LA-UR -81-1575

TITLE: THE EFFECT OF GEOMETRIC SHAPE ON TWO-DIMENSIONAL FINITE ELEMENTS

AUTHOR(S): W. A. Cook

MASTER

SUBMITTED TO: The 6th International Seminar on Computational Aspects of the Finite Element Method (CAFEM-6)

University of California

LOS ALAMOS SCIENTIFIC LABORATORY
Post Office Box 1663 Los Alamos, New Mexico 87545
An Affirmative Action/Equal Opportunity Employer

THE EFFECT OF GEOMETRIC SHAPE ON TWO-DIMENSIONAL FINITE ELEMENTS

by

W. A. Cook

ABSTRACT

Three quadrilateral elements are defined. These are an eight-nodal-point serendipity element (QUAD8s), a nine-nodal-point serendipity element (QUAD9s), and a nine-nodal-point quadrilateral element composed of two six-nodal-point triangular elements (QUAD9t). The effect that the geometric shape of the element has on the approximation function of each element is discussed. Two beam problems demonstrate that when the shape of the elements becomes skewed, the QUAD9t element significantly improves the calculated results. Finally, a recommendation is made for the QUAD8s and QUAD9t to be used together for the most efficient and accurate results.

I. INTRODUCTION AND SUMMARY

The serendipity elements are used extensively in stress analysis finite element computer programs. One such program is ADINA, which is described in Ref. [1] and is used extensively at the Los Alamos National Laboratory. The theory for the serendipity elements was first presented in Ref. [2] and then presented more completely in Ref. [3]. The ADINA computer program was used for this study.

In 1977 J. A. Strickland and his associates at Texas A&M showed that the eight-nodal-point serendipity quadrilateral two-dimensional isoparametric element (QUAD8s) becomes very stiff when its geometric shape is distorted [4].

He explained that others had also seen this effect and suggested using another element, the six-nodal-point linear strain triangle (subparametric element). This element has straight sides and will not accurately model problems with curved boundaries. L. N. Gifford [5] showed in 1979 that this same loss of accuracy occurs for twelve-nodal-point serendipity elements, and J. Backlund (in 1978) showed how reduced integration improves the accuracy of skewed elements [6].

I will show that a marked improvement is made in the serendipity quadrilateral two-dimensional isoparametric element (QUAD9s) by adding a center nodal point. However, even with a very careful placement of the center nodal point, this element is affected by the geometric shape of the element. C. M. Stone of Sandia National Laboratories, Albuquerque, suggested this element.*

I will also present a nine-nodal-point quadrilateral two-dimensional element that consists of two six-nodal-point triangular isoparametric elements (QUAD9t). When the sides of these triangular elements are straight, they are equivalent to the elements suggested by Strickland, et al. [4].

Three plane-stress beam problems are used to evaluate the QUAD8s, QUAD9s, and QUAD9t elements. The first problem is a longitudinal beam with constant strain. All three elements solve this problem exactly using three elements for both rectangular and skewed-shaped elements. This problem proved that the QUAD9t element must be integrated as two triangles. Solving this problem with triangular integration gave exact displacements, but solving with quadrilateral integration gave meaningless displacements. The two other beam problems were solved with both quadrilateral and triangular integration of QUAD9t, and the

---

* C. M. Stone, Sandia National Laboratories, Albuquerque, personal communication, August 1979.

displacements were nearly identical. The second problem, a cantilevered beam with a shear load, was used by Strickland, et al. [4]. All three elements solve this problem very accurately using three elements when the quadrilaterals are rectangular; however, when the elements are skewed, the errors are 56% for the maximum displacement for QUAD8s, 17% for QUAD9s, and 4% for QUAD9t. The third problem is a circular beam problem. Again, all three elements solve this problem accurately using six elements when the quadrilaterals are uniform circular segments. When these elements are skewed, the errors in the maximum radial displacement are 23% for QUAD8s, 15% for QUAD9s, and 3% for QUAD9t.

I conducted this study for two reasons. First, in nonlinear analyses that use a deforming configuration technique (updated Lagrangian in ADINA), the finite elements may become skewed and complicate the solution technique. Second, three-dimensional problems with complicated geometries are limited in the number of elements that may be used for solution in a reasonable amount of calculation time and cost. Therefore the most efficient finite elements possible are needed for these analyses.

II. DEFINITIONS OF TRIANGULAR AND SERENDIPITY ELEMENTS

I will define two types of elements by presenting the shape functions used in each element and describing the numerical integration. These two element types are the triangular isoparametric elements and the serendipity isoparametric elements. These two types of elements are more general than the actual elements that are compared later in the paper. The difference is the midside nodal points, which are optional in the presentation but are used for the triangular element QUAD9t and the two serendipity elements QUAD8s and QUAD9s discussed later.

A. Triangular Elements

The triangular element shape functions are $h_j^i(r,s)$, where the superscript i refers to the element this shape function represents and the subscript j refers to the nodal point in the element that this shape function represents. These shape functions are derived for triangular-shaped regions with three to six nodal points. The corner nodal points are required, whereas the midside nodal points are optional. These triangular elements are used in pairs to model quadrilateral regions. Quadrilaterals can be formed from triangles in two ways: by placing the diagonal between nodal points 2 and 4 (shown in Fig. 1), and by placing the diagonal between nodal points 1 and 3 (shown in Fig. 2).

