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TITLE: EXTENSIONS TO A NONLINEAR FINITE-ELEMENT AXISYMMETRIC SHELL MODEL BASED ON REISSNER'S SHELL THEORY

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EXTENSIONS TO A NONLINEAR FINITE ELEMENT AXISYMMETRIC SHELL MODEL BASED ON REISSNER'S SHELL THEORY

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ABSTRACT

Extensions to shell analysis not usually associated with shell theory are described in this paper. These extensions involve thick shells, nonlinear materials, a linear normal stress approximation, and a changing shell thickness.

A finite element shell-of-revolution model has been developed at the Los Alamos National Laboratory by the author to analyze nuclear material shipping containers under severe impact conditions. I presented the formulation for this model at the 5th SMiRT, and the formulation is based on a nonlinear shell theory presented by E. Reissner in 1972. This model is capable of large strain, large rotations, nonlinear material analysis, and a changing shell thickness.

To establish the limits for this shell model, I studied the basic assumptions used in its development; these are listed in this paper. Several extensions were evident from the study of these limits: a thick shell, a plastic hinge, and a linear normal stress.

The Reissner shell theory includes shear deformation, which becomes important for thick shells. Also, the approximations for curvature and additional terms resulting from changing shell thickness in the strains can be added for thick shells. These extensions appear as modifications to the constitutive matrix, and thus the constitutive matrix becomes a function of the deforming geometry. The limiting condition is that planes remain approximately planar.

When plastic conditions exist through the thickness (plastic hinge), the cross coupling terms between stress resultants and bending strains and between bending moments and membrane strains are nonzero, but equal. Thus the constitutive matrix remains symmetric but is dependent on loading conditions.

The usual assumption for normal stress is that it is zero. This approximation can be extended to a linear normal stress. Again, this extension enters the theory through the constitutive relation and becomes important only when the pressure is comparable in magnitude to moduli. This approximation modifies the force vector as an integral quantity.

Three example problems have been used to verify this shell model: an axially loaded cylinder with strains to 100% and thickness changes of 25%, an expanding hemisphere with 100% strains and a thickness change of 37%, and a cylinder loaded to deform into a spherical shape with strain to 67% and thickness changes to 30%. The limits that have been established for this model were checked for these three problems and were accurate to within 7%.

1. Introduction

Analytically modeling the response of shells to severe accident conditions requires a nonlinear shell model that accounts for both geometric and material nonlinearities. The Los Alamos National Laboratory has developed an axisymmetric, shell-of-revolution finite element model capable of analyzing nonlinear failures. Cook [1 - 2] describes this model. The nonlinea shell-of-revolution theory presented by E. Reissner in [3] and [4] was used to develop the model.

Orginally, our model was developed to analyze nuclear material shipping containers with the NONSAP code, which uses the Newton method for solving nonlinear problems and is described by Bathe et al. [5]. Later the model was added to the ADINA code [6], and it has been used with the Broyden-Fletcher-Goldfarb-Shanno (BFGS) matrix update solution method. Use of the BFGS solution method has improved the performance of the model.

This model is capable of large strains, large rotations, nonlinear material, and changing shell thicknesses. To use this model for nonlinear failure analysis, we had to determine what assumptions were used in the development of the theory on which this model is based. In determining these assumptions, we found that we could extend this model to include thick shells, plastic hinges, and linear normal stress and thermal stresses.

2. Equilibrium and Strain Equations

The equilibrium equations may be written in terms of undeformed quantities as

$$[illegible]$$

$$[illegible] \qquad (1)$$

$$[illegible]$$

These equations are equivalent to the equilibrium equations presented by Reissner in references [3] and [4]. The shell strains he derived are

$$[illegible]$$

$$[illegible]$$

$$\kappa_{\theta\theta} = \frac{\sin\phi - \sin\phi_0}{r_0} \ , \quad \text{and} \quad \lambda_0 = \frac{\cos\phi - \cos\phi_0}{r_0} \ .$$

u_r and u_z are the horizontal (radial) and vertical (axial) displacements, and ω is the rotation $\phi - \phi_0$.

