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Methods for Hyperbolic Systems with Stiff Relaxation

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Abstract

Three methods are analyzed for solving a linear hyperbolic system that contains stiff relaxation. We show that the semi-discrete discontinuous Galerkin method, with a linear basis, is accurate when the relaxation time is unresolved (asymptotically preserving — AP). A recently developed central method is shown to be non-AP. To discriminate between AP and non-AP methods, we argue that one must study problems that are diffusion dominated.

1 Introduction

Hyperbolic systems with stiff relaxation terms remain a challenge for numerical methods [2, 5, 8]. Such systems can be found, for example, in combustion, multiphase flow, rarefied gas dynamics, and radiation hydrodynamics. In an effort to better understand the behavior of numerical methods for these systems, this study focuses on a simple model problem [3, 2]:

$$\partial_t u + \partial_x v = 0, \tag{1.1a}$$

$$\partial_t v + \partial_x u = (ru - v)/\varepsilon, \tag{1.1b}$$

where $\varepsilon \geq 0$ and $|r| < 1$. The system (1.1) is in nondimensional form, so that the frozen ($\varepsilon \rightarrow \infty$) wavespeeds are ± 1 . We assume that boundary and initial conditions are such that the appropriate time and length scales are 1. For $\varepsilon \ll 1$, it can be shown that away from the initial layer, the system (1.1) reduces to an advection–diffusion equation:

$$\partial_t u + r\partial_x u - \varepsilon(1 - r^2)\partial_x^2 u = O(\varepsilon^2). \tag{1.2}$$

The system (1.1) is the simplest we know of that presents many of the numerical difficulties that are shared with more complicated systems.

We seek so-called *asymptotically preserving* (AP) methods for (1.1), which are also sometimes referred to as ‘uniformly accurate.’ An AP method can be defined roughly as follows:

Definition 1 *Let E_h be an error norm of the numerical solution that uses a mesh with spacing h . Assume that the timestep is proportional to h . For $h \ll 1$, independent of ε , an AP method satisfies $E_h = O(h^p)$, where $p > 0$. In particular, it may be that $h \gg \varepsilon$.*

Note that standard error analyses typically apply only when $h \ll \varepsilon$. But if $\varepsilon \ll 1$ so that (1.2) holds, requiring $h \ll \varepsilon$ is unnecessary and impractical, even with adaptive mesh refinement. An AP method only requires that the exact solution be resolved ($h \ll 1$).

Following [3], a useful technique to help determine whether a method is AP is to study the small- ε asymptotics of the modified (or ‘equivalent’) equation for (1.1). For $\varepsilon \ll h \ll 1$ (the ‘thick’ regime [4]), the methods we have studied have a modified equation of the form

$$\partial_t u + r\partial_x u - \varepsilon(1 - r^2)\partial_x^2 u = T.E.(u, \varepsilon, r, h), \tag{1.3}$$

where $T.E.(u, \varepsilon, r, h)$ is the truncation error. All of the methods in this study satisfy $T.E. = O(h^2)$, but may be non-AP as a result of $O(h^3)$ terms. We will not review the modified equation or asymptotic analyses here, as both techniques are very well known [4, 3]. To keep this report concise, we will also omit the steps in deriving (1.3) for each method. When appropriate, we will also mention our results for the ‘intermediate regime’ analysis ($h = O(\varepsilon)$; $h \ll 1$; see [4]). For the ‘thin regime’ ($h \ll \varepsilon$; $h \ll 1$), standard error analyses apply, and the methods we consider here are second-order accurate for smooth data.

This study analyzes two semi-discrete methods and one fully-discrete method. For the semi-discrete methods, we do not analyze any particular time integrator, and demonstrate that it is the spatial operator that dominates the asymptotic behavior. We actually prefer fully-discrete methods, but have concentrated on semi-discrete methods because their asymptotic analysis is *much* easier. It also removes doubt regarding any particular choice of time integrator. On the other hand, the spatial operator from a non-AP semi-discrete method, when used with a clever choice of predictor step(s), may result in an AP method [1]. We leave the investigation of such methods for future work.

