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PROPOSED NUMERICAL METHOD FOR CALCULATION OF SHOCKS, II

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Abstract

The method, proposed in an earlier report (LA-671), for treatment of shocks in a stepwise, numerical integration of the equations of hydrodynamics, has been tested by a series of trial calculations. The results are presented and analysed in this report. They lead to the conclusion that the mock dissipative terms that have been introduced (loc. cit.) can be chosen in such a way that the integration is stable and at the same time the integral properties of the shock are in close agreement with the Hugoniot theory, without appreciably blurring the shock (that is, the thickness given to it by the dissipative terms of the same order of magnitude as the elementary distance appearing in the mesh used for the numerical integration). An analytical treatment of the stability question is given, which is believed to be more quantitative than the one given previously. It indicates that when the coefficients of a partial differential equation are variable, it is sometimes permissible to violate the Courant condition locally in a small region, because errors that develop in this region tend to diffuse out into other regions where they are quenched. However, this effect is small, and in practice one cannot do much better than to satisfy the Courant condition everywhere. These results are also in accord with the trial calculations, which were not, however, sufficiently extensive to delineate in great detail the set of conditions

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under which the integration is stable. Further exploration of the method is planned.

Proposed Numerical Method for Calculation of Shocks, II

I. Outline of the trial calculations.

In report LA-671, a method was proposed for treatment of shocks in numerical hydrodynamical calculations. It was hoped that the method would combine the advantage of the "VonNeumann" method (that no special shock-fitting calculations are necessary) with the advantage of giving acceptably accurate results even when strong shocks are involved.

For the special case of a one-dimensional motion of an ideal gas with Lagrangean coordinate x; Eulerian coordinate X(x,t), where t is the time; pressure p (x,t); internal energy per unit mass E(x,t); specific volume V(x,t); temperature T(x,t); the method is based on the equations

$$T = pV \qquad (1)$$

$$E = \frac{pV}{\gamma-1} \qquad \frac{T}{\gamma-1} \qquad (2)$$

$$V = V_o \quad \frac{\partial X}{\partial x} \qquad (3)$$

$$\frac{\partial E}{\partial t} + p \quad \frac{\partial V}{\partial t} = \quad \varepsilon \left| \frac{\partial V}{\partial t} \right|^3 \qquad (4)$$

$$\frac{1}{V_o} \quad \frac{\partial^2 X}{\partial t^2} = - \frac{\partial p}{\partial x} + \quad \varepsilon \frac{\partial}{\partial x} \quad \left[\frac{\partial V}{\partial t} \quad \left| \frac{\partial V}{\partial t} \right| \right] \qquad (5)$$

The terms containing ε represent a fictitious dissipation which is chosen as to have no appreciable influence on the ordinarily continuous part of the motion but cause any shock discontinuity to be replaced by a rapid continuous variation. The thickness w given to the shock in this way is nearly independent of the strength of the shock, and is given approximately by the equation

$$\mathcal{E} = \frac{\gamma+1}{2} \; w^2/V_0^3 \, , \qquad (6)$$

according to a rough estimate given in the report cited above. In any case, we call w, as given by the above equation the nominal shock thickness.

By means of difference equations derived from (4) and (5) it should be possible to make hydrodynamical calculations without resorting to special (and usually complicated) "shock-fitting" calculations at each discontinuity.

It seems clear on general grounds that the nominal shock-thickness w should be chosen as a small multiple of the interval Δ_x used for the numerical calculation. If $w \ll \Delta_x$ the dissipative terms will have no effect and it is known that under these circumstances only weak shocks are even approximately correctly represented without shock-fitting. On the other hand, if $w \gg \Delta_x$ the difference equations used will almost certainly be unstable. The crucial question to be answered by the trial calculations was whether w can be chosen large enough to produce the desired result without introducing serious instability.

In writing the difference equations, we represent the value of a function at a mesh point by indices; for example, $T(x_\ell, t^n) = T_\ell^{\,n}$. The mesh used had ten points in the x-direction, with spacing Δ_x equal to unity, so that $x_1 = 1, x_2 = 2, \ldots x_{10} = 10$. Furthermore we set $V_o = 1$, $\gamma = 1.4$. The difference equations are:

$$p^{\,n}_{\ell+\frac{1}{2}} = T^{\,n}_{\ell+\frac{1}{2}} \big/ V^{\,n}_{\ell+\frac{1}{2}} \tag{7}$$

$$\frac{X^{\,n+1}_{\ell} - 2\,X^{\,n}_{\ell} + X^{\,n-1}_{\ell}}{(\Delta t)^2} = p^{\,n}_{\ell-\frac{1}{2}} - p^{\,n}_{\ell+\frac{1}{2}} \tag{8}$$

$$+ \; a \left[A^{\,n-\frac{1}{2}}_{\ell+\frac{1}{2}} \left| A^{\,n-\frac{1}{2}}_{\ell+\frac{1}{2}} \right| - A^{\,n-\frac{1}{2}}_{\ell-\frac{1}{2}} \left| A^{\,n-\frac{1}{2}}_{\ell-\frac{1}{2}} \right| \right]$$

(where $a = \varepsilon/(\Delta t)^2$)

$$V^{\,n+\frac{1}{2}}_{\ell+\frac{1}{2}} = X^{\,n+1}_{\ell+1} - X^{\,n+1}_{\ell} \tag{9}$$

$$A^{\,n+\frac{1}{2}}_{\ell+\frac{1}{2}} = V^{\,n+\frac{1}{2}}_{\ell+\frac{1}{2}} - V^{\,n}_{\ell+\frac{1}{2}} \tag{10}$$

$$\frac{5}{2}\left(T^{\,n+1}_{\ell+\frac{1}{2}} - T^{\,n}_{\ell+\frac{1}{2}}\right) = \left(\frac{3}{2}\,p^{\,n}_{\ell+\frac{1}{2}} - \frac{1}{2}\,p^{\,n-1}_{\ell+\frac{1}{2}}\right)\left(-A^{\,n+\frac{1}{2}}_{\ell+\frac{1}{2}}\right) + a \left| A^{\,n+\frac{1}{2}}_{\ell+\frac{1}{2}} \right|^{3} \tag{11}$$

For the simple plane-shock problem discussed here, the difference equations could have been written in a simpler form, but the above are a close analogy to equations that have been found useful in more complicated physical problems.

As initial condition, the gas was assumed to be initially at rest with $\Delta_X = 1$, $p = 1$, $V = 1$, $T = 1$, and a shock was sent into the gas by applying a constant pressure $p_1 > 1$ at the left boundary while maintaining the original pressure $p_{10\frac{1}{2}} = 1$ at the right boundary. The difference equations were then solved numerically * in a number of cases. The parameters of the problem in terms of which the several cases are differentiated are as follows. η = nominal compression ratio (i.e. the compression ratio that would correspond to shock pressure $p_{\frac{1}{2}}$ according to the Hugoniot equation $p_{\frac{1}{2}} = \frac{6\eta - 1}{6 - \eta}$). w or w/Δ_x is the nominal shock thickness, in units of Δ_x. C_f is the nominal value of the "Courant number" (sound speed times $\Delta t/\Delta x$) behind the shock i.e.,

$$C_f = \sqrt{\gamma \frac{p_{\frac{1}{2}}}{\eta}} \ \frac{\Delta t}{\Delta X} = \frac{\sqrt{\gamma p_{\frac{1}{2}} \eta}}{\eta^2} \ \frac{\Delta t}{\Delta x} \qquad (12)$$

The constants required in the calculation ($p_{\frac{1}{2}}$, Δt, a) can be computed from values of η, w, and C_f. The cases that were considered in the series of calculations are:

Case:	I	II	III	IV	V
η =	3	3	3	3	1.5
w =	2	4	2	4	4
C_f =	½	½	1/5	1/5	½

The numerical results are listed in the tables. Case I was found to be stable and was carried through twenty cycles. Case II was found to be definitely unstable and was discontinued at the fifth cycle. Case III was then skipped, because it should

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* This work was carried out by Irene Stegun of the New York Office of the Computation Laboratory of the National Bureau of Standards.

be more stable, by any theory, than case I, which had been found to be stable. Case IV was found to be stable and was carried through eleven cycles. Case V was found to be unstable and was discontinued at the fifth cycle.