Assuming that planes must remain planes, we may derive the following strains. The normal strain is

$$\epsilon_{no} = \frac{\frac{h}{\cos\gamma} - h_o}{h_o} = \frac{h}{h_o \cos\gamma} - 1 \quad . \tag{3}$$

The meridional strain is

$$\epsilon_{so} = \frac{\Delta S(\zeta) - \Delta S_o(\zeta)}{\Delta S_o(\zeta)} = \frac{\epsilon_{so} + \zeta \frac{h_o}{2} \kappa_{so} + \zeta \frac{h_o}{2R} \frac{q}{q_o} \epsilon_{no} \cos\gamma}{1 + \zeta \frac{h_o}{2R_o}} \quad . \tag{4}$$

The hoop strain is

$$\epsilon_{\theta\zeta} = \frac{r(\zeta) - r_o(\zeta)}{r_o(\zeta)} = \frac{\epsilon_{\theta o} + \zeta \frac{h_o}{2} \kappa_{\theta o} + \zeta \frac{h_o \sin\phi}{2 r_o} \epsilon_{no}}{1 + \zeta \frac{h_o \sin\phi_o}{2 r_o}} \quad . \tag{[illegible]}$$

From Reissner [3] and [4], we get

$$\epsilon_{s\zeta} = - \frac{\zeta \; [illegible]}{1 + \zeta \frac{h_o}{2 R_o}} \tag{[illegible]}$$

5. <u>Pressure and Shear Loads</u>

Instead of assuming that the pressure and shear loads are reacted at the neutral surface, let them be reacted at the shell boundaries. Then $p_\zeta r_o$ and $S_\zeta r_o$ in the force equilibrium equations in eq. (1) are replaced by

$$p_\zeta r_o = p_{io} r_o \left(1 - \frac{h_o}{2} \frac{\sin\phi_o}{r_o}\right)\left(1 - \frac{h_o}{2 R_o}\right) \; p_{eo} r_o \left(1 + \frac{h_o}{2} \frac{\sin\phi_o}{r_o}\right)\left(1 + \frac{h_o}{2 R_o}\right) \text{ and}$$

$$S_\zeta r_o = S_{io} r_o \left(1 - \frac{h_o}{2} \frac{\sin\phi_o}{r_o}\right)\left(1 - \frac{h_o}{2 R_o}\right) \; S_{eo} r_o \left(1 + \frac{h_o}{2} \frac{\sin\phi_o}{r_o}\right)\left(1 + \frac{h_o}{2 R_o}\right) \; . \tag{[illegible]}$$

Also, a term is added to the rotational equilibrium equation:

$$S_{io} r_o \frac{h_o}{2} \left(1 - \frac{h_o}{2} \frac{\sin\phi_o}{r_o}\right)\left(1 - \frac{h_o}{2 R_o}\right) + S_{eo} r_o \frac{h_o}{2} \left(1 + \frac{h_o}{2} \frac{\sin\phi_o}{r_o}\right)\left(1 + \frac{h_o}{2 R_o}\right) \; . \tag{8}$$

4. Stress Resultants, Bending Moments, and Constitutive Equations

The stress resultants and bending moments can be derived as

$$N_{so} = \frac{h_o}{2} \int_{-1}^{1} \sigma_{so} \left(1 + \xi \frac{h_o \sin\phi_o}{2\, r_o}\right) d\xi \ ,$$

$$N_{\theta o} = \frac{h_o}{2} \int_{-1}^{1} \sigma_{\theta o} \left(1 + \xi \frac{h_o}{2\, R_o}\right) d\xi \ .$$

$$Q_o = \frac{h_o}{2} \int_{-1}^{1} \tau_o \left(1 + \xi \frac{h_o \sin\phi_o}{2\, r_o}\right) d\xi \ , \qquad (9)$$

$$M_{so} = \frac{h_o^2}{4} \int_{-1}^{1} \sigma_{so}\, \xi \left(1 + \xi \frac{h_o \sin\phi_o}{2\, r_o}\right) d\xi \ , \quad \text{and}$$

$$M_{\theta o} = \frac{h_o^2}{4} \int_{-1}^{1} \sigma_{\theta o}\, \xi \left(1 + \xi \frac{h_o}{2\, R_o}\right) d\xi \ .$$