2 Accuracy of Methods for $r = 0$

Methods can be eliminated from consideration by first studying the $r = 0$ case. This special case simplifies the analysis considerably and elucidates why certain methods fail. For $r = 0$, equation (1.2) reduces to the heat equation,

$$\partial_t u - \varepsilon \partial_x^2 u = O(\varepsilon^2). \quad (2.1)$$

The Fourier transform solution of (2.1) shows that data of wavenumber k is damped as $\exp(-dt)$, where d is the damping rate, given by $d = 4\varepsilon\pi^2 k^2$. Let d_h be the damping rate for a particular numerical method. In order to measure the performance of a method, we define

$$N \equiv \frac{\text{mesh cells}}{\text{wavelength}} \text{ required for } \left| \frac{d_h}{d} - 1 \right| = 0.01. \quad (2.2)$$

Following the definition in the Introduction, an AP method is one in which N is independent of ε . After all, if instead of discretizing (1.1) we discretize (2.1) directly, then any reasonable method will yield an N that is independent of ε .

2.1 A High-Resolution Godunov Method (HR)

In this section, we give an example of a non-AP method which was first analyzed in [3]. Consider a semi-discrete, high-resolution Godunov method that uses a central-difference slope reconstruction [9]. A slope limiter can also be applied, but is not needed for the purposes of this study. We use the ‘frozen’ Riemann problem (RP) for the flux solver, by which we mean that we do not account for effects of the source term in (1.1) when computing the interface flux.

For $\varepsilon \ll h \ll 1$, the modified equation for the HR method reduces to

$$\partial_t u - \varepsilon \partial_x^2 u = - \left(\frac{1}{6} \varepsilon h^2 + \frac{1}{8} h^3 \right) \partial_x^4 u + O(\varepsilon^2, \varepsilon h^4, h^5). \quad (2.3)$$

Because $h \gg \varepsilon$, the $O(h^3)$ term may dominate the damping, so this method is non-AP. The damping rate for (2.3) is given by

$$d_h = 4\varepsilon\pi^2 k^2 + 2\pi^4 k^4 h^2 \left(h + \frac{4}{3}\varepsilon \right). \quad (2.4)$$

A good estimate for N can be obtained by ignoring the εh^2 term (such as in [3]). Using the fact that $h = 1/(kN)$, we obtain

$$N = \left(\frac{50\pi^2}{\varepsilon k} \right)^{1/3}. \quad (2.5)$$

Note that N increases with decreasing ε , so again, the method is non-AP. Choosing $k = 2$ and $\varepsilon = 10^{-5}$ requires $N = 292$ cells/wavelength to resolve the damping rate to within 1%.

Increasing the spatial order of accuracy may lower the exponent in (2.5), but we suspect that the resulting method will be non-AP. Note that we have shown previously that for steady linear transport, the HR method with any slope reconstruction that is independent of the source term is non-AP [8]. Another option is to replace the frozen RP by the generalized RP, which accounts for the source term when computing the flux [2]. However, the generalized RP reduces to the frozen RP as $\Delta t/\varepsilon \rightarrow 0$, and therefore the analysis above holds in this limit. There are other fixes proposed in [3] which should also be considered, but are beyond the scope of this study.

2.2 Liotta, Romano, & Russo Method (LRR)

The LRR method is a central scheme (extended Nessyahu & Tadmor) that is derived in [5]. This method uses a uniformly nonoscillatory (UNO) procedure to compute certain derivatives; the analysis here holds for the UNO method and also any other second-order approximation. For $\varepsilon \ll h \ll 1$, the modified equation for the LRR method reduces to

$$\partial_t u - \varepsilon \partial_x^2 u = \left(\frac{5}{24} \varepsilon h^2 - \frac{3}{128} \frac{h^3}{\nu} \right) \partial_x^4 u + O(\varepsilon^2, \varepsilon h^4, h^5), \quad (2.6)$$

where $\nu = \Delta t/h$ and $\nu < 1/2$ for stability. Just as with the HR method, the $O(h^3)$ term results in a non-AP method. Note that Reference [5, end of §5.1] drops $O(h^3)$ terms in their analysis. From the modified equation, a good estimate for N is

$$N = \left(\frac{75\pi^2}{8\varepsilon\nu k} \right)^{1/3}. \quad (2.7)$$

For $k = 2$, $\varepsilon = 10^{-5}$, and $\nu = 1/2$ we obtain $N = 210$ cells/wavelength, which is an improvement over the HR method. However, because of N 's dependence on ν , the LRR method is less accurate than the HR method when $\nu \lesssim 3/16$ and $\varepsilon \ll h \ll 1$.

