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Title: R-MATRIX FOR A GEODESIC FLOW ASSOCIATED WITH A NEW INTEGRABLE PEAKON EQUATION

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Submitted to: Proceedings of Nonlinear Evolution Equations Dynamics Systems (NEEDS) 2001
Cambridge, UK
July 24-31, 2001

Los Alamos
NATIONAL LABORATORY

R-matrix for a geodesic flow associated with a new integrable peakon equation

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November 27, 2001

Abstract

We use the r-matrix formulation to show the integrability of geodesic flow on an N-dimensional space with coordinates q_k, with $k = 1, ..., N$, equipped with the co-metric $g^{ij} = e^{-|q_i-q_j|}(2 - e^{-|q_i-q_j|})$. This flow is generated by a symmetry of the integrable partial differential equation (pde) $m_t + um_x + 3mu_x = 0,\ m = u - \alpha^2 u_{xx}$ (α is a constant), which was recently proven to be completely integrable and possess peakon solutions by Degasperis, Holm and Hone. The isospectral eigenvalue problem associated with this integrable pde is used to find a new Lax representation for its N-peakon solution dynamics. By employing this Lax matrix we obtain the r-matrix for the integrable geodesic flow.

Keywords Peakon equation, Lax representation, Hamiltonian, Lax matrix, r-matrix structure.

AMS Subject: 35Q53; 58F07; 35Q35
PACS: 03.40.Gc; 03.40Kf; 47.10.+g

1 Introduction

The $b = 3$ peakon equation and its isospectral problem

We begin with the case $b = 3$ of the b−weighted peakon equation. This is the evolutionary equation defined on the real line as,

$$m_t + um_x + b\,mu_x = 0\,, \quad m = u - \alpha^2 u_{xx}\,, \quad \lim_{|x|\to\infty} m = 0\,, \tag{1.1}$$

in which the subscripts denote partial derivatives with respect to the independent variables x and t. For any values of the dimensionless constant b and constant lengthscale α, this equation admits exact N-peakon solutions

$$u(x,t) = \sum_{i=1}^{N} p_i(t)\, e^{-|x-q_i(t)|/\alpha}\,, \tag{1.2}$$

in which the $2N$ time-dependent functions $p_i(t)$ and $q_i(t)$, $i = 1, 2, \ldots, N$, satisfy a system of ordinary differential equations whose character depends on the value of the bifurcation parameter b. The case $b = 2$ is the dispersionless limit of the integrable Camassa-Holm (CH) equation that was discovered for shallow water waves in [2]. As shown in [3], the CH equation with dispersion is one full order more accurate in asymptotic approximation beyond Korteweg-de Vries (KdV) for shallow water waves, yet it still preserves KdV's soliton properties such as complete integrability via the inverse scattering transform (IST) method.

An equation equivalent to the case $b = 3$ of the peakon equation (1.1) was singled out for special attention among a family of related equations by Degasperis and Procesi in [1]. The peakon equation (1.1) was shown to be completely integrable for the case $b = 3$ by Degasperis, Holm and Hone [4], who found the following Lax pair consisting of a third order eigenvalue problem and a second-order evolutionary equation for the eigenfunction,

$$\psi_{xxx} = \frac{1}{\alpha^2}\psi_x - \lambda m\psi, \tag{1.3}$$

$$\psi_t = -\frac{1}{\alpha^2\lambda}\psi_{xx} - u\psi_x + (u_x + \frac{2}{3\lambda})\psi. \tag{1.4}$$

Compatibility $\psi_{xxxt} = \psi_{txxx}$ implies Eq. (1.1) with $b = 3$ provided $d\lambda/dt = 0$. Thus, Eq. (1.1) with $b = 3$ is integrable by the inverse spectral transform for the isospectral eigenvalue problem (1.3).