Assume the following constitutive equations:

$$\sigma_{so} = C_s\, \epsilon_{so} + C_{s\theta}\, \epsilon_{\theta o} + C_{sn}\, \epsilon_{no} - \sigma_{st} \ ,$$

$$\sigma_{\theta o} = C_{s\theta}\, \epsilon_{so} + C_{\theta}\, \epsilon_{\theta o} + C_{\theta n}\, \epsilon_{no} - \sigma_{\theta t} \ , \qquad (10)$$

$$\sigma_{no} = C_{sn}\, \epsilon_{so} + C_{\theta n}\, \epsilon_{\theta o} + C_n\, \epsilon_{no} - \sigma_{nt} \ , \quad \text{and}$$

$$\tau_o = G\, \gamma_{sno} \ .$$

Using the strains in Sec. 2 and the stress resultants and bending moments in this section, the shell constitutive equations are

$$N_{so} = C_s\, \epsilon_{so} + C_{s\theta}\, \epsilon_{\theta o} + T_s\, \kappa_{so} + T_{s\theta}\, \kappa_{\theta o} + N_{st} \ ,$$

$$N_{\theta o} = C_{\theta s}\, \epsilon_{so} + C_{\theta}\, \epsilon_{\theta o} + T_{\theta s}\, \kappa_{so} + T_{\theta}\, \kappa_{\theta o} + N_{\theta t} \ ,$$

$$M_{so} = T_s\, \epsilon_{so} + T_{s\theta}\, \epsilon_{\theta o} + D_s\, \kappa_{so} + D_{s\theta}\, \kappa_{\theta o} + M_{st} \ , \qquad (11)$$

$$M_{\theta o} = T_{\theta s}\, \epsilon_{so} + T_{\theta}\, \epsilon_{\theta o} + D_{\theta s}\, \kappa_{so} + D_{\theta}\, \kappa_{\theta o} + M_{\theta t} \ ,$$

$$Q_o = G\, \gamma_o , \text{ and } P_o = D_\lambda\, \kappa_o \ .$$

These equations are not symmetric, and C_s, T_s, D_s, N_{st}, and M_{st}, will be defined next.

We define the following functions:

$$g_s(\xi) = \frac{\dfrac{c_s \ \xi \dfrac{h_0}{2\,R} \dfrac{q}{q_0} \cos\gamma}{1 + \xi \dfrac{h_0}{2\,R_0}} + \dfrac{c_{se} \ \xi \dfrac{h_0 \sin\phi}{2\,r_0}}{1 + \xi \dfrac{h_0 \sin\phi_0}{2\,r_0}} + c_{sn}}{\dfrac{c_{sn} \ \xi \dfrac{h_0}{2\,R} \dfrac{q}{q_0} \cos\gamma}{1 + \xi \dfrac{h_0}{2\,R_0}} + \dfrac{c_{en} \ \xi \dfrac{h_0 \sin\phi_0}{2\,r_0}}{1 + \xi \dfrac{h_0 \sin\phi_0}{2\,r_0}} + c_n} \ , \tag{12}$$

$$g_{se}(\xi) = \frac{1 + \xi \dfrac{h_0 \sin\phi_0}{2\,r_0}}{1 + \xi \dfrac{h_0}{2\,R_0}} \ ,$$

$$\dot{g}_s(\xi) = \left(1 + \xi \frac{h_0 \sin\phi_0}{2\,r_0}\right) \left\{\left(\sigma_{no} + \sigma_{nt}\, g_s(\xi)\right) - \sigma_{st}\right\} \ , \quad \text{and}$$