II. <u>Comparison of results with steady-state theory.</u>

The development and propagation of the shock are exhibited graphically in Fig. 1, in which the specific volume V is plotted as a function of (Lagrangean) position at various instants of time, for case I, which is the only case in which the numerical calculations were carried far enough to allow the shock tobecome thoroughly detached from the piston and approach of state of steady propagation through the gas.

It is of interest to compare the speed and structure of the shock, in the late cycles of this calculation, with the speed and structure as given by a simpler theory. For if the assumption of a steady state is made at the outset (i.e. the gas flow is assumed stationary in a frame of reference moving with the shock) the partial differential equations (3) to (5) reduce to ordinary differential equations that can be solved explicitly. The solution thus obtained is of course the correct solution of the steady state problem, whereas the one obtained from the difference equations, even after a steady state is reached, may be in error because the finite difference, Δx, is comparable with the shock thickness, so that the dependent variables may change by non-small fractions of themselves in a single step (e.g. the specific volume increases by about 50% in some of the steps shown in Fig. 1). We shall find that the two methods agree (in case I) with an accuracy that is adequate for most problems, and we surmise that the difference equations (when they are stable) give a good representation of the differential equations also in problems not permitting a steady state to be reached.

For the steady state problem it is assumed that the dependent variables depend on x and t only through the combination

$$\xi = x - St$$

where S is the speed of the shock relative to the gas ahead. Equation (3), (4) and (5) then become:

$$V = V_o \frac{dX}{d\xi} \tag{13}$$

$$-S\frac{dE}{d\xi} - pS\frac{dV}{d\xi} = \varepsilon S^3 \left|\frac{dV}{d\xi}\right|^3 \tag{14}$$

$$\frac{S^2}{V_o^2}\frac{dV}{d\xi} = -\frac{dp}{d\xi} + S^2\varepsilon\frac{d}{d\xi}\left[-\frac{dV}{d\xi}\left|\frac{dV}{d\xi}\right|\right] \tag{15}$$

We look for a solution having $dV/d\xi > 0$, as will be the case for a shock going to the right. We assume that far ahead of the shock the specific volume has the value V_o and we call the corresponding pressure p_o. Integrating (15) with respect

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to ξ gives

$$\frac{s^2}{V_o^2}\,(V-V_o) = p_o-p - \varepsilon s^2 \left(\frac{dV}{d\xi}\right)^2 \tag{16}$$

If the expression for p, obtained by solving equation (16) is substituted into (14), we can integrate again with respect to ξ and obtain a relation between E, p and V. Using the " γ law" (equation (2)), the relation can be written as

$$\frac{pV-p_oV_o}{\gamma-1} = \frac{s^2}{2V_o^2}\,(V-V_o)^2 - p_o(V-V_o) \tag{17}$$

Let p_f and V_f be the asymptotic values of p and V at large distances behind the shock, where $dV/d\xi$ vanishes. Then (16) and (17) lead to the Hugoniot results:

$$s^2 = V_o^2 \quad \frac{p_f-p_o}{V_o-V_f} \tag{18}$$

$$\frac{p_f}{p_o} = \frac{\frac{\gamma+1}{\gamma-1}\,\eta - 1}{\frac{\gamma+1}{\gamma-1} - \eta} \tag{19}$$

where $\eta = V_o/V_f$.

Furthermore, we can eliminate p between (16) and (17) to obtain a differential equation of V. The coefficients of this differential equation are simplified by means of (18) and (19). We also use equation (6) and define a new independent variable $y = \xi/w$. The differential equation is then

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$$\left(\frac{d}{dy}\;\frac{V}{V_o}\right)^2 = \frac{V_o}{V}\left(1 - \frac{V}{V_o}\right)\left(\frac{V}{V_o} - \frac{1}{\eta}\right) \qquad (20)$$

so that the structure of the shock can be found by integrating the equation

$$\frac{dy}{dV} = \sqrt{\frac{V}{(1-V)(V-\frac{1}{\eta})}} \qquad (21)$$

Wherein the units have been so chosen as to make V_o equal to unity.

Equation (21) was integrated numerically for $\eta = 3$(which applies to case I), and the result is plotted in Fig. 2, where it is referred to as the "exact steady-state solution", and where comparison is made with the results (circles) of cycle 20 of the trial calculation for case I with the difference equations (7) to (11).

The agreement between the two methods is seen to be good, except for the two or three points at the left of the graph. It is believed that this discrepancy would have disappeared * if the trial calculation had been carried further and that it was caused by the fact that in the early cycles the shock is much steeper than in the steady state (it was in fact assumed to be infinitely sharp at the beginning of the calculation). This causes the energy dissipation to be abnormal when the shock passes the first few points so that thereafter

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* By this is meant that the mass-points which occupy corresponding positions relative to the shock at later times will not show such a discrepancy.

the entropy of the gas is abnormal near the piston. To test this explanation, the temperature T was calculated from equations (1) and (17), using (18) to eliminate S. The result is plotted in Fig. 3, where comparison is again made with the results of the trial calculation. It is seen that indeed the first three points have too high temperature. Finally, in Fig. 4 is plotted the quantity $TV^{\gamma-1}$, which is a function of the entropy, and it is seen that the first three points have acquired considerably more entropy than they would have in a steady state shock.

The speed of the shock can be obtained from the horizontal separation of the curves in Fig. 1. In a steady state these curves would be parallel and equidistant. This is seen to be approximately true for the last three curves, except near the bottom, and the speed obtained from the upper portions of these three curves is S 2.68, plus or minus perhaps 0.05, in good agreement with the correct value, from equations (18) and (19), which, for $\gamma = 3$, is $\sqrt{7}$ or ≈ 2.65.

The evidence of this section supports the view that, at least under the conditions of case I, equations (7) to (11) give a good description of shocks.

