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INVARIANT FUNCTIONAL FORMS FOR $K(\rho,P)$ TYPE EQUATIONS OF STATE FOR HYDRODYNAMICALLY DRIVEN FLOW

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A complete Lie group analysis of the general 1D hydrodynamic flow problem for a general $K(\rho,P)$ type equation of state is presented. Since many equations of state (e.g., Vinet, Birch-Murnaghan, Shaker, and Tait) are of this form, one can use the techniques presented here to study their behavior. Making use of the full transformation group, the hyperbolic conservation laws for mass, momentum, and energy reduce to non-linear, ordinary differential equations. These equations describe the density, particle velocity, and pressure behind the shock as functions of the shock front Mach number. Note that unlike the classical result that defines the flow only at the strong shock limit, in this analysis the Mach number is allowed to be one or greater. This allows the behavior of a shocked material to be described down to the acoustic limit. This technique is illustrated using the Tait EOS for a shock moving through NaCl. Finally, the ratios of the group parameters are shown to have definite physical meanings defined in terms of the equation of state and the physical conditions occurring during the event that has produced the shock wave.

INTRODUCTION

Group analysis of the general 1D hydrodynamic flow problem has only been superficially interpreted and applied. Traditional work has centered on the classical self-similar solution of the first kind. The typical application being the generation of analytic test problems of a physically degenerate, asymptotic flow useful in benchmarking general computer codes.

However, this represents a small and limited class of solutions. A much richer interpretation exists if one considers the full group transformation.

The goal of this paper is to provide a general technique that allows the researcher to accurately describe the hydrodynamic flow behind the shock front for Mach numbers at the shock front greater than or equal to one. This is done via a similarity transformation using the full Lie group and is ap-

plicable for 1D hydrodynamic flow with a $K(\rho,P)$ closure/EOS relation.

GROUP STRUCTURE

The 1-D flow equations for this problem are as follows;

$$\rho_t + \rho u_r + u \rho_r + j u \rho r^{-1} = 0 \quad (1)$$

$$u_t + u u_r + u \rho_r + \rho^{-1} P_r = 0 \quad (2)$$

$$P_t + u P_r + K(\rho, P) \left\{ u_r + j u r^{-1} \right\} = 0 \quad (3)$$

Where t and r are the temporal and spatial variables, ρ is the density, u is the particle velocity, P is the pressure, $K(\rho,P)$ is the EOS/closure relation, and $j=0, 1, \text{ or } 2$ for rectangular, cylindrical, and spherical geometry respectively.

Application of the Lie algorithm leads to the following five-parameter group generator;⁽³⁾

$$\hat{U} = c_1 r \partial_r + (c_2 t + c_3) \partial_t + c_5 \rho \partial_\rho + (c_1 - c_2) \mu \partial_\mu + ((c_5 + 2(c_1 - c_2))P + c_4) \partial_P \quad (4)$$

Dividing by c_2 , these five group parameters can be reduced to the four values, $\lambda = c_1/c_2$ the expansion rate of the shock wave, $\sigma = c_3/c_2$ the time delay till the onset of shock formation, $\psi = c_4/c_2$ the material resistance ratio, and $\mu = c_5/c_2$ the material uniformity in front of the shock. The problem is further restricted by the invariance condition that dictates the structure of the bulk modulus K .

$$\mu \rho \frac{\partial K}{\partial \rho} + [(\mu + 2(\lambda - 1))P + \psi] \frac{\partial K}{\partial P} = (\mu + 2(\lambda - 1))K \quad (5)$$

From the group generator a set of characteristic equations can be written that can be used to generate various coordinate transformations of the flow equations

$$\frac{dr}{\lambda r} = \frac{dt}{t + \sigma} = \frac{d\rho}{\mu \rho} = \frac{du}{(\lambda - 1)\mu} = \frac{dP}{\{\mu + 2(\lambda - 1)\}P + \psi} \quad (6)$$

Given this set of equations, it is then possible to construct the following general class of coordinate transformations. Starting with the similarity variable.