C_s and T_s material properties may be defined as

$$C_s = \frac{h_0}{2} \int_{-1}^{1} f_s(\xi)\, d\xi \quad \text{and} \quad T_s = \frac{h_0^2}{4} \int_{-1}^{1} \xi\, f_s(\xi)\, d\xi \ , \tag{13}$$

where

$$f_s(\xi) = q_{se}(\xi)\, (c_s - g_s(\xi)\, c_{sn}) \ .$$

Also,

$$I_s = \frac{h_0^3}{8} \int_{-1}^{1} \xi^2 f_s(\xi)\, d\xi \ . \tag{14}$$

The initial stress resultants and bending moments, N_{si} and M_{si}, are

$$N_{si} = \frac{h_0}{2} \int_{-1}^{1} \dot{g}_s(\xi)\, d\xi \quad \text{and} \quad M_{si} = \frac{h_0^2}{4} \int_{-1}^{1} \xi\, \dot{g}_s(\xi)\, d\xi \ . \tag{15}$$

For a linear normal stress in the normal coordinate, we derived

$$\sigma_{no} = -\left(\frac{p_{eo} + p_{io}}{2}\right) - \xi \left(\frac{p_{eo} - p_{io}}{2}\right) \tag{16}$$

This extension can be included in the shell model by adding it to the thermal stress σ_{nt}. When this shell is joined to continuum elements, σ_{no} must be calculated from a reaction load that would have to be calculated after each iteration.

Another natural extension is to let the temperature field, and consequently σ_{st}, $\sigma_{\theta t}$, and σ_{nt}, vary linearly in the normal coordinate.

5. Reissner's Shell Theory Assumptions

In this section, I describe the assumptions used for Reissner's shell model.

1) Planes are assumed to remain planes, and thus shear deformations are included. The usual shell theory approximation is that normals to the neutral surface remain normal.

2) Meridional and hoop stresses are approximated as linear functions and shear as an average value through the thickness. These are the usual shell theory approximations; however, in Reissner's theory these approximations are for both deformed and undeformed configuration. σ_{so}, $\sigma_{\theta o}$, and τ_o are Lagrangian stresses. This assumption is used in Sec. 4.

3) γ has no mathematical limit; however, from a practical point of view, it should be less than $\pi/4$.

4) The normal strain, ε_{no}, is assumed to be constant in the normal coordinate as described in Sec. 2. When the deformed thickness is needed, ε_{no} can be found from the normal constitutive equation of eq. (10) ($\sigma_{no} = -p_o$ should be included) and substituted into eq. (3). The material properties needed for this evaluation of ε_{no} are secant material properties.

5) The thin shell assumptions

$$\frac{h_o \sin\phi_o}{2 r_o} \ll 1 \quad \text{and} \quad \frac{h_o}{2 R_o} \ll 1 \;, \quad \frac{h \sin\phi}{2 r \cos\gamma} \ll 1 \;, \quad \text{and} \quad \frac{h}{2 R \cos\gamma} \ll 1 \qquad (17)$$

are the usual thin shell assumptions and are used in Sec. 3 to justify the pressure being reacted at the neutral surface. The shear moment term (eq. (8)) must be small compared with the terms in the rotational equilibrium equation of eq. (1).