III. Stability of the difference equations.

The report referred to above (LA-671) contained a discussion of the stability of the system of difference equations

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(7) to (11). It was indicated that from the point of view of stability, the differential equations of the problem are equivalent to a single equation of the diffusion-equation type. Specifically, if one imagines going to a sufficiently fine mesh in the x-t plane, keeping the parameters of the problem fixed (including ε), the dominant terms, from the point of view of stability, are the left member and the last term of the right member of equation (5) (or the corresponding terms of equation (8)). Introducing the material velocity $u = \frac{\partial x}{\partial t}$ and noting that $\frac{\partial V}{\partial t} = V_o \quad \frac{\partial u}{\partial x}$, these dominant terms can be written as

$$\frac{1}{V_o} \quad \frac{\partial u}{\partial t} = 2\varepsilon V_o^2 \quad \left| \frac{\partial u}{\partial x} \right| \quad \frac{\partial^2 u}{\partial x^2} \tag{22}$$

or

$$\frac{\partial u}{\partial t} = \sigma(x,t) \quad \frac{\partial^2 u}{\partial x^2} \tag{23}$$

where

$$\sigma(x,t) \quad 2\varepsilon V_o^3 \quad \left| \frac{\partial u}{\partial x} \right| \tag{24}$$

By examination of (8) it is seen that the difference equation corresponding to (24) is to be written as

$$\frac{u_\ell^{n+\frac{1}{2}} - u_\ell^{n-\frac{1}{2}}}{\Delta t} = \sigma_\ell^{n-\frac{1}{2}} \quad \frac{u_{\ell+1}^{n-\frac{1}{2}} - 2u_\ell^{n-\frac{1}{2}} \quad u_{\ell-1}^{n-\frac{1}{2}}}{(\Delta x)^2} \tag{25}$$

In a terminology frequently used, the difference equation (25) is of the explicit type. In problems in which the diffusion coefficient σ is constant, the condition for stability is that the quantity $C = \frac{2\sigma\Delta t}{(\Delta x)^2}$ should not exceed

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unity. In the present problem σ is large in the shock, but small elsewhere. To investigate qualitatively the effect of such a variation, we suppose that σ is a given function of x, which varies continuously and has a single maximum at $x = 0$. Then

$$u_{\ell}^{n+\frac{1}{2}} - u_{\ell}^{n-\frac{1}{2}} = \frac{1}{2}\, C_{\ell} \left[u_{\ell+1}^{n-\frac{1}{2}} - 2u_{\ell}^{n-\frac{1}{2}} + u_{\ell-1}^{n-\frac{1}{2}} \right] \tag{26}$$

where

$$C(x) = \frac{2\,\sigma(x)\,\Delta t}{(\Delta x)^2} \tag{27}$$

$C(x)$ is supposed slowly varying and positive.

It is now supposed that superposed on a desired solution u there is a small disturbance having the form

$$u = g(x)e^{\alpha t} \tag{28}$$

where α is a constant. If for every solution of this form, without restriction on $g(x)$ except boundedness, the real part of α is non-positive, the difference equation is stable; and otherwise not. Substituting (28) into (26) and dividing through by $u_{\ell}^{n-\frac{1}{2}}$, we get

$$\xi - 1 = \frac{1}{2}\, C_{\ell} \left[y_{\ell+\frac{1}{2}} - 2 + \frac{1}{y_{\ell-\frac{1}{2}}} \right] \tag{29}$$

where

$$\xi = e^{\alpha \Delta t} \tag{30}$$

and

$$y\left(x + \frac{\Delta x}{2}\right) = \frac{g(x + \Delta x)}{g(x)} \tag{31}$$

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Since C(x) is a slowly varying function, equation (29) determines y(x) as a slowly varying function. (This of course does not imply that g(x) is slowly varying.) The requirement of stability is that the absolute value of ξ should not exceed unity for any solution y(x) leading to a bounded g(x).

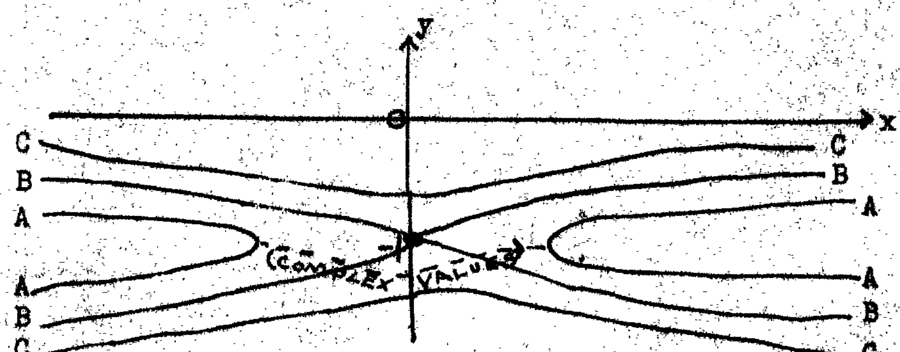
To solve (29) approximately for y(x), we consider a sequence of approximations, as follows: As the first approximation, we set

$$\xi - 1 = \frac{C}{2}\left[\, y - 2 + \frac{1}{y} \,\right] \tag{32}$$

or

$$y \quad 1 - \frac{1-\xi}{C} \pm \sqrt{\left(\frac{1-\xi}{C}\right)^2 - 2\,\frac{1-\xi}{C}} \tag{33}$$

From the assumed character of C(x), $(1-\xi)/C$ has a minimum at $x = 0$; we assume further that, for large positive or negative x, $\frac{1-\xi}{C}$ has an approximately constant value, which will be generally greater than 2. Therefore, to the far left and right, y(x) has two branches, as in the three typical cases sketched in the diagram:



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Type A, B or C occurs according to whether the minimum value of $\frac{1-\xi}{C(x)}$ is less than, equal to or greater than 2. At the far left and right, the two branches have values in the intervals (0, -1) and (-1, $-\infty$), respectively.

Although this is only the first approximation, it is expected that further approximations will have the same character. Before obtaining the next approximation, let us indicate how the question stability is related to these curves. From (31) it is seen that the boundedness of g(x) as x goes to plus or minus infinity requires that for large negative x we use the lower branch of the function y(x) and that for large positive x we use the upper branch. Therefore disturbances of type (28) can arise only if it is possible to get continuously from the lower to the upper branch at intermediate values of x. The limiting case is that in which the two branches cross at x   0, and the corresponding value of ξ determines the most rapid rate of growth (or slowest decay) of errors of the difference equation (26). This value of ξ must have absolute magnitude not greater than unity for stability. For example, the condition for crossing, in the first approximation, is that $\frac{1-\xi}{C_o} = 2$, where C_o is the maximum of C(x), and the condition for stability is that $C_o \leqslant 1$.

To obtain a second approximation, we write

$$y_{\ell+\frac{1}{2}} = y_\ell \pm \frac{\Delta x}{2} \quad y'_\ell \quad \cdots\cdots\cdots, \qquad (34)$$

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where primes denote differentiation with respect to x, and approximate (29) as

$$\xi - 1 \approx \frac{C_\ell}{2}\left[y_\ell - 2 + \frac{1}{y_\ell} + \frac{\Delta x}{2}\left(y'_\ell + \frac{y'_\ell}{(y_\ell)^2}\right)\right] \tag{35}$$

and in this equation the coefficient of $\Delta x/2$ is to be evaluated from the first approximation. This again determines a function $y(x)$ which, to the far left and far right, has two branches. To find, in second approximation, the value of ξ corresponding to the critical rate of growth or decay of errors, the two branches must be made to cross, which means that we equate to zero the discriminant of (35), regarded as an equation for y_ℓ, for $x = 0$. To do this we require the value of $y'(0)$, from the first approximation, for the rising curve to type B, for use in the last term of the bracket of (35). Differentiate (32) twice with respect to x:

$$0 = \frac{C}{2}\left[y'' - \frac{y''}{y^2} + 2\frac{(y')^2}{y^3}\right] + C'\left[y' - \frac{y'}{y^2}\right] + \frac{C''}{2}\left[y - 2 + \frac{1}{y}\right] \tag{36}$$

and in this equation set $x = 0$, noting that at $x = 0$, in first approximation, $y = -1$, $C'(0) = 0$, $C(0) - 1$;

$$0 = -\left[y'(0)\right]^2 - 2C''(0) \tag{37}$$

Therefore

$$y'(0) = \sqrt{-2C''(0)} = \sqrt{\frac{-2\sigma''(0)}{\sigma(0)}} = \sqrt{-2(\ln\sigma)''_{x=0}} \quad (38$$