Equation (1.1) with $b = 3$,

$$m_t + um_x + 3\,mu_x = 0\,, \quad m = u - \alpha^2 u_{xx}\,, \tag{1.5}$$

was shown to be integrable by the inverse spectral transform and to possess an infinite sequence of conservation laws in Degasperis et al. [4]. The first few of these are, in the notation of [4],

$$\begin{aligned} H_{-1} &= \tfrac{1}{6}\int u^3\,dx\,, & H_0 &= \int m\,dx\,, \\ H_1 &= \tfrac{1}{2}\int (v_{xx}^2 + 5v_x^2 + 4v^2)\,dx\,, & H_2 &= \int m^{1/3}\,dx\,. \end{aligned} \tag{1.6}$$

We shall pay special attention to the quadratic conservation law H_1, in which the quantity v is defined as

$$v := (4 - \partial_x^2)^{-1}u \equiv (4 - \partial_x^2)^{-1}(1 - \partial_x^2)^{-1}m. \tag{1.7}$$

5

Lax matrix for N-peakon dynamics

Substituting the N-peakon solution,

$$u(x,t)=\sum_{j=1}^{N}p_j(t)e^{-|x-q_j(t)|/\alpha},\quad m(x,t)=2\sum_{j=1}^{N}p_j(t)\delta(x-q_j(t)), \tag{1.8}$$

into the isospectral eigenvalue problem (1.3) yields [4]

$$\frac{1}{\alpha^2\lambda}\psi(x,t)=\frac{1}{2}\sum_{j=1}^{N}\left[1+sgn(x-q_j(t))\left(1-e^{-|x-q_j(t)|/\alpha}\right)\right]p_j\psi(q_j(t)). \tag{1.9}$$

Setting $\psi(q_i(t),t)=\psi_i(t)$ then gives the following matrix eigenvalue problem,

$$\frac{2}{\alpha^2\lambda}\psi_i=\sum_{j=1}^{N}\tilde{L}_{ij}\psi_j, \tag{1.10}$$

where

$$\tilde{L}_{ij}=\left[1+sgn(q_i-q_j)\left(1-e^{-|q_i-q_j|/\alpha}\right)\right]p_j\,. \tag{1.11}$$

Let $\tilde{L}$ denote the $N\times N$ matrix $\tilde{L}_{ij}$. In Ref. [4], the authors used the two conserved quantities $\mathrm{tr}\tilde{L}$ and $\mathrm{tr}\tilde{L}^2$ to solve the 2-peakon subdynamics of the the N-peakon dynamics q_k, p_k, with $k=1,...,N,\ \alpha=1$, satisfying

$$\begin{aligned}\dot{p}_j &= 2\textstyle\sum_{k=1}^{N}p_jp_k\,sgn(q_j-q_k)e^{-|q_j-q_k|},\\ \dot{q}_j &= \textstyle\sum_{k=1}^{N}p_ke^{-|q_j-q_k|}\,.\end{aligned} \tag{1.12}$$

Amongst other results, the authors in [4] discovered the two-peakon collision rules for $N=2$ and gave explicit formulas for its phase shifts as functions of the asymptotic speeds of the two peakons.

2 A geodesic flow associated $b = 3$ peakons

The quantity used for determining the two-peakon $N = 2$ collision laws in [4],

$$H_1 = \frac{1}{2}\mathrm{tr}\tilde{L}^2 = \frac{1}{2}\sum_{i,j=1}^{N} p_i p_j e^{-|q_i - q_j|}\left(2 - e^{-|q_i - q_j|}\right), \tag{2.1}$$

is also the quadratic conservation law H_1 in (1.6) for the $b = 3$ peakon equation (1.5), when H_1 is evaluated on the $N-$peakon solution (1.8) with $\alpha = 1$.

The canonical Hamiltonian dynamics generated by H_1 is geodesic motion on an N-dimensional space with co-metric $g^{ij} = e^{-|q_i-q_j|}(2 - e^{-|q_i-q_j|})$. As we shall show by finding its $r-$matrix structure in the remainder of the present paper, the geodesic motion canonically generated by the conservation law $H_1 = Tr\tilde{L}^2$ in (2.1) provides a new $2N$-dimensional integrable system,

$$\dot{q}_k = \frac{\partial H_1}{\partial p_k} = \sum_{j=1}^{N} p_j e^{-|q_k - q_j|}\left(2 - e^{-|q_k - q_j|}\right), \tag{2.2}$$

$$\dot{p}_k = -\frac{\partial H_1}{\partial q_k} = -2p_k \sum_{j=1}^{N} p_j sgn(q_j - q_k) e^{-|q_k - q_j|}\left(1 - e^{-|q_k - q_j|}\right). \tag{2.3}$$

These geodesic H_1-dynamics for p_k, q_k, are not the same as the $N-$peakon dynamics in (1.12). Rather, we are studying the restriction to the peakon sector of the H_1-flow in the hierarchy of integrable equations associated with the isospectral problem for equation (1.5).