2.3 Discontinuous Galerkin (DG)

In this section we analyze a semi-discrete DG method. Within each cell- j , the solution is approximated using a linear basis:

$$\mathbf{u}(x) = (1 - \xi)\mathbf{u}_1 + \xi\mathbf{u}_2, \quad \xi = (x - x_{j-1/2})/h, \quad (2.8)$$

where $\mathbf{u} = (u, v)^T$ and $(\mathbf{u}_1, \mathbf{u}_2)$ are computed in each cell. For a linear system, DG in cell- j can then be written as

$$\partial_t \mathbf{u}_1 + \frac{1}{h} [-4\mathbf{f}_{j-1/2} - 2\mathbf{f}_{j+1/2} + 3\mathbf{f}(\mathbf{u}_1) + 3\mathbf{f}(\mathbf{u}_2)] = \mathbf{s}(\mathbf{u}_1), \quad (2.9)$$

$$\partial_t \mathbf{u}_2 + \frac{1}{h} [4\mathbf{f}_{j+1/2} + 2\mathbf{f}_{j-1/2} - 3\mathbf{f}(\mathbf{u}_1) - 3\mathbf{f}(\mathbf{u}_2)] = \mathbf{s}(\mathbf{u}_2). \quad (2.10)$$

where $\mathbf{s}(\mathbf{u})$ is the source term, $\mathbf{f}(\mathbf{u}) = (v, u)^T$, and the interface flux $\mathbf{f}_{j+1/2}$ is computed via the frozen Riemann problem. More information on this particular DG implementation can be found in [6, 7].

For $\varepsilon \ll h \ll 1$, the modified equation for DG reduces to

$$\partial_t u - \varepsilon \partial_x^2 u = -\frac{1}{12} \varepsilon h^2 \partial_x^4 u + O(\varepsilon^2, \varepsilon h^4), \quad (2.11)$$

which yields $N = 10\pi/\sqrt{3} \approx 19$ cells/wavelength, *independent* of ε . A straightforward analysis shows that DG also satisfies (1.2) to $O(\varepsilon^2)$ in the intermediate regime. Therefore, at least through $O(h^4)$ and $r = 0$, semi-discrete DG is AP. A disadvantage of DG is that it requires twice as many unknowns per cell as the other methods in this study.

2.4 Numerical Results for $r = 0$

In this section, we demonstrate that the truncation error estimates above are in good agreement with numerical results. For the semi-discrete methods, we use a predictor–corrector time integrator. The predictor can be written as

$$\frac{\mathbf{u}^{n+1/2} - \mathbf{u}^n}{\Delta t/2} = D(\mathbf{u}^n) + S(\mathbf{u}^{n+1/2}). \quad (2.12)$$

where the operator D corresponds to differential terms and S corresponds to the source term. For the corrector, we used a lumped–linear DG method for the source term, which involves solving the following coupled system:

$$\frac{(\mathbf{u}^{n+1} + \mathbf{u}^*)/2 - \mathbf{u}^n}{\Delta t/2} = D(\mathbf{u}^{n+1/2}) + S(\mathbf{u}^*), \quad (2.13a)$$

$$\frac{\mathbf{u}^{n+1} - (\mathbf{u}^{n+1} + \mathbf{u}^*)/2}{\Delta t/2} = D(\mathbf{u}^{n+1/2}) + S(\mathbf{u}^{n+1}), \quad (2.13b)$$

where \mathbf{u}^* is an intermediate state. This integrator is point-implicit, L-stable, has positive amplification for all $\Delta t/\varepsilon$, and is second-order accurate when $\Delta t/\varepsilon$ is small.

Table (1) shows results from the three methods analyzed above. For each method, we Fourier transformed its modified equation in order to analytically estimate the error with respect to the exact solution of the heat equation. This estimate is denoted as $L_2^{T.E.}(u)$. Also tabulated is $L_2(u)$, which is the measured error in u from the numerical simulation with respect to the exact solution of (1.1). The values of $L_2^{T.E.}(u)$ and $L_2(u)$ are in good agreement for all of the methods, which is a good indicator that our modified equation analysis and code implementation are correct. It also shows that the time integrator did not significantly affect the modified equation for the semi-discrete methods. The order of accuracy is computed from $L_2(u)$. DG shows second-order accuracy, while the other methods don't show second-order convergence until the exact solution is over-resolved.