$$\xi = r(t + \sigma)^{-\lambda} \quad (7)$$

and the new dependent variables for the density

$$\rho(\xi, t) = (t + \sigma)^\mu f(\xi) \quad (8)$$

and the velocity

$$u(\xi, t) = (t + \sigma)^{\lambda-1} g(\xi) \quad (9)$$

and finally the pressure

$$P(\xi, t) = (\mu + 2(\lambda - 1))^{-1} \times \left[(t + \sigma)^{(\mu + 2\lambda - 2)} h(\xi) - \psi \right] \quad (10)$$

Note that these variables are expressed in terms of the independent similarity variable ξ , time, the group parameter ratios as described above, and the reduced functions $f(\xi)$, $g(\xi)$, and $h(\xi)$.

REDUCTION OF THE FLOW EQUATIONS

The conservation equations (1-3) can be reduced by the substitution of the similarity variable and the transformed dependent variables (eqns. 7 - 10). The continuity equation is then

$$f[\xi](\mu \xi + j \lambda g[\xi]) + \lambda \xi (g[\xi] - \xi) f'[\xi] + \lambda \xi f[\xi] g'[\xi] = 0 \quad (11)$$

The conservation of linear momentum equation becomes

$$(\lambda - 1) f[\xi] g[\xi] + \lambda f[\xi] (g[\xi] - \xi) g'[\xi] + \lambda h[\xi] = 0 \quad (12)$$

and the conservation of energy is written

$$((2(\lambda - 1) + \mu) \xi + \lambda j \kappa g[\xi]) h[\xi] + \lambda \kappa \xi h[\xi] g'[\xi] + \lambda \xi (g[\xi] - \xi) h'[\xi] = 0 \quad (13)$$

Where $\kappa = K/P$.

Reduction of this coupled set via Cramer's rule yields $f[\xi] = \Delta[\xi]/\Theta[\xi]$, $g[\xi] = \Phi[\xi]/\Theta[\xi]$, and $h[\xi] = \Omega[\xi]/\Theta[\xi]$, where

$$\Theta[\xi] = \lambda \xi (f[\xi] (\xi - g[\xi])^2 - \kappa h[\xi]) \quad (14)$$

$$\Delta[\xi] = f[\xi] (f[\xi] (\xi - g[\xi]) (\mu \xi^2 + (\lambda - \mu) \xi g[\xi] - j \lambda g[\xi]^2) + (\lambda - \mu) \xi g[\xi] - j \lambda g[\xi]^2) + (\lambda - \mu) \xi h[\xi] / (\xi - g[\xi]) \quad (15)$$

$$\Phi[\xi] = (\lambda - 1) \xi f[\xi] (\xi - g[\xi]) g[\xi] + (\lambda - 1) \xi f[\xi] (\xi - g[\xi]) h[\xi] \quad (16)$$

and

$$\Omega[\xi] = f[\xi] (\lambda \xi^2 + (\kappa \lambda - \lambda) \xi g[\xi] - j \kappa \lambda g[\xi]^2) h[\xi] \quad (17)$$

Note that $\chi = -1 + \lambda + j \lambda$ and $\vartheta = -2 + 2\lambda + \mu$.

INVARIANCE CONDITIONS FOR THE BULK MODULUS

The invariance condition on the bulk modulus (5) allows the following four mathematical forms;

- The full form with material resistance and non-uniformity ($\psi \neq 0$ and $\mu \neq 0$)

$$\frac{K(\rho, P)}{\rho^{(-2+2\lambda+\mu)/\mu}} = C \left[\frac{P + \frac{\psi}{(-2+2\lambda+\mu)}}{\rho^{(-2+2\lambda+\mu)/\mu}} \right] \quad (18)$$

- no material resistance $\psi = 0$, but non-uniform material $\mu \neq 0$

$$\frac{K(\rho, P)}{\rho^{(-2+2\lambda+\mu)/\mu}} = C [\rho^{-(2+2\lambda+\mu)/\mu} P] \quad (19)$$