R-matrix results for the geodesic H_1-dynamics

To find the r-matrix structure for these H_1-dynamics for p_k, q_k, we now introduce an alternative Lax matrix for the peakon dynamics of Eqn. (1.5),

$$L = \sum_{i,j=1}^{N} L_{ij} E_{ij}, \tag{2.4}$$

where

$$
\begin{aligned}
L_{ij} &= \sqrt{p_i p_j} A_{ij}, && (2.5)\\
A_{ij} &= A(q_i - q_j) = \sqrt{(2 - e^{-|q_i - q_j|}) e^{-|q_i - q_j|}}. && (2.6)
\end{aligned}
$$

The Lax matrix (2.4) also satisfies,

$$
H_1 = \frac{1}{2}\mathrm{tr}L^2 = \frac{1}{2}\sum_{i,j=1}^{N} p_i p_j e^{-|q_i - q_j|}\left(2 - e^{-|q_i - q_j|}\right), \qquad (2.7)
$$

which is the Hamiltonian for the canonical dynamics in Eqs. (2.2) and (2.3). In Eq. (2.6), we have

$$
A(x) = \sqrt{(2 - e^{-|x|}) e^{-|x|}}, \qquad (2.8)
$$

and the function $A(x)$ satisfies the following relations,

$$
\begin{aligned}
A'(x) &= -sgn(x)\frac{1 - e^{-|x|}}{2 - e^{-|x|}} A(x), && (2.9)\\
A_{ij} &= A_{ji}, \quad A_{ii} = 1, && (2.10)\\
A'_{ij} = A'(q_i - q_j) &= -A'(q_j - q_i) = -A'_{ji}, \quad A'_{ii} = 0, && (2.11)\\
\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) A(x)A(y) &= A'(x)A(y) + A(x)A'(y) \\
&= -A(x)A(y)\left[sgn(x)\frac{1 - e^{-|x|}}{2 - e^{-|x|}} + sgn(y)\frac{1 - e^{-|y|}}{2 - e^{-|y|}}\right], && (2.12)\\
\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) A(x)A(-x) &= 0. && (2.13)
\end{aligned}
$$

We shall work in the canonical matrix basis E_{ij},

$$
(E_{ij})_{kl} = \delta_{ik}\delta_{jl}, \quad i, j, k, l = 1, ..., N.
$$

To find the r-matrix structure for the H_1-dynamics in Eqs. (2.2) and (2.3), we consider the so-called fundamental Poisson bracket [6]:

$$
\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2], \qquad (2.14)
$$

where

$$
\begin{aligned}
L_1 &= L \otimes \mathbf{1} = \sum_{i,j=1}^{N} L_{ij} E_{ij} \otimes \mathbf{1}, \\
L_2 &= \mathbf{1} \otimes L = \sum_{k,l=1}^{N} L_{kl} \mathbf{1} \otimes E_{kl}, \\
r_{12} &= \sum_{i,j,s,t}^{N} r_{ij;st} E_{ij} \otimes E_{st}, \\
r_{21} &= \sum_{i,j,s,t}^{N} r_{ij;st} E_{st} \otimes E_{ij}, \\
\{L_1, L_2\} &= \sum_{i,j,k,l=1}^{N} \{L_{ij}, L_{kl}\} E_{ij} \otimes E_{kl}.
\end{aligned}
$$

Here $\{L_{ij}, L_{kl}\}$ is the standard Poisson bracket of two functions, $\mathbf{1}$ is the $N \times N$ unit matrix, and the quantities $r_{ij;st}$ are to be determined. In Eq. (2.14), $[\cdot,\cdot]$ denotes the usual commutator of matrices.

After a lengthy calculation for both sides of Eq. (2.14), we obtain the following key equalities (whose detailed verification is given in the Appendix):