3 Accuracy for $r \neq 0$

In this section, we show that DG retains the AP property for $r \neq 0$. The modified equation for DG and $\varepsilon \ll h \ll 1$ becomes

$$\partial_t u + r \partial_x u - \varepsilon(1 - r^2) \partial_x^2 u = -\frac{1}{36} \varepsilon(1 - r^2)(3h^2 \partial_x^4 + 2rh^3 \partial_x^5)u - \boxed{\frac{1}{72} r^2 h^3 \partial_x^4 u} + O(\varepsilon^2, \varepsilon h^4, rh^4). \quad (3.1)$$

Table 1: Results for $r = 0$, $\varepsilon = 10^{-5}$, $u(x, 0) = \cos(2\pi x)$, $v(x, 0) = 0$, periodic domain, final time is 1000. $L_2(u)$ is the measured error, from which the order of accuracy is computed on successive meshes. $L_2^{T.E.}(u)$ is an analytical estimate from the truncation error. Note that the fully-damped solution $u = 0$ corresponds to $L_2(u) = 0.4765$.

Method	Cells/Wavelength	$L_2^{T.E.}(u)$	$L_2(u)$	Order
HR($\nu = 0.8$)	10	4.765e-01	4.765e-01	—
	20	4.765e-01	4.765e-01	0.00
	40	4.538e-01	4.535e-01	0.07
	80	1.509e-01	1.506e-01	1.59
LRR($\nu = 0.4$)	10	4.765e-01	4.765e-01	—
	20	4.765e-01	4.765e-01	0.00
	40	3.619e-01	3.615e-01	0.40
	80	7.763e-02	7.760e-02	2.22
DG($\nu = 0.3$)	10	6.342e-03	6.821e-03	—
	20	1.557e-03	1.587e-03	2.10
	40	3.874e-04	3.887e-04	2.03
	80	9.673e-05	9.653e-05	2.01

Note that (3.1) reduces to (2.11) for $r = 0$. Following our analysis of non-AP methods for $r = 0$, a skeptic might conclude that because $\max(r) = 1$, the boxed term may dominate the diffusion, and therefore, DG is non-AP for $r \neq 0$. But the boxed term dominates only if $Pe \gg 1$, where Pe is the Peclet number, defined here as

$$Pe = \frac{r}{\varepsilon(1 - r^2)}. \quad (3.2)$$

If one insists on resolving the diffusion in the advection-dominated case, then certainly a fine mesh is required.

A more practical argument is to compare (3.1) with a second-order discretization of (1.2). A semi-discrete discretization of (1.2), using a central-difference slope reconstruction, the upwind flux solver, and a three-point central discretization for the diffusion term has a modified equation given by

$$\partial_t u + r \partial_x u - \varepsilon(1 - r^2) \partial_x^2 u = \frac{1}{12} r h^2 \partial_x^3 u - \frac{1}{12} \varepsilon(1 - r^2) h^2 \partial_x^4 u - \boxed{\frac{1}{8} r h^3 \partial_x^4 u} + O(h^4). \quad (3.3)$$

The boxed term here suffers from the same problem as that in equation (3.1). For small ε , one cannot expect that a method for (1.1) be better than can be obtained by directly discretizing (1.2). We claim that DG also satisfies the intermediate regime analysis, so that semi-discrete DG is AP.

The discussion above implies that convergence tests should be run at a fixed Pe . In fact, we have found that all of the methods in this study appear second-order for very large Pe , presumably because in this case the errors in advection dominate the convergence rate. Figure (1) compares L_2 -errors from the DG and LRR methods for three values of the Peclet number. Each plot shows results that are roughly in the thick ($\varepsilon = 10^{-5}$, 10^{-4} , 10^{-3}), intermediate ($\varepsilon = 0.02$), and thin ($\varepsilon = 10^5$) regimes. The problem's initial condition was $u(x, 0) = \cos(2\pi x)$, $v(x, 0) = ru(x, 0)$, with periodic boundary conditions. The final time was chosen so that the

equilibrium wave propagates 1 wavelength. The DG method shows second order accuracy, independent of ε . Both methods perform similarly in the intermediate and thin regimes, but the LRR method generally does poorly in the thick regime. However, if Pe is large enough, second-order accuracy is observed even in the thick regime. The results of the HR method (not shown) are very similar to the those of LRR.