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(The last equality follows from the vanishing of $\sigma'(0)$)

Then, at $x = 0$, (35) becomes

$$\xi - 1 \approx \frac{C_o}{2} \left[y_o - 2 \quad \frac{1}{y_o} + \Delta x \sqrt{-2(\ln\sigma)''_o} \right] \tag{39}$$

To obtain the limiting condition for stability, we now put $\xi = -1$, and equate the discriminant of (39) to zero:

$$\left[\frac{4}{C_o} - 2 \quad \Delta x \sqrt{-2(\ln\sigma)''_0} \right]^2 - 4 = 0 \tag{40}$$

Or,

$$\frac{4}{C_o} - 2 + \Delta x \sqrt{-2(\ln\sigma)''_o} = \pm 2 \tag{41}$$

Clearly we must take the plus sign on the right, because C_o is positive. This gives the limiting condition for stability, and general condition for stability is

$$C_o \leq 1 + \frac{\Delta x}{4} \sqrt{-2(\ln\sigma)''_o} \tag{42}$$

Therefore, in the region of maximum σ, we are permitted to violate slightly the previous stability condition that C should not exceed unity. However, the gain is not very great, because in normal problems the last term of (42) is small compared to unity.

For application to the problem at hand it would have been more realistic to consider a problem in which the diffusion coefficient, σ, is a function of x - St (where S is the speed of the shock) rather than on x alone. Although

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we have not worked this case out, it seems clear on intuitive grounds that the restriction on C_o (for stability) would be even more lenient. We reason in this way; the result obtained above may be interpreted by saying that although errors tend to multiply in the neighborhood of $x = 0$, they tend to diffuse outward into regions where $C(x)$ is less than 1 and there be quenched. In the present case there is an added effect that the errors tend to be swept out of the dangerous region by the flow of the material through it. But again the gain is presumably small.

To get a rough idea of the magnitude of these effects we use equation (42) as it stands. From (24), (21), we may write:

$$\ln \sigma = \ln \frac{dV}{dt} + \text{const.} \tag{43}$$

$$\ln \sigma = \tfrac{1}{2} \ln (1-V) + \tfrac{1}{2} \ln \left(V - \frac{1}{\gamma}\right) - \tfrac{1}{2} \ln V + \text{const.} \tag{44}$$

$$(\ln \sigma)'' \quad \frac{d^2}{dx^2} \ln \sigma = \frac{1}{w^2} \quad \frac{d^2}{dy^2} \ln \sigma \tag{45}$$

$$\frac{dV}{dy} = \sqrt{\frac{(1-V)\left(V - \frac{1}{\gamma}\right)}{V}} \tag{46}$$

From these equations it is readily found that $\sigma(x)$ attains its maximum value when the specific volume reaches the value $V = 1/\sqrt{\gamma}$. This is to be taken as the point $x = 0$; for this point it is found that

$$(\ln \sigma)'' \quad \frac{-\sqrt{\gamma}}{w^2} \tag{47}$$

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Therefore (42) becomes

$$C_o \leq 1 + \frac{\Delta x}{w} \quad \frac{\gamma^{\frac{1}{4}}}{2\sqrt{2}} \qquad (48)$$

Lastly the peak value of $C(x)$ can now be obtained from (21), (24), (27), and is:

$$C_o \quad \frac{24}{\sqrt{5(6\gamma - 1)}} \quad \frac{w}{\Delta x} \left(1 - \frac{1}{\sqrt{\gamma}}\right) \quad C_f \qquad (49)$$

We now tabulate the numerical values of the two members of the inequality (48) for various trial cases together with the findings relative to their stability:

Case	C_o	$1 + \frac{\Delta x}{w} \frac{\gamma^{\frac{1}{4}}}{2\sqrt{2}}$	Stability
I	1.10	1.233	stable
II	2.20	1.116	unstable
III	.442	1.233	(not investigated)
IV	.883	1.116	stable
V	1.39	1.098	unstable

It is to be noted that the findings are in conformity with the theory; also that they would not have been, in case I, had we stopped with the first approximation.

IV. Further trial calculations.

The findings of this report indicate that the proposed method of treating shocks should give satisfactory results, so long as one does not depart too much from the conditions of case I. But in complex problems of physical interest,

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the speed and strength of the shock and the conditions of the material ahead of it will in general be changing during the course of the phenomenon under investigation, and it may happen that no one fixed choice of the coefficient of the fictitious dissipation term will provide conditions close to those of case I at all times or even for all shocks (if there are two or more in the problem) at any one time. It is therefore of interest to know more accurately than we do now the range of conditions under which the method is usable, and to know how well the method works under unusual circumstances, such as when a shock crosses an interface. So far as stability is concerned, we may perhaps place considerable reliance in formula (48), but the question of accuracy remains to be answered.

A completely satisfactory exploration of the method would involve a rather extended series of trial calculations, and for each of them the calculation should probably be carried considerably further than any of the examples described above. It is planned to initiate such a program, with the aid of the Los Alamos IBM group, in the near future.

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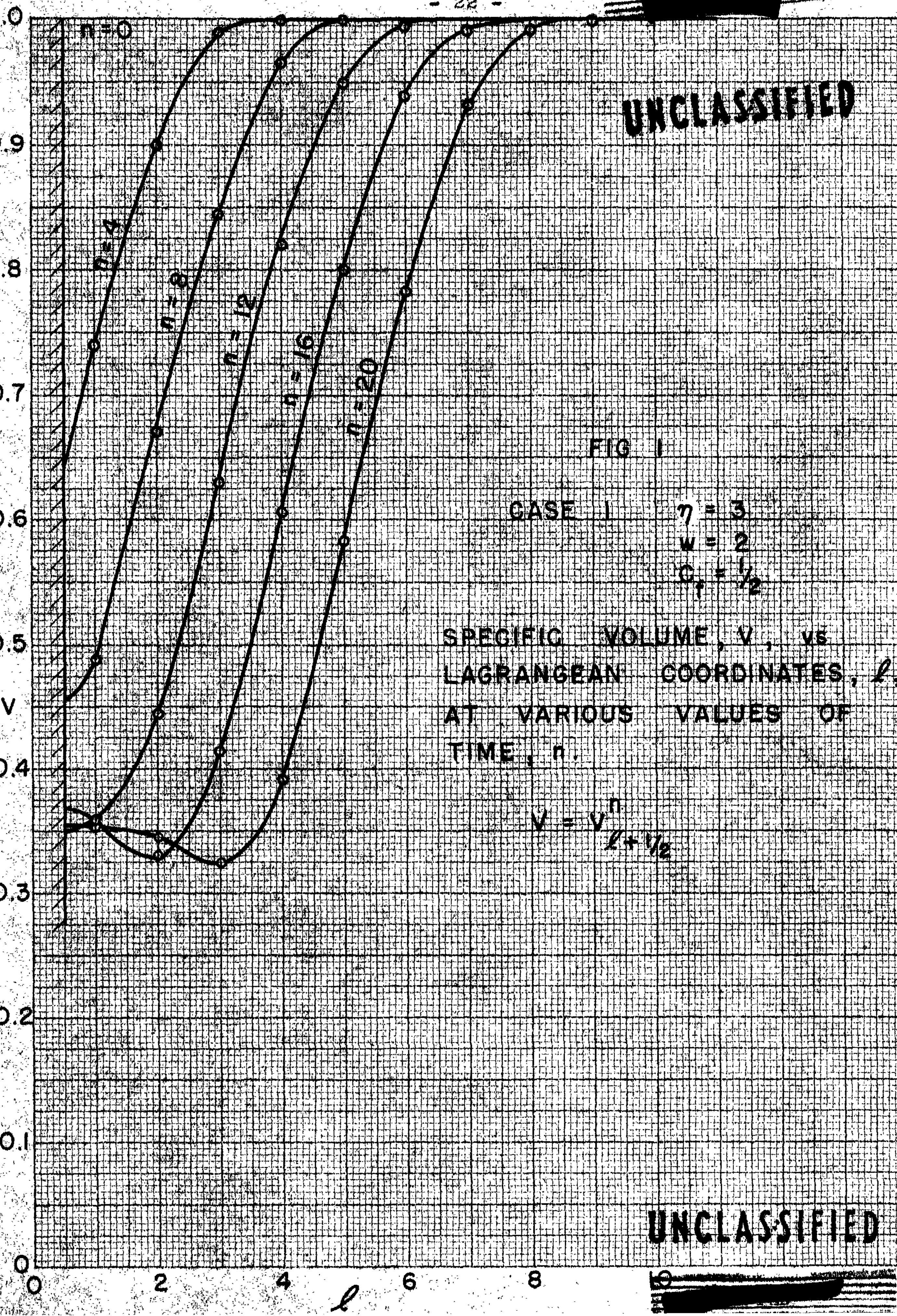