$$
\begin{aligned}
[r_{ll}, L]_{ll} &= 0, \\
[r_{jj}, L]_{ll} &= [r_{ll}, L]_{jj}, \ j \neq l, \\
[r_{jl}, L]_{lj} &= [r_{lj}, L]_{lj} = 0, \\
[r_{ll}, L]_{lj} &= [r_{ll}, L]_{jl} = -\sqrt{p_j p_l} A'_{jl}, \\
[r_{lj}, L]_{ll} &= [r_{jl}, L]_{ll} = -\sqrt{p_j p_l} A'_{jl}, \\
[r_{ll}, L]_{kj} &= [r_{jl}, L]_{kk} = 0, \ j \neq l, k; k \neq l, \\
[r_{jl}, L]_{lk} &= [r_{lj}, L]_{kl} = \frac{1}{2}\sqrt{p_k p_j}(A_{jl} A_{lk})', \ j \neq l, k; k \neq l, \\
[r_{jl}, L]_{kl} &= [r_{lj}, L]_{lk} = \frac{1}{2}\sqrt{p_k p_j}(A_{jl} A_{lk})', \ j \neq l, k; k \neq l,
\end{aligned}
$$

$$[r_{st}, L]_{jl} = 0, \text{ for different } s, t, j, l.$$

where $r_{jl} = \sum_{k,m} r_{km,jl} E_{km}$, $r_{ll} = \sum_{k,m} r_{km,ll} E_{km}$, are two $N \times N$ matrices whose entries are to be determined, L is the Lax matrix, and $[\cdot, L]_{kl}$ stands for the k-th row and the l-th colum element of $[\cdot, L]$.

In matrix notation, all the above equalities can be rewritten as

$$[r_{jl}, L] = B^{jl}, \quad j \neq l, \tag{2.15}$$

$$[r_{ll}, L] = B^{ll}, \tag{2.16}$$

where B^{jl}, B^{ll} are the following two $N \times N$ matrices:

$$B^{jl} = \begin{pmatrix} 0 & \cdots & \frac{1}{2}\sqrt{p_1 p_l}(A_{lj}A_{j1})' & \cdots & \frac{1}{2}\sqrt{p_1 p_j}(A_{jl}A_{l1})' & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2}\sqrt{p_1 p_l}(A_{lj}A_{j1})' & \cdots & -\sqrt{p_j p_l}A'_{lj} & \cdots & 0 & \cdots & \frac{1}{2}\sqrt{p_N p_l}(A_{lj}A_{jN})' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{2}\sqrt{p_1 p_j}(A_{jl}A_{l1})' & \cdots & 0 & \cdots & -\sqrt{p_j p_l}A'_{jl} & \cdots & \frac{1}{2}\sqrt{p_N p_j}(A_{jl}A_{lN})' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{2}\sqrt{p_N p_l}(A_{lj}A_{jN})' & \cdots & \frac{1}{2}\sqrt{p_N p_j}(A_{jl}A_{lN})' & \cdots & 0 \end{pmatrix},$$

and

$$B^{ll} = \begin{pmatrix} 0 & \cdots & -\sqrt{p_1 p_l}A'_{1l} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -\sqrt{p_1 p_l}A'_{1l} & \cdots & 0 & \cdots & -\sqrt{p_N p_l}A'_{Nl} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\sqrt{p_N p_l}A'_{Nl} & \cdots & 0 \end{pmatrix}.$$

By solving Eqs. (2.15) and (2.16), we have the following r-matrix structure:

$$\begin{aligned} r_{12} = & \sum_{j,l=1}^{N} \left(\frac{A'_{lj}}{A_{lj}} E_{jl} \otimes (E_{jl} + E_{lj}) + \frac{A'_{lj}}{A_{lj}} E_{ll} \otimes E_{jj} \right) \\ & + \frac{1}{2} \sum_{j,k,l=1}^{N} \sqrt{\frac{p_k}{p_j}} \left(\frac{A'_{kj}A_{kl}}{A_{kj}A_{lj}} + \frac{(A_{jk}A_{lj})'}{A_{lj}} \right) E_{ll} \otimes E_{jk}. \end{aligned} \tag{2.17}$$

Perhaps not unexpectedy, this non-contant r-matrix for the geodesic H_1-dynamics differs from the constant r-matrix associated with the CH equation ($b = 2$) discovered by Ragnisco and Bruschi in [5].

Concluding remarks

In this paper, we found the r-matrix formulation for the integrable geodesic motion generated canonically by the quadratic quantity H_1 in (2.1). This quantity arises by restriction to the peakon sector of a quadratic conservation law in the hierarchy of integrable equations associated with the isospectral problem for equation (1.5). The quadratic quantity H_1 is also conserved for the 2-peakon dynamics of the 1+1 integrable partial differential equation (1.5) that was singled out in [1] and was proven to be completely integrable by the isospectral transform in [4]. We also introduced a new Lax matrix L for the N-peakon flows of the integrable equation (1.5) that facilitated the r-matrix calculations and for which $H_1 = \frac{1}{2}\mathrm{tr}L^2$. In later work, we shall discuss additional flows in the hierarchy of integrable equations associated with the isospectral problem for equation (1.5) and study their relationships to classical finite-dimensional integrable systems [7].