6) The constitutive equations are assumed in eq. (10) of Sec. 4. For the shell constitutive equations (eq. (11)) to be independent of shell geometry and be symmetric,

$$\frac{c_s}{c_{sn}} \frac{h_o}{2R} \frac{q}{q_o} \cos\gamma + \frac{c_{s\theta}}{c_{sn}} \frac{h_o \sin\phi}{2 r_o} \ll 1 \;, \quad \frac{c_{s\theta}}{c_{sn}} \frac{h_o}{2R} \frac{q}{q_o} \cos\gamma + \frac{c_\theta}{c_{sn}} \frac{h_o \sin\phi}{2 r_o} \ll 1 \;, \quad \text{and} \qquad (18)$$

$$\frac{c_{sn}}{c_n} \frac{h_o}{2R} \frac{q}{q_o} \cos\gamma + \frac{c_{\theta n}}{c_n} \frac{h_o \sin\phi}{2 r_o} \ll 1 \;.$$

With these assumptions, which are limits on deformed as well as undeformed quantities, the $C_s \ldots G$ reduce to simple integrals that can be integrated easily. However, if a plastic hinge exists in the shell the material properties ($c_s \ldots c_\theta$) vary through the thickness and require integration for the evaluation of $C_s \ldots G$.

Cook [2] indicates how the shell constitutive equations are used for incremental quantities, and thus the material properties ($c_s \ldots c_\theta$) are tangential quantities.

7) The normal stress assumptions are

$$\left| \frac{\dfrac{c_{sn}}{c_n}\,\sigma_{no}}{\left(c_s - \dfrac{c_{sn}^2}{c_n}\right)\epsilon_{so} + \left(c_{s\theta} - \dfrac{c_{sn}\,c_{\theta n}}{c_n}\right)\epsilon_{\theta o}} \right| \ll 1 \quad \text{and}$$

$$\left| \frac{\dfrac{c_{\theta n}}{c_n}\,\sigma_{no}}{\left(c_{s\theta} - \dfrac{c_{\theta n}\,c_{sn}}{c_n}\right)\epsilon_{so} + \left(c_\theta - \dfrac{c_{\theta n}^2}{c_n}\right)\epsilon_{\theta o}} \right| \ll 1 \quad . \tag{19}$$

I described a method for including normal stress in Sec. 4.

8) A constant change in temperature in the normal coordinate is a basic assumption. Thus σ_{st}, $\sigma_{\theta t}$, and σ_{nt} are constants for the integration in Sec. 4. Table I shows how well the assumptions in this section are satisfied for the three example problems that were used to verify this shell model. The three nonlinear example problems in reference [illegible] were solved very accurately by this model.

6. Extensions to the Shell Model

Thick shells are included in this model by 1) integrating each iteration for $C_s \ldots M_{\theta t}$. This integration would include the geometric terms, the linear normal stress terms, and the linear thermal stress terms described in Sec. 4, which would make the stiffness matrix nonsymmetric, and 2) reacting the pressure and shear terms at the shell boundaries. (This is discussed in Sec. 3 and would affect the force vector calculation.)

The assumptions for this extended shell model would then be the same as for items 1—4 in Sec. 5; however, the remaining items 5 and 6 would be less restrictive as follows.

5) The normal stress is assumed to be linear in the normal coordinate as described in Sec. 4, eq. (16).

6) The temperature field is assumed to be linear in the normal coordinate as described in Sec. 4; thus σ_{st}, $\sigma_{\theta t}$, and σ_{nt} are assumed to be linear in the normal coordinate.

7. Conclusions

In this paper I have presented the assumptions used to develop this shell model. These assumptions are more restrictive than those used for linear shell theory. In Table I, I show that the assumptions are valid for the three example problems used to verify this computer program. These three example problems have large strains and displacements, large rotations, and shell thickness changes of 25%, 30%, and 37%.

The changes necessary to extend this model to thick shells are discussed. The greatest concern is that the shell constitutive equations are no longer symmetric. Extending the model to thick shells is accomplished by

1. changing the solution technique to solve nonsymmetric equations,
2. modifying the force vector by integrating temperatures and adding pressure and shear at the boundaries, and
3. calculating the shell material matrix for geometric changes in each iteration.