FIG 1

CASE I $\quad \eta = 3$
$w = 2$
$C_1 = 1/2$

SPECIFIC VOLUME, V, vs LAGRANGEAN COORDINATES, ℓ, AT VARIOUS VALUES OF TIME, n.

$$V = V^{n}_{\ell + 1/2}$$

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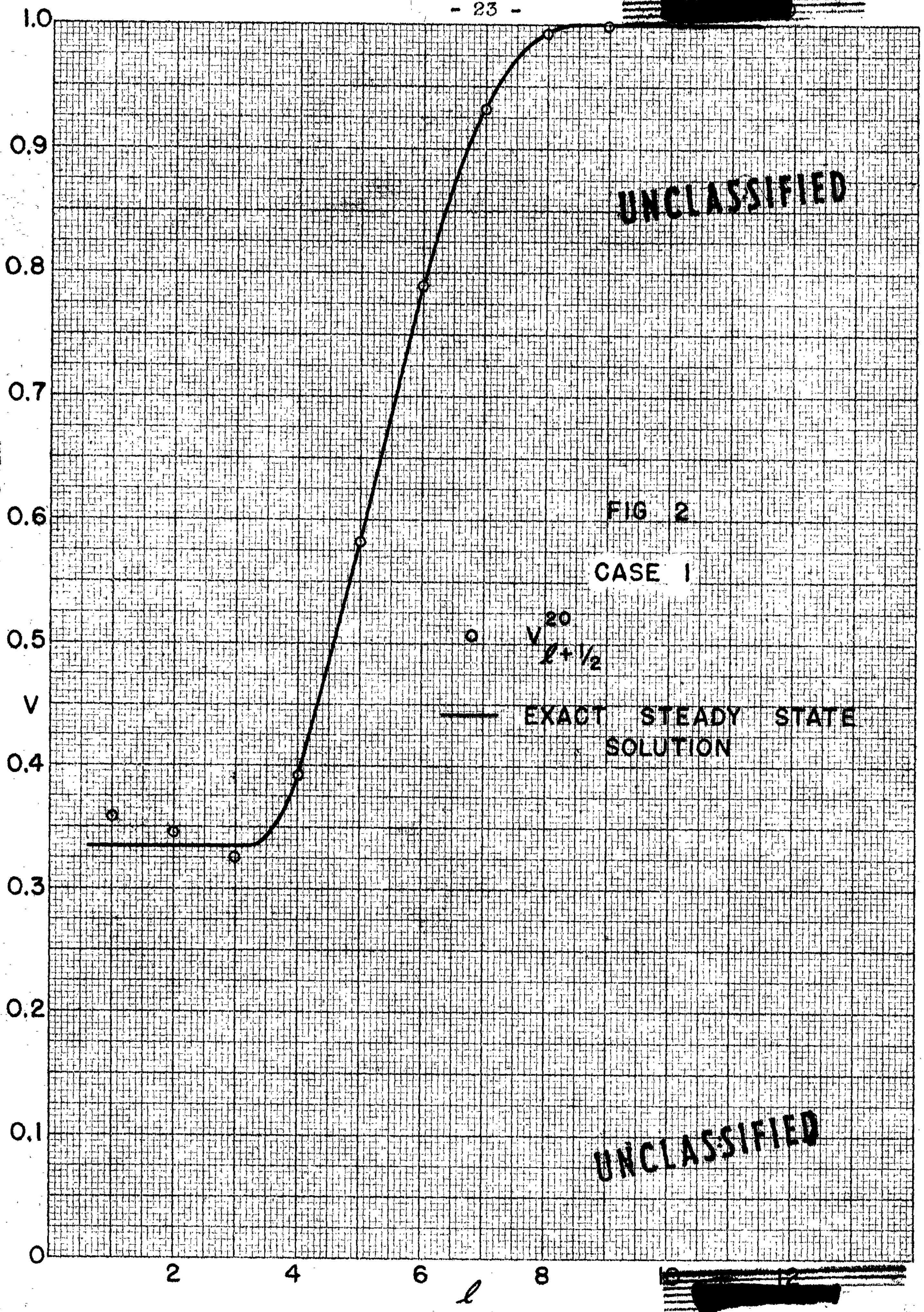