Acknowledgments

D. D. Holm is grateful to A. Degasperis and A. N. W. Hone for their help and collaboration in many discussions that parallel the work reported here. Z. Qiao would like to express his sincere thanks to J. M. Hyman and L. G. Margolin for their warm invitations and friendly academic discussions. This work was supported by the U.S. Department of Energy under contracts W-7405-ENG-36 and the Applied Mathematical Sciences Program KC-07-01-01; and also the Special Grant of National Excellent Doctorial Dissertation of China.

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Appendix

The following computations are needed in verifying Eq. (2.14).
First, we calculate the left hand side of Eq. (2.14).

$$\begin{aligned}
\frac{\partial L_{ij}}{\partial q_m} &= \sqrt{p_i p_j} A'_{ij}(\delta_{im} - \delta_{jm}), \\
\frac{\partial L_{kl}}{\partial p_m} &= \frac{A_{kl}}{2\sqrt{p_k p_l}}(p_l \delta_{km} + p_k \delta_{lm}),
\end{aligned}$$

$$\begin{aligned}
\{L_{ij}, L_{kl}\} &= \sum_{m=1}^{N} \left(\frac{\partial L_{ij}}{\partial q_m}\frac{\partial L_{kl}}{\partial p_m} - \frac{\partial L_{kl}}{\partial q_m}\frac{\partial L_{ij}}{\partial p_m} \right) \\
&= \frac{1}{2}\sum_{m=1}^{N} \left[\sqrt{p_i p_j} A'_{ij} \frac{A_{kl}}{\sqrt{p_k p_l}} (\delta_{im} - \delta_{jm})(p_l \delta_{km} + p_k \delta_{lm}) \right. \\
&\quad \left. - \sqrt{p_k p_l} A'_{kl} \frac{A_{ij}}{\sqrt{p_i p_j}} (\delta_{km} - \delta_{lm})(p_j \delta_{im} + p_i \delta_{jm}) \right] \\
&= \frac{1}{2}\left[\sqrt{p_i p_j}(-A'_{ij}A_{kj} + A'_{kj}A_{ij})\delta_{jl} + \sqrt{p_l p_j}(A'_{ij}A_{il} - A'_{il}A_{ij})\delta_{ik} \right. \\
&\quad \left. - \sqrt{p_i p_l}(A_{ij}A_{jl})'\delta_{kj} + \sqrt{p_k p_j}(A_{ij}A_{ki})'\delta_{il} \right] \\
&= \frac{1}{2}\left[\sqrt{p_k p_i}(A_{kj}A_{ji})'\delta_{jl} + \sqrt{p_l p_j}(A_{li}A_{ij})'\delta_{ik} \right. \\
&\quad \left. + \sqrt{p_i p_l}(A_{lj}A_{ji})'\delta_{kj} + \sqrt{p_k p_j}(A_{ki}A_{ij})'\delta_{il} \right],
\end{aligned}$$

where the superscript $\prime$ means Eq. (2.12) with the argument.

Thus, we obtain the following formula,

$$\begin{aligned}
\{L_1, L_2\} &= \sum_{i,j,k,l=1}^{N} \{L_{ij}, L_{kl}\} E_{ij} \otimes E_{kl} \\
&= \frac{1}{2}\sum_{j,k,l=1}^{N} \left[\sqrt{p_k p_j}(A_{kl}A_{lj})' E_{jl} \otimes E_{kl} + \sqrt{p_k p_j}(A_{kl}A_{lj})' E_{lj} \otimes E_{lk} \right.
\end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{p_k p_j}(A_{kl}A_{lj})' E_{jl} \otimes E_{lk} + \sqrt{p_k p_j}(A_{kl}A_{lj})' E_{lj} \otimes E_{kl}] \\
=\ & \frac{1}{2} \sum_{j,k,l=1}^{N} \sqrt{p_k p_j}(A_{kl}A_{lj})' \left[E_{jl} \otimes E_{kl} + E_{lj} \otimes E_{lk} + E_{jl} \otimes E_{lk} + E_{lj} \otimes E_{kl}\right] \\
=\ & \frac{1}{2} \sum_{j,k,l=1, j\neq k,l; k\neq l}^{N} \sqrt{p_k p_j}(A_{kl}A_{lj})' \left(E_{jl} + E_{lj}\right) \otimes \left(E_{kl} + E_{lk}\right) \\
& + \sum_{k,l=1}^{N} \sqrt{p_k p_l} A'_{kl} \left(E_{ll} \otimes (E_{kl} + E_{lk}) - (E_{kl} + E_{lk}) \otimes E_{ll}\right). \qquad (2.18)
\end{aligned}
$$