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TABLE I

NONLINEAR EXAMPLE PROBLEMS

	C_s, C_θ, C_n (psi)	C_{sn}, $C_{s\theta}$, $C_{\theta n}$ (psi)	$\frac{h_o}{\bar{R}}$	ε_{sn}	$\varepsilon_{\theta n}$	$\frac{h_o \sin\phi_o}{2 r_o}$	$\frac{h_o}{2 R_o}$	$\frac{h \sin\phi}{2 r \cos\gamma}$	$\frac{h}{2 R \cos\gamma}$	A	B	C	D	E
Axially Loaded Cylinder	$\frac{6}{5}10^6$	$\frac{2}{5}10^6$	1.333	1.0	-0.25	0.005	0.00000	0.005	0.000000	0.005	0.015	0.001667	0.0000[illegible]	0.000[illegible]
Expanding Hemisphere	$\frac{6}{5}10^6$	$\frac{2}{5}10^6$	1.579	1.0	1.0	0.01667	0.01667	0.005271	0.005[illegible]	0.06667	0.0[illegible]	0.01111	0.01667	0.0166
Cylinder Deforms into a Sphere Hinged end	$\frac{6}{5}10^5$	$\frac{2}{5}10^5$	1.060	0.1591	5.0000	0.01667	0.00[illegible]	0.009430	0.0094[illegible]	0.04478	0.04159	0.00[illegible]197	0.009195	[illegible]
Free end			1.433	0.1591	0.6667	0.0166[illegible]	0.00000	0.00[illegible]477	0.00[illegible]	0.05144	0.06159	0.009419	0.0316[illegible]	0.01467

Equation (15)

$$A = \frac{C_s}{C_{sn}} \frac{h_o}{2R} \frac{q}{q_n} \cos\gamma + \frac{C_{s\theta}}{C_{sn}} \frac{h_o}{2} \frac{\sin\phi}{r_o} \qquad B = \frac{C_{s\theta}}{C_{sn}} \frac{h_o}{2R} \frac{q}{q_o} \cos\gamma + \frac{C_\theta}{C_{sn}} \frac{h_o}{2} \frac{\sin\phi}{r_o}$$

$$C = \frac{C_{sn}}{C_n} \frac{h_o}{2R} \frac{q}{q_o} \cos\gamma + \frac{C_{\theta n}}{C_n} \frac{h_o}{2} \frac{\sin\phi}{r_o}$$

Equation (16)

$$D = \frac{\frac{C_{sn}}{C_n}\,\varepsilon_{nn}}{C_s - \frac{C_{sn}^2}{C_n}\,\varepsilon_{sn} + C_{s\theta} - \frac{C_{sn} C_{\theta n}}{C_n}\,\varepsilon_{\theta n}} \qquad E = \frac{\frac{C_{\theta n}}{C_n}\,\varepsilon_{nn}}{C_{s\theta} - \frac{C_{\theta n} C_{sn}}{C_n}\,\varepsilon_{sn} + C_\theta - \frac{C_{\theta n}^2}{C_n}\,\varepsilon_{\theta n}}$$

NOMENCLATURE

- C_{ij} - Continuum material properties
- C, D, G, T - Shell material properties
- E - A continuum strain measure
- h - Shell thickness
- M, P - Bending moments
- N, Q - Stress resultants
- p - Pressure load
- q - Meridional variable
- r - Radial coordinate
- R - Meridional radius of curvature
- S - Shear load
- s - Meridional coordinate
- u - Displacement
- β - Rotation, $\phi - \phi_o$
- γ - Shear angle
- ε, γ - Shell strains
- κ, λ - Shell bending strains
- ζ - Normalized thickness coordinate
- σ, τ - Lagrangian stresses
- ϕ - Azimuthal coordinate

Subscripts

- e - External to shell
- i - Internal or initial
- n - Normal coordinate
- o - Undeformed shell configuration
- t - Thermal
- θ - Circumferential (hoop) coordinate