FIG 2

CASE 1

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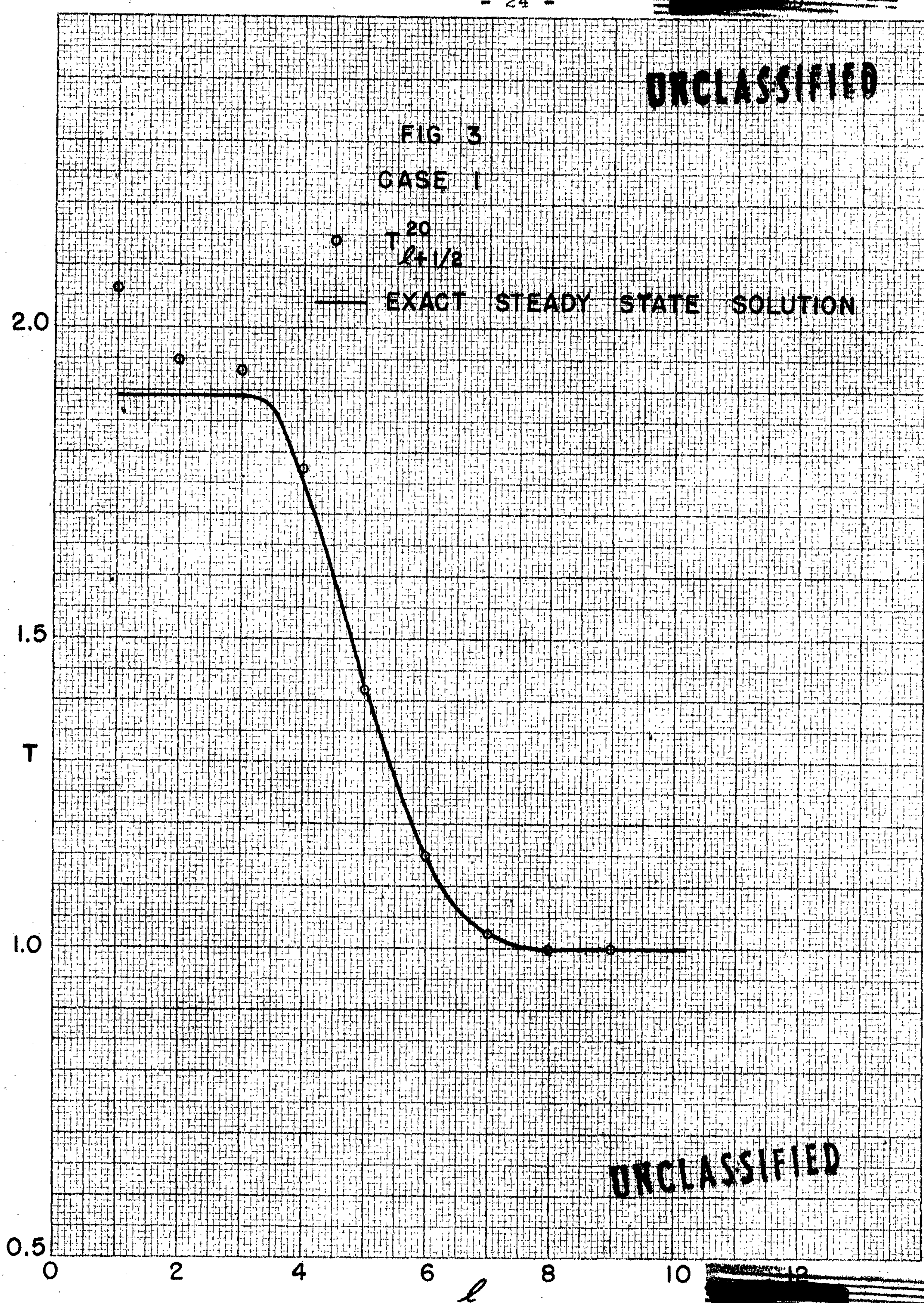
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FIG 3
CASE 1
$T^{20}_{\ell+1/2}$
EXACT STEADY STATE SOLUTION
2.0
1.5
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[redacted]
ℓ
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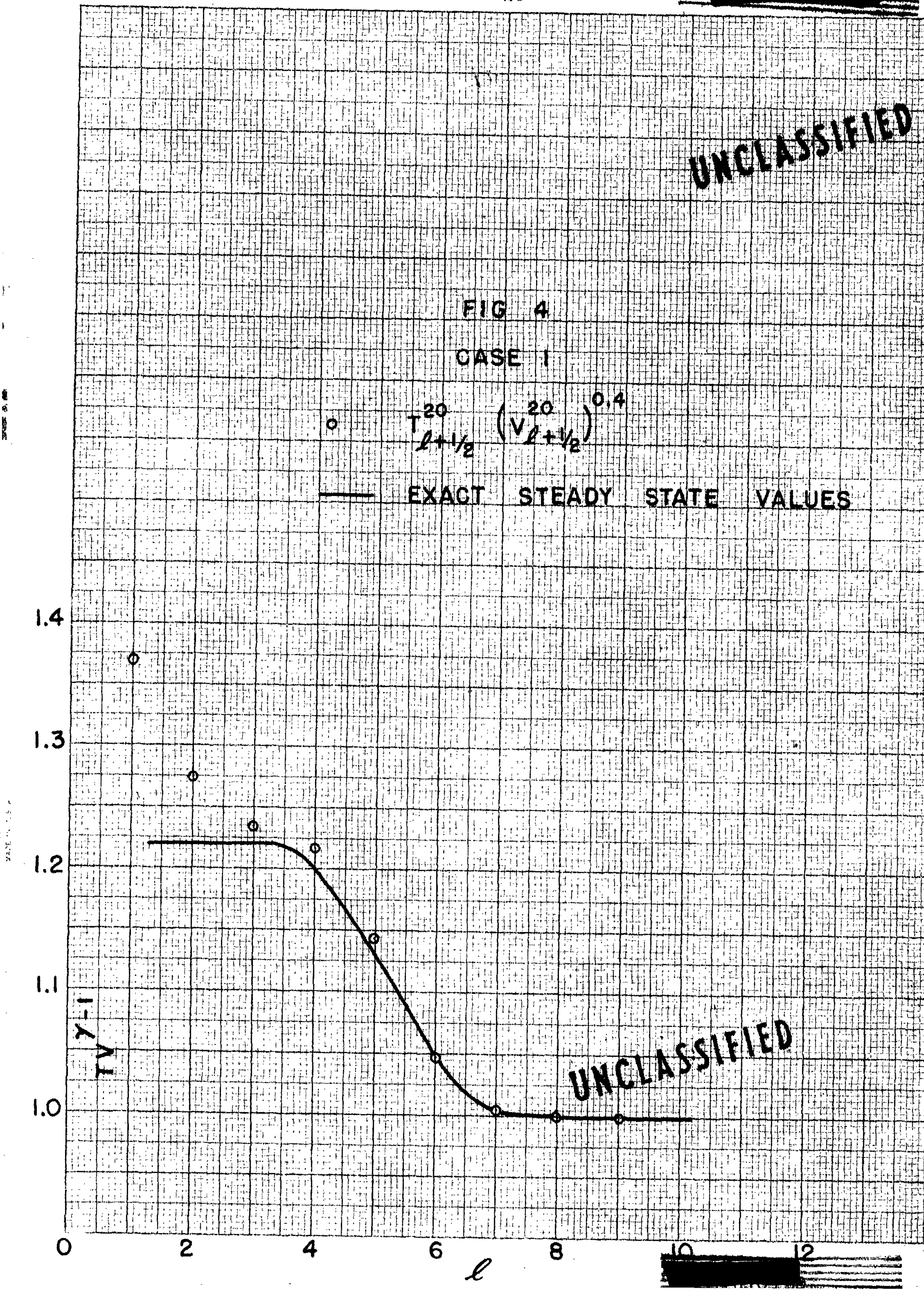

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Table I

Proposed Hydrodynamical Calculations

Table of $V^n_{\ell+\frac{1}{2}}$, $T^n_{\ell+\frac{1}{2}}$

Case I $\gamma = 3$, $w = 2$, $C_f = \frac{1}{2}$

ℓ	$V^1_{\ell+\frac{1}{2}}$	$T^1_{\ell+\frac{1}{2}}$	$V^2_{\ell+\frac{1}{2}}$	$T^2_{\ell+\frac{1}{2}}$
1	.95098 0392	1.04113 8024	.87800 0928	1.14552 7188
2	1	1	.98747 0124	1.00537 1516
3	1	1	1	1
4	1	1	1	1
5	1	1	1	1
6	1	1	1	1
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1

ℓ	$V^3_{\ell+\frac{1}{2}}$	$T^3_{\ell+\frac{1}{2}}$	$V^4_{\ell+\frac{1}{2}}$	$T^4_{\ell+\frac{1}{2}}$
1	.81258 8707	1.23356 8584	.74172 4465	1.34466 9367
2	.94806 2845	1.03274 8878	.89944 2157	1.07563 5516
3	.99905 5985	1.00037 7760	.98975 5947	1.00425 2183
4	1	1	.99998 1825	1.00000 7270
5	1	1	1	1
6	1	1	1	1
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1