Next, we compute the right hand side of Eq. (2.14),

$$
\begin{aligned}
& [r_{12}, L_1] - [r_{21}, L_2] \\
=\ & \sum_{i,j,s,t,k,l=1}^{N} r_{ij;st} L_{kl} \left[(E_{ij} \otimes E_{st})(E_{kl} \otimes \mathbf{1}) - (E_{kl} \otimes \mathbf{1})(E_{ij} \otimes E_{st})\right. \\
& \left. -(E_{st} \otimes E_{ij})(\mathbf{1} \otimes E_{kl}) + (\mathbf{1} \otimes E_{kl})(E_{st} \otimes E_{ij})\right] \\
=\ & \sum_{i,j,s,k,l=1}^{N} \left[r_{ij;sk} \left(L_{jl}(E_{il} \otimes E_{sk}) - L_{li}(E_{lj} \otimes E_{sk})\right) - r_{ij;ss} L_{kl} \left((E_{ss} \otimes E_{ij}E_{kl}) - (E_{ss} \otimes E_{kl}E_{ij})\right)\right] \\
=\ & \sum_{i,j,s,k,l=1}^{N} \left[r_{ij;sk} \left(L_{jl}(E_{il} \otimes E_{sk}) - L_{li}(E_{lj} \otimes E_{sk})\right)\right] \\
& + \sum_{i,j,s,l=1}^{N} \left[-r_{ji;ss} L_{il}(E_{ss} \otimes E_{jl}) + r_{ij;ss} L_{li}(E_{ss} \otimes E_{lj})\right] \\
=\ & \sum_{i,j,s,t,l, s\neq t, j\neq l} \left[r_{ji;st} L_{il}(E_{jl} \otimes E_{st}) - r_{ij;st} L_{li}(E_{lj} \otimes E_{st})\right] \\
& + \sum_{i,j,s,l, s\neq j,l, j\neq l} \left[r_{ji;ss} L_{il}(E_{jl} \otimes E_{ss} - E_{ss} \otimes E_{jl}) - r_{ij;ss} L_{li}(E_{lj} \otimes E_{ss} - E_{ss} \otimes E_{lj})\right. \\
& \left. + (r_{si;lj} L_{is} - r_{is;lj} L_{si}) E_{ss} \otimes E_{lj})\right] \\
& + \sum_{i,j,l, j\neq l} \left[(r_{li;lj} L_{il} - r_{il;lj} L_{li} + r_{ij;ll} L_{li} - r_{li;ll} L_{ij}) E_{ll} \otimes E_{lj}\right. \\
& + (r_{li;jl} L_{il} - r_{il;jl} L_{li} - r_{ji;ll} L_{il} + r_{il;ll} L_{ji}) E_{ll} \otimes E_{jl}
\end{aligned}
$$

$$
\begin{aligned}
&+ (r_{ji;ll}L_{il} - r_{il;ll}L_{ji})(E_{jl} \otimes E_{ll} - E_{ll} \otimes E_{jl}) - (r_{ij;ll}L_{li} - r_{li;ll}L_{ij})(E_{lj} \otimes E_{ll} - E_{ll} \otimes E_{lj})] \\
&+ \sum_{i,j,l,j\neq l} [(r_{li;jj}L_{il} - r_{il;jj}L_{li} + r_{ij;ll}L_{ji} - r_{ji;ll}L_{ij})E_{ll} \otimes E_{jj} \\
&+ (r_{li;jj}L_{il} - r_{il;jj}L_{li})(E_{ll} \otimes E_{jj} - E_{jj} \otimes E_{ll})] \\
&+ \sum_{i,l} 4(r_{li;ll}L_{il} - r_{il;ll}L_{li})E_{ll} \otimes E_{ll} \qquad (2.19)
\end{aligned}
$$