4 Summary

We have shown that semi-discrete DG is asymptotically preserving (AP) for a model problem. To discriminate between AP and non-AP methods, we have argued that one must study problems that have a small Peclet number. It is insufficient to fix the equilibrium wavespeed(s) and vary ε . In other work [6, 7], we have obtained good results for DG for nonlinear extensions, such as for the Broadwell model of gas kinetics and problems in radiation hydrodynamics. Following our analysis, an obvious requirement for the AP property is that $T.E.(u, \varepsilon, r, h) = 0$ whenever $\varepsilon = 0 = r$. The failure of non-AP methods is often the result of higher-order terms in h . Moreover, the terms that cause failure may be traced back to the discretization and possibly remedied (*e.g.*, see [3]). This analysis is left for future work.

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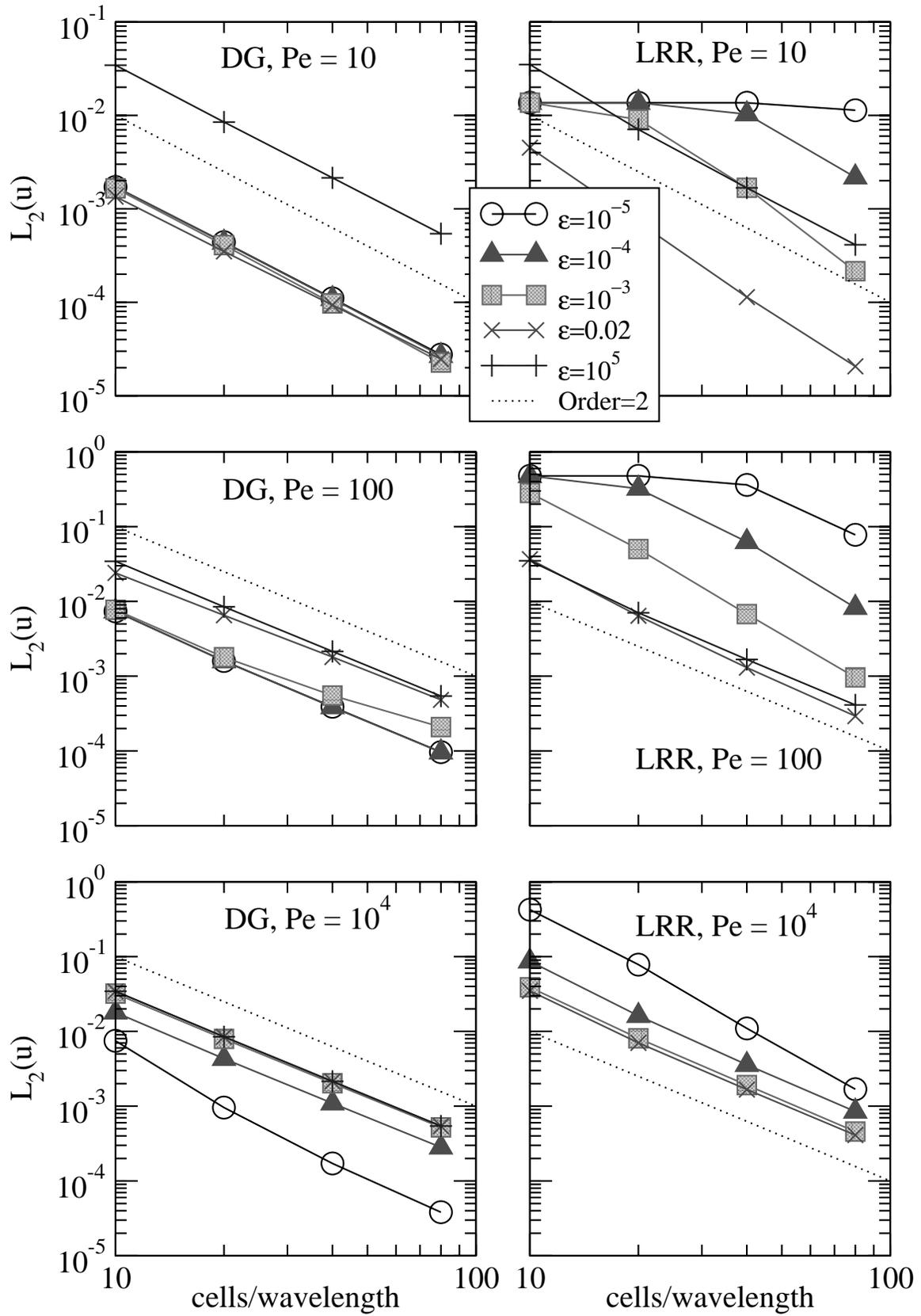


Figure 1: DG($\nu = 0.3$) and LRR($\nu = 0.4$) errors for various Peclet numbers.