ℓ	$V^5_{\ell+\frac{1}{2}}$	$T^5_{\ell+\frac{1}{2}}$	$V^6_{\ell+\frac{1}{2}}$	$T^6_{\ell+\frac{1}{2}}$
1	.67372 2226	1.45547 0312	.60734 1267	1.57091 0192
2	.84441 8785	1.13357 6519	.78852 9609	1.19713 7231
3	.96615 8896	1.01489 6448	.93396 0741	1.03680 7933
4	.99939 5185	1.00024 1972	.99609 0239	1.00157 2212
5	.99999 9731	1.00000 0108	.99998 8918	1.00000 4433
6	1	1	.99999 9996	1.00000 0002
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1

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Table II

Proposed Hydrodynamical Calculations

Table of $V^{n}_{\ell+\frac{1}{2}}$, $T^{n}_{\ell+\frac{1}{2}}$

Case I $\eta = 3$, $w = 2$, $C_f = \frac{1}{2}$

ℓ	$V^{7}_{\ell+\frac{1}{2}}$		$T^{7}_{\ell+\frac{1}{2}}$		$V^{8}_{\ell+\frac{1}{2}}$		$T^{8}_{\ell+\frac{1}{2}}$	
1	.54713	6593	1.67821	7363	.49122	3704	1.78414	3510
2	.72891	6918	1.27415	5687	.66980	0536	1.35596	9995
3	.89274	6374	1.06841	5416	.84638	3000	1.10962	9951
4	.98622	5371	1.00572	4536	.96748	7745	1.01462	3736
5	.99986	8193	1.00005	2724	.99907	6684	1.00036	9503
6	.99999	9829	1.00000	0069	.99999	7659	1.00000	0937
7	1		1		.99999	9997	1	
8	1		1		1		1	
9	1		1		1		1	

ℓ	$V^{9}_{\ell+\frac{1}{2}}$		$T^{9}_{\ell+\frac{1}{2}}$		$V^{10}_{\ell+\frac{1}{2}}$		$T^{10}_{\ell+\frac{1}{2}}$	
1	.44406	2457	1.87716	0284	.40513	6435	1.95839	4972
2	.60951	7090	1.44816	2365	.55221	4340	1.54104	1718
3	.79745	6851	1.15781	2581	.74434	2263	1.21754	4522
4	.93954	2617	1.03049	7106	.90479	2185	1.05374	7540
5	.99612	1302	1.00155	8559	.98859	8784	1.00466	8065
6	.99997	8958	1.00000	8417	.99986	1618	1.00005	5355
7	.99999	9960	1.00000	0015	.99999	9612	1.00000	0154
8	1		1		1		1	
9	1		1		1		1	

ℓ	$V^{11}_{\ell+\frac{1}{2}}$		$T^{11}_{\ell+\frac{1}{2}}$		$V^{12}_{\ell+\frac{1}{2}}$		$T^{12}_{\ell+\frac{1}{2}}$	
1	.37771	0745	2.01852	2627	.36052	6751	2.05793	6680
2	.49628	8313	1.64008	1619	.44608	7991	1.73473	5004
3	.68957	0033	1.28543	0018	.63202	8379	1.36578	8162
4	.86443	1453	1.08511	4660	.82023	8479	1.12388	2084
5	.97443	3151	1.01097	6580	.95246	9278	1.02212	1655
6	.99930	6110	1.00027	7651	.99741	6824	1.00103	5627
7	.99999	7174	1.00000	1129	.99998	3132	1.00000	6746
8	.99999	9994	1.00000	0002	.99999	9940	1.00000	0024
9	1		1		1		1	

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Table III

Proposed Hydrodynamical Calculations

Table of $V^n_{\ell+\frac{1}{2}}$, $T^n_{\ell+\frac{1}{2}}$

Case I $\eta = 3$, $w = 2$, $C_f = \frac{1}{2}$

ℓ	$V^{13}_{\ell+\frac{1}{2}}$	$T^{13}_{\ell+\frac{1}{2}}$	$V^{14}_{\ell+\frac{1}{2}}$	$T^{14}_{\ell+\frac{1}{2}}$
1	.35362 8038	2.07425 0501	.35377 7845	2.07389 4298
2	.40180 7814	1.82464 4858	.36774 0432	1.89819 7685
3	.57495 9245	1.45247 9717	.51769 3569	1.54885 5589
4	.77139 9885	1.17306 4159	.71934 5146	1.23207 6968
5	.92335 4585	1.03933 7643	.88787 3694	1.06384 9485
6	.99255 4047	1.00301 1396	.98275 0103	1.00716 0078
7	.99991 4343	1.00003 4263	.99962 3940	1.00015 0448
8	.99999 9638	1.00000 0145	.99999 8064	1.00000 0775
9	1	1	.99999 9995	1.00000 0002

ℓ	$V^{15}_{\ell+\frac{1}{2}}$	$T^{15}_{\ell+\frac{1}{2}}$	$V^{16}_{\ell+\frac{1}{2}}$	$T^{16}_{\ell+\frac{1}{2}}$
1	.35776 6507	2.06455 5848	.36122 4655	2.05664 4316
2	.34451 0822	1.95133 5123	.33378 7337	1.97693 3224
3	.46397 4331	1.64648 1017	.41479 7743	1.74349 9227
4	.66348 9565	1.30434 2045	.60632 0003	1.38634 4784
5	.84742 6077	1.09591 7722	.80189 0652	1.13759 6153
6	.96626 0151	1.01478 6652	.94247 7584	1.02735 6914
7	.99862 3112	1.00055 1255	.99583 9585	1.00167 1539
8	.99999 0609	1.00000 3757	.99995 8346	1.00001 6662
9	.99999 9962	1.00000 0015	.99999 9786	1.00000 0085

ℓ	$V^{17}_{\ell+\frac{1}{2}}$	$T^{17}_{\ell+\frac{1}{2}}$	$V^{18}_{\ell+\frac{1}{2}}$	$T^{18}_{\ell+\frac{1}{2}}$
1	.36189 0669	2.05513 7857	.35993 2718	2.05958 1092
2	.33308 6517	1.97862 9849	.33855 6348	1.96564 3609
3	.37466 9367	1.82803 1904	.34537 7108	1.89375 2519
4	.54794 7991	1.47982 8508	.49157 6743	1.57813 6687
5	.75219 2984	1.18948 0014	.69805 9430	1.25447 9204
6	.91156 7999	1.04647 8324	.87463 4761	1.07307 4539
7	.98959 1917	1.00423 4722	.97788 6691	1.00930 0164
8	.99982 8608	1.00006 8562	.99936 1270	1.00025 5578
9	.99999 8999	1.00000 0400	.99999 5640	1.00000 1744

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Table IV

Proposed Hydrodynamical Calculations

Table of $V^n_{\ell+\frac{1}{2}}$, $T^n_{\ell+\frac{1}{2}}$

Case I $\gamma = 3$, $w = 2$, $C_f = \frac{1}{2}$

ℓ	$V^{19}_{\ell+\frac{1}{2}}$	$T^{19}_{\ell+\frac{1}{2}}$	$V^{20}_{\ell+\frac{1}{2}}$	$T^{20}_{\ell+\frac{1}{2}}$
1	.35785 7201	2.06435 1223	.35795 3009	2.06412 9259
2	.34387 3427	1.95346 5642	.34495 2757	1.95104 0326
3	.32969 5495	1.93074 5833	.32612 2233	1.93939 0952
4	.43849 8254	1.67904 1096	.39300 9924	1.77154 5525
5	.64140 2002	1.33089 5091	.58274 1574	1.41973 8417
6	.83212 8926	1.10864 2336	.78493 1767	1.15400 6626
7	.95923 0473	1.01825 3912	.93300 4733	1.03284 0967
8	.99791 9670	1.00083 3470	.99424 2090	1.00231 9376
9	.99998 1961	1.00000 7216	.99992 8168	1.00002 8734