• uniform material $\mu=0$, but resistance $\psi \neq 0$
 $K(\rho, P) = (2P(\lambda - 1) + \psi)C[\rho]$ (20)

• uniform material $\mu=0$, and no resistance $\psi=0$
 $K(\rho, P) = PC[\rho]$ (21)

These forms can be used to represent the bulk modulus in various ways in the reduced equations (11-13). Note that the material resistance ψ , does not appear in the solution of the reduced functions f, g, and h, but rather is seen in the expression for the pressure (10)

The expression for the pressure without this material resistance turn out to be different from the one where the material resistance is incorporated into the analysis. Estimates of the effect of including ψ will be considered in a future paper.

ILLUSTRATIVE EXAMPLE

As an example, consider the Tait Equation of State⁽⁴⁾

$$e = \frac{P}{-1 + \exp\left[\left(1 - \frac{\rho_o}{\rho}\right)[1 + K'_o]\right]} \times \left(\frac{1}{\rho} + \frac{\exp\left[\left(1 - \frac{\rho_o}{\rho}\right)[1 + K'_o]\right]}{\rho_o[1 + K'_o]}\right) \quad (22)$$

$$P = \frac{K_o \left(-1 + \exp\left[\left(1 - \frac{\rho_o}{\rho}\right)[1 + K'_o]\right]\right)}{1 + K'_o} \quad (23)$$

$$K(\rho, P) = \frac{\rho_o}{\rho} \exp\left[\left(1 - \frac{\rho_o}{\rho}\right)[1 + K'_o]\right] \quad (24)$$

where ρ_o is the initial density.

If we start with the general form (18) we can write the Tait bulk modulus as follows

$$K(\rho, P) = P + \frac{\psi}{(-2 + 2\lambda + \mu)} \quad (25)$$

This yields the material resistance function ψ

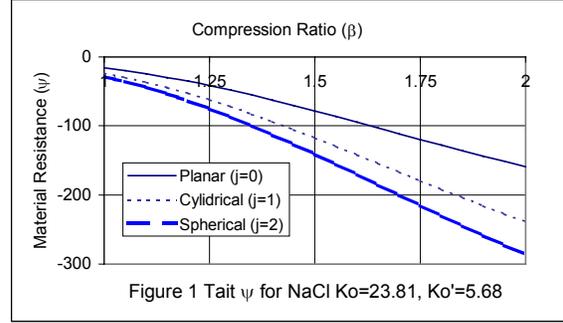
$$\psi = \frac{(-2 + 2\lambda + \mu)K_o}{\beta(1 + K'_o)} \times (\beta + \exp\left[\left(1 - \frac{1}{\beta}\right)[1 + K'_o](1 - \beta + K'_o)\right]) \quad (26)$$

Where β is the compression ratio and K_o and K'_o are the bulk modulus and its first derivative.

The variation of ψ for NaCl⁽⁵⁾ is shown in Figure 1. Note that $\mu=2 - (3+j)\lambda$ in order to maintain invariance of the energy integral.

The Rankine-Hugoniot relations

$$\rho_1 u_1 = \rho_0 u_0 \quad (27)$$



$$P_1 = u_0^2 \rho_0 \left(1 - \frac{\rho_0}{\rho_1}\right) \quad (28)$$

$$e_1 - e_0 = \frac{P_1}{2} \left(\frac{1}{\rho_0} - \frac{1}{\rho_1}\right) \quad (29)$$

are used to generate the boundary conditions at the shock front ($\xi=1$). From the conservation of energy relation (29), an estimate of the compression ratio for the particular equation of state (i.e., Tait) can be defined in terms of the Mach Number and the first derivative of the bulk modulus K'_o .