The first term of Eq. (2.19) is:

$$
\begin{aligned}
& \sum_{i,j,s,t,l,s\neq t,j\neq l} (r_{ji;st}L_{il} - r_{il;st}L_{ji})\, E_{jl} \otimes E_{st} \\
= & \sum_{i,j,s,t,l;s\neq t,j,l;t\neq j,l;j\neq l} (r_{ji;st}L_{il} - r_{il;st}L_{ji})\, E_{jl} \otimes E_{st} \\
&+ \sum_{i,j,k,l,j\neq k,l;k\neq l} [(r_{li;kl}L_{ij} - r_{ij;kl}L_{li})\, E_{lj} \otimes E_{kl} + (r_{ki;lj}L_{il} - r_{il;lj}L_{ki})\, E_{kl} \otimes E_{lj} \\
&+ (r_{ji;kl}L_{il} - r_{il;kl}L_{ji})\, E_{jl} \otimes E_{kl} + (r_{li;lk}L_{ij} - r_{ij;lk}L_{li})\, E_{lj} \otimes E_{lk}] \\
&+2 \sum_{i,j,l,j\neq l} [(r_{li;jl}L_{ij} - r_{ij;jl}L_{li})\, E_{lj} \otimes E_{jl} + (r_{li;lj}L_{ij} - r_{ij;lj}L_{li})\, E_{lj} \otimes E_{lj}]\,.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& [r_{12}, L_1] - [r_{21}, L_2] \\
= & \sum_{i,j,s,t,l;s\neq t,j,l;t\neq j,l;j\neq l} (r_{ji;st}L_{il} - r_{il;st}L_{ji})\, E_{jl} \otimes E_{st} \\
&+ \sum_{i,j,k,l,j\neq k,l;k\neq l} [(r_{li;kl}L_{ij} - r_{ij;kl}L_{li})\, E_{lj} \otimes E_{kl} + (r_{ji;lk}L_{il} - r_{il;lk}L_{ji})\, E_{jl} \otimes E_{lk} \\
&+ (r_{ji;kl}L_{il} - r_{il;kl}L_{ji})\, E_{jl} \otimes E_{kl} + (r_{li;lk}L_{ij} - r_{ij;lk}L_{li})\, E_{lj} \otimes E_{lk}] \\
&+ \sum_{i,j,k,l;k\neq j,l,j\neq l} [(-r_{ji;kk}L_{il} + r_{il;kk}L_{ji} + r_{ki;jl}L_{ik} - r_{ik;jl}L_{ki})E_{kk} \otimes E_{jl} \\
&+ (r_{ji;kk}L_{il} - r_{il;kk}L_{ji})E_{jl} \otimes E_{kk}] \\
&+ \sum_{i,j,l;j\neq l} [(r_{li;lj}L_{il} - r_{il;lj}L_{li} + 2\,(r_{ij;ll}L_{li} - r_{li;ll}L_{ij}))E_{ll} \otimes E_{lj}
\end{aligned}
$$

$$
\begin{aligned}
& + \left(r_{li;jl} L_{il} - r_{il;jl} L_{li} + 2(-r_{ji;ll} L_{il} + r_{il;ll} L_{ji})\right) E_{ll} \otimes E_{jl} \\
& + \left(r_{li;ll} L_{ij} - r_{ij;ll} L_{li}\right) E_{lj} \otimes E_{ll} + 2 \left(r_{li;jl} L_{ij} - r_{ij;jl} L_{li}\right) E_{lj} \otimes E_{jl} \\
& + \left(r_{ji;ll} L_{il} - r_{il;ll} L_{ji}\right) E_{jl} \otimes E_{ll} + 2 \left(r_{li;lj} L_{ij} - r_{ij;lj} L_{li}\right) E_{lj} \otimes E_{lj}] \\
& +2 \sum_{i,j,l,j \neq l} \left(r_{li;jj} L_{il} - r_{il;jj} L_{li} - r_{ji;ll} L_{ij} + r_{ij;ll} L_{ji}\right) E_{ll} \otimes E_{jj} \\
& +4 \sum_{i,l} (r_{li;ll} L_{il} - r_{il;ll} L_{li}) E_{ll} \otimes E_{ll} \\
= \; & \{L_1, L_2\} \quad \text{by Eqs. (2.18) and (2.19). This finishes the proof of Eq. (2.14).}
\end{aligned}
$$