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Table V

Proposed Hydrodynamical Calculations

Table of $V^n_{\ell+\frac{1}{2}}$, $T^n_{\ell+\frac{1}{2}}$

Case II $\quad \gamma = 3$, $w = 4$, $C_f = \frac{1}{2}$

ℓ	$V^1_{\ell+\frac{1}{2}}$	$T^1_{\ell+\frac{1}{2}}$	$V^2_{\ell+\frac{1}{2}}$	$T^2_{\ell+\frac{1}{2}}$
1	.95098 0392	1.10572 8567	.94863 1970	1.10690 6694
2	1	1	.95215 4603	1.09921 7161
3	1	1	1	1
4	1	1	1	1
5	1	1	1	1
6	1	1	1	1
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1

ℓ	$V^3_{\ell+\frac{1}{2}}$	$T^3_{\ell+\frac{1}{2}}$	$V^4_{\ell+\frac{1}{2}}$	$T^4_{\ell+\frac{1}{2}}$
1	.85694 2015	1.71336 7666	1.08675 2784	10.36510 1602
2	.99039 8611	1.12127 2141	.76489 2748	9.60678 1873
3	.95604 7705	1.07965 9651	1.01568 6919	1.20627 0952
4	1	1	.96155 1221	1.05693 6668
5	1	1	1	1
6	1	1	1	1
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1

ℓ	$V^5_{\ell+\frac{1}{2}}$
1	-1.67891 2222
2	
3	
4	
5	
6	
7	
8	
9	

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Table VI

Proposed Hydrodynamical Calculations

Table of $V^n_{\ell+\frac{1}{2}}$, $T^n_{\ell+\frac{1}{2}}$

Case IV $\gamma = 3$, $w = 4$, $C_f = 1/5$

ℓ	$V^1_{\ell+\frac{1}{2}}$	$T^1_{\ell+\frac{1}{2}}$	$V^2_{\ell+\frac{1}{2}}$	$T^2_{\ell+\frac{1}{2}}$
1	.99215 6863	1.00534 1945	.97887 7427	1.02146 0425
2	1	1	.99879 6581	1.00048 9332
3	1	1	1	1
4	1	1	1	1
5	1	1	1	1
6	1	1	1	1
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1

ℓ	$V^3_{\ell+\frac{1}{2}}$	$T^3_{\ell+\frac{1}{2}}$	$V^4_{\ell+\frac{1}{2}}$	$T^4_{\ell+\frac{1}{2}}$
1	.96464 2012	1.04067 0536	.95018 9069	1.06080 5240
2	.99419 5564	1.00277 9499	.98641 3097	1.00808 4087
3	.99996 9346	1.00001 2262	.99951 7909	1.00019 7053
4	1	1	.99999 9910	1.00000 0036
5	1	1	1	1
6	1	1	1	1
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1

ℓ	$V^5_{\ell+\frac{1}{2}}$	$T^5_{\ell+\frac{1}{2}}$	$V^6_{\ell+\frac{1}{2}}$	$T^6_{\ell+\frac{1}{2}}$
1	.93510 5776	1.08333 5095	.91961 3526	1.10763 5859
2	.97618 8890	1.01606 7245	.96620 3132	1.02598 9602
3	.99787 6776	1.00087 4359	.99451 4285	1.00239 8684
4	.99999 4765	1.00000 2094	.99993 2885	1.00002 6858
5	1	1	.99999 9987	1.00000 0052
6	1	1	1	1
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1

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Table VII

Proposed Hydrodynamical Calculations

Table of $V^n_{\ell+\frac{1}{2}}$, $T^n_{\ell+\frac{1}{2}}$

Case IV $\eta = 3$, $w = 4$, $C_f = 1/5$

ℓ	$V^7_{\ell+\frac{1}{2}}$	$T^7_{\ell+\frac{1}{2}}$	$V^8_{\ell+\frac{1}{2}}$	$T^8_{\ell+\frac{1}{2}}$
1	.90391 4042	1.13302 5979	.88810 7550	1.15915 2227
2	.95494 0607	1.03735 0327	.94314 0776	1.05004 2889
3	.98934 4647	1.00511 9357	.98267 1917	1.00919 9358
4	.99964 1062	1.00014 3717	.99881 4279	1.00047 7247
5	.99999 9743	1.00000 0098	.99999 7019	1.00000 1188
6	1	1	1	1
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1

ℓ	$V^9_{\ell+\frac{1}{2}}$	$T^9_{\ell+\frac{1}{2}}$	$V^{10}_{\ell+\frac{1}{2}}$	$T^{10}_{\ell+\frac{1}{2}}$
1	.87221 4006	1.18596 0272	.85624 9540	1.21340 9832
2	.93085 8392	1.06404 6090	.91817 3218	1.07924 9143
3	.97491 9947	1.01452 9685	.96641 3173	1.02090 6858
4	.99711 9078	1.00117 9111	.99432 5787	1.00240 1914
5	.99997 8386	1.00000 8641	.99989 7850	1.00004 0859
6	.99999 9993	1	.99999 9928	1.00000 0026
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1

ℓ	$V^{11}_{\ell+\frac{1}{2}}$	$T^{11}_{\ell+\frac{1}{2}}$
1	.84024 6427	1.24139 2930
2	.90517 8194	1.09547 1010
3	.95732 6919	1.02820 2875
4	.99037 4866	1.00428 0140
5	.99965 4586	1.00013 8250
6	.99999 9498	1.00000 0198
7	1	1
8	1	1
9	1	1

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Table VIII

Proposed Hydrodynamical Calculations

Table of $V^n_{\ell+\frac{1}{2}}$, $T^n_{\ell+\frac{1}{2}}$

Case V $\gamma = 1.5$, $w = 4$, $C_f = \frac{1}{2}$

ℓ	$V^1_{\ell+\frac{1}{2}}$	$T^1_{\ell+\frac{1}{2}}$	$V^2_{\ell+\frac{1}{2}}$	$T^2_{\ell+\frac{1}{2}}$
1	.94791 6667	1.03703 7037	.96050 8243	1.03151 9072
2	1	1	.94162 0879	1.04617 0334
3	1	1	1	1
4	1	1	1	1
5	1	1	1	1
6	1	1	1	1
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1

ℓ	$V^3_{\ell+\frac{1}{2}}$	$T^3_{\ell+\frac{1}{2}}$	$V^4_{\ell+\frac{1}{2}}$	$T^4_{\ell+\frac{1}{2}}$
1	.85195 8579	1.22440 3896	1.33924 4936	14.17887 4372
2	1.02707 7322	1.07786 8276	.47069 0843	21.05824 1104
3	.92712 8926	1.07352 8041	1.21621 9707	3.70140 1314
4	1	1	.88747 0231	1.20843 7573
5	1	1	1	1
6	1	1	1	1
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1

ℓ	$V^5_{\ell+\frac{1}{2}}$	$T^5_{\ell+\frac{1}{2}}$
1	-14.93216 9900	
2		
3		
4		
5		
6		
7		
8		
9		

[redacted]