, obtained therefrom in analogy with the system (27) to (31), of the form

$$\text{(38)} \quad \begin{cases} \delta u = u_1 \, e^{i\beta\chi} \, e^{\alpha t} \\ \delta V = V_1 \, e^{i\beta\chi} \, e^{\alpha t} \\ \delta p = p_1 \, e^{i\beta\chi} \, e^{\alpha t} \end{cases}$$

Substitution of these expressions into the difference equations leads to a set of simultaneous linear equations for u_1, p_1 and V_1. From this we eliminate u_1, p_1, and V_1 and obtain an equation relating α to β, and containing the interval sizes $\Delta\chi$ and Δt. From this equation we determine what relation between $\Delta\chi$ and Δt must be maintained in order that the real part of α be non-positive for all real β, as this is clearly the condition that disturbances of all sorts decrease, not increase, as the calculation proceeds. The various terms of this equation contain various combinations of $\Delta\chi$ and Δt, and since we are interested in a fine mesh, we drop all terms except those of lowest order in the small quantities $\Delta\chi$ and Δt. The only terms which remain are those that derive from the first term, left, and the last term, right, in equation (37), so that we are dealing in effect with a stability problem of the same type as that met in connection with diffusion or heat flow. This is in accord with the general principle that stability is affected only by those terms of a partial differential equation containing derivatives of

[redacted]

[redacted]

highest order. But when, as in [redacted] we deal with a system of equations in several dependent variables, it is perhaps not immediately evident which terms are to be considered those of highest order. For that reason we went through the perturbation calculation outlined above and found that for stability considerations we can drop the first term, left, in (37).

With respect to the remaining terms of (37) our difference equations are of the "explicit" type (see vonNeumann and Richtmyer, A Proposed Numerical Method for Solving Partial Differential Equations of Parabolic Type, LA report No. 657), and the condition for stability is that the quantity

$$(39) \qquad C_2 = 2\epsilon V_0^3 \left|\frac{\partial u}{\partial \chi}\right| \frac{2\Delta t}{\Delta \chi}$$

should be less than unity throughout the calculation.

We shall compare this quantity with the "Courant number" C_1 applicable in the continuous part of the motion, and find that the two are of the same order of magnitude, so that satisfying the condition $C_2 < 1$ in the shock should not be much more difficult than satisfying the condition $C_1 < 1$ for the continuous part of the motion.

We rewrite (39) approximately by means of

$$(40) \qquad \left|\frac{\partial u}{\partial \chi}\right| \to \left|\frac{\partial u}{\partial \chi}\right|_{max} = \frac{1}{V_0}\left|\frac{\partial V}{\partial t}\right|_{max} \approx \frac{S}{V_0}\,\frac{V_i - V_f}{w} ,$$

and

$$(41) \qquad \epsilon = \frac{\gamma+1}{2}\,\frac{w^2}{V_0^2 V_i} ,$$

and find that

$$(42) \qquad C_2 = 2(\gamma+1)\frac{w}{\Delta\chi}\,\frac{V_i - V_f}{V_i}\,\frac{s\Delta t}{\Delta\chi} .$$

We shall compare this with the value C_{1f}, assumed by C_1 behind the shock, which is the most critical region for the ordinary Courant condition. We have:

[redacted]

$$(43)\qquad C_{1f} = \sqrt{\gamma p_f V_f}\ \frac{\;}{\Delta x} = V_0\sqrt{\frac{\;}{V_f}}\ \frac{\;}{\Delta \mu}$$

Eliminating $\Delta t/\Delta \mu$ from (42) and (43) gives the relation between C_2 and C_{1f}. We also substitute for S and p_f from (5) and (6), and call

$$(44)\qquad \eta = \frac{V_1}{V_f}.$$

η is the compression ratio and is a measure of the shock strength. The result is

$$(45)\qquad C_2 = 2\sqrt{2(\gamma+1)}\ \frac{w}{\Delta \mu}\ \frac{\eta-1}{\eta}\left(\eta - \frac{\gamma-1}{\gamma+1}\right)^{-\frac{1}{2}} C_{1f}.$$

Since we don't know the shock strength in general in advance, we maximise (45) with respect to η. We do know that for real shocks η is always between 1 and $\frac{\gamma+1}{\gamma-1}$, and in this interval the maximum comes at

$$(46)\qquad \eta = \eta_0 = \frac{3 + \sqrt{9 - 8\frac{\gamma-1}{\gamma+1}}}{2}$$

We observe that η_0 lies between 2 and 3. We can therefore find a simple inequality for C_2 by observing that of the two factors containing η in (45) the first is an increasing and the second is a decreasing function of η, so that by putting η equal to 3 in the first factor and 2 in the second, we find

$$C_2 < \frac{4\sqrt{2}}{3}\ \frac{\gamma+1}{\sqrt{\gamma+3}}\ \frac{w}{\Delta \mu}\ C_{1f}$$

To illustrate; if $C_{1f} = \frac{1}{3}$, $\frac{w}{\Delta \mu} = 3$, and $\gamma = 1.375$, C_2 might rise to something of the order of 3 at the center of the shock for a shock of worst value of η (slightly under 3). This will lead to temporary amplification of errors in the shock. If we assume that this situation will persist effectively for about 5 cycles of the calculation in the neighborhood of a given point x, and that the average value of C_2 over those five cycles can be taken as 2, the amplification factor for errors of worst possible wavelength, $k\Delta \mu = \pi$, will be $(2C_2 - 1)^5$ or about 240. This is probably tolerable, if the irregularities of the motion were sufficiently small prior to the

[redacted]

[redacted]

arrival of the shock. But if $w/\Delta x$ were assumed to be of the order of 6 instead of 3 at the center of the shock, the amplification factor might well be of the order of 10^6.

It seems likely that the success or failure of this method of treating shocks will depend on whether we can choose $w/\Delta x$ large enough to make the dissipative terms accomplish what was intended without incurring serious instability.

[redacted]

[redacted]

UNCLASSIFIED [redacted]

UNCLASSIFIED

DOCUMENT ROOM

REC. FROM Ed. Div.

DATE 3-12-48

REC. ... NO. ...

[redacted]

UNCLASSIFIED

UNCLASSIFIED

[redacted]