$$(1 - \beta^2)(1 + K'_o) + \exp\left[\left(1 - \frac{1}{\beta}\right)[1 + K'_o]\right](\beta - 1) \times (-1 - \beta + K'_o(\beta - 1)) - \quad (30)$$

$$\frac{2\beta^2[2 + K'_o] \exp\left(-\frac{1 + K'_o}{\beta}\right)}{M^2[1 + K'_o]} \times$$

$$\left(\exp\left[\frac{1 + K'_o}{\beta}\right] - \exp[1 + K'_o]\right) = 0$$

where we have made use of the fact that $K=c^2b$ and the speed of the shock wave $R=cM$, where c is the speed of sound through the medium and M is the Mach Number. For NaCl ($K_o=23.81$ GPa and $K'_o=5.68$) the compression ratio asymptotically approaches 1.6823 (see Table 1). Note that this

table shows how the speed of the shock can only produce a well defined (via eq. 30) compression or it will not conserve energy across the shock front.

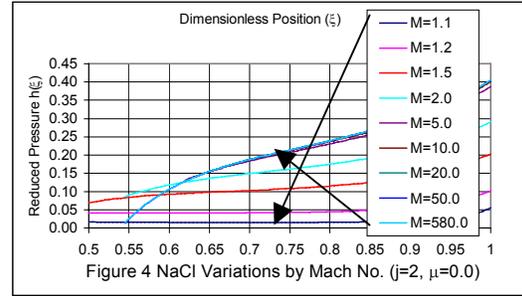
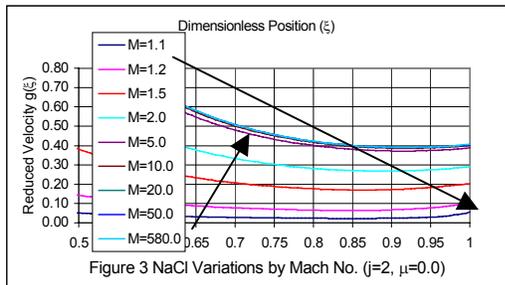
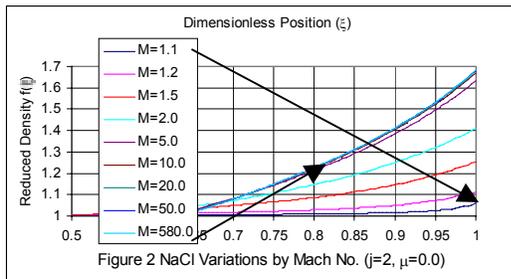
Table 1. Energy Conservation Across The Shock Front for NaCl

Mach No. (M)*	Compression Ratio (β)
1.1	1.0584
1.2	1.1135
1.5	1.2547
2.0	1.4101
5.0	1.6329
10.0	1.6697
20.0	1.6791
50.0	1.6818
580.0 ($\rightarrow\infty$)	1.6823

*Note that the high Mach flow ($M > 5$) are only presented to show the onset of asymptotic behavior. Using the Mach-Compression relationship (30), and numerically integrating (11-13) for the Tait κ equation

$$\kappa = \frac{K}{P} = \frac{(1+K'_o) \exp[1+K'_o]}{(\exp[\frac{1+K'_o}{\beta}] - \exp[1+K'_o]) \beta} \quad (31)$$

(14-17) yield solutions to (11-13). Figures 2 - 4 illustrate how the density, particle velocity, and pressure profiles vary from the shock front ($\xi=1$) back towards the center of initiation ($\xi=0$) for a spherically expanding shock wave in NaCl.



Note that in figures 2-4, the lowest curves correspond to $M=1.1$ and that the curves for $M \geq 5$ are merging at the asymptotic, strong shock limit.

DISCUSSION

The profiles vary with Mach number and should approach an acoustic wave solution as $M \rightarrow 1$. As $M \rightarrow \infty$, the profiles asymptotically converge as a consequence of the conservation of energy requirement at the shock front (30) and generate the strong shock, self-similar solution.

CONCLUSION

Group theoretical methods have been shown to be a powerful method in the evaluation of flow behavior behind a shock front. Using the full group transformation and the Tait EOS, the passage of a shock through a block of NaCl has been simulated as a function of Mach number at the shock front.

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