For the configuration in which the diagonal is between nodal points 2 and 4 (Fig. 1) the element I shape functions are

$$h_1^1 = \frac{1}{2}(r + s) - \frac{1}{2}h_6^1 - \frac{1}{2}h_8^1 \ ,$$

$$h_2^1 = \frac{1}{2}(1 - r) - \frac{1}{2}h_6^1 - \frac{1}{2}h_9^1 \ ,$$

$$h_4^1 = \frac{1}{2}(1 - s) - \frac{1}{2}h_8^1 - \frac{1}{2}h_9^1 \ ,$$

$$h_6^1 = (1 - r)(r + s) \text{ or } 0 \text{ (if this node does not exist)},$$

$$h_8^1 = (r + s)(1 - s) \text{ or } 0 \ , \text{ and}$$

$$h_9^1 = (1 - r)(1 - s) \text{ or } 0 \ . \tag{1}$$

Similarly, element II (also shown in Fig. 1) has the shape functions

$$h_2^2 = \frac{1}{2}(1 + s) - \frac{1}{2}h_6^2 - \frac{1}{2}h_9^2 \ ,$$

$$h_3^2 = -\frac{1}{2}(r + s) - \frac{1}{2} h_6^2 - \frac{1}{2} h_7^2 \quad ,$$

$$h_4^2 = \frac{1}{2}(1 + r) - \frac{1}{2} h_7^2 - \frac{1}{2} h_9^2 \quad ,$$

$$h_6^2 = -(r + s)(1 + s) \text{ or } 0 \quad ,$$

$$h_7^2 = -(r + s)(1 + r) \text{ or } 0 \quad , \text{ and}$$

$$h_9^2 = (1 + r)(1 + s) \text{ or } 0 \quad . \qquad (2)$$

For a configuration where the diagonal is between nodal points 1 and 3 (as shown in Fig. 2), the element III shape functions are

$$h_1^3 = \frac{1}{2}(1 + r) - \frac{1}{2} h_5^3 - \frac{1}{2} h_9^3 \quad ,$$

$$h_2^3 = \frac{1}{2}(-r + s) - \frac{1}{2} h_5^3 - \frac{1}{2} h_6^3 \quad ,$$

$$h_3^3 = \frac{1}{2}(1 - s) - \frac{1}{2} h_6^3 - \frac{1}{2} h_9^3 \quad ,$$

$$h_5^3 = (-r + s)(1 + r) \text{ or } 0 \quad ,$$

$$h_6^3 = (-r + s)(1 - s) \text{ or } 0 \quad , \text{ and}$$

$$h_9^3 = (1 + r)(1 - s) \text{ or } 0 \quad . \qquad (3)$$

Similarly, element IV (shown in Fig. 2) has shape functions

$$h_1^4 = \frac{1}{2}(1 + s) - \frac{1}{2} h_8^4 - \frac{1}{2} h_9^4 \quad ,$$

$$h_3^4 = \frac{1}{2}(1 - r) - \frac{1}{2} h_7^4 - \frac{1}{2} h_9^4 \quad ,$$

$$h_4^4 = \frac{1}{2}(r - s) - \frac{1}{2} h_7^4 - \frac{1}{2} h_8^4 \quad ,$$

$$h_7^4 = (r - s)(1 - r) \text{ or } 0 \quad ,$$

$$h_8^4 = (r - s)(1 + s) \text{ or } 0 \quad , \text{ and}$$

$$h_9^4 = (1 - r)(1 + s) \text{ or } 0 \quad . \tag{4}$$

(r,s) are natural coordinates that vary as

$$-1 \leq r \leq 1 \quad \text{and} \quad -1 \leq s \leq 1 \quad .$$

The number of integration points used for the triangular elements is either one or three. These are listed in Table I. The single integration point is exact integration for a linear approximation function, whereas the three integration points give exact integration for a quadratic approximation function.

The criterion used for dividing the quadrilateral region into triangular elements is the shortest diagonal. Thus if the distance between the nodal point 2 and nodal point 4 is shorter than the distance between nodal points 1 and 3 then the triangular elements are I and II (as shown in Fig. 1). Conversely, if the diagonal distance between nodal points 1 and 3 is the shortest, then the triangular elements are III and IV (as shown in Fig. 2). If the two diagonal distances are the same, the user may specify elements I and II or III and IV or average all four elements. However, averaging all four is a poor choice unless the elements are rectangles.

Thus the QUAD9t element consists of a quadrilateral region with nine nodal points (shown in Figs. 1 and 2), which consists of two six-nodal-point triangular elements. All midside nodal points are centered, and the ninth nodal point is centered on the diagonal.

Consider a region to be analyzed. This region has many elements; some are in the interior of the region to be analyzed, and some have boundaries that are on the boundary of the region to be analyzed. The midside nodal points, which are in the interior of the region being analyzed, are centered on straight lines, whereas those midside nodal points on the boundary of the region being analyzed are centered on the boundary, which may or may not be curved.

B. Serendipity Element

The serendipity element shape functions are $h_j(r,s)$, where the subscript j designates the nodal point that this shape function represents. This is a four-to-nine-nodal-point element. The corner nodes are required, but the midside nodal points and center nodal point are optional.

$$
\begin{aligned}
h_1 &= \frac{1}{4}(1+r)(1+s) - \frac{1}{2}h_5 - \frac{1}{2}h_8 - \frac{1}{4}h_9 \quad , \\
h_2 &= \frac{1}{4}(1-r)(1+s) - \frac{1}{2}h_5 - \frac{1}{2}h_6 - \frac{1}{4}h_9 \quad , \\
h_3 &= \frac{1}{4}(1-r)(1-s) - \frac{1}{2}h_6 - \frac{1}{2}h_7 - \frac{1}{4}h_9 \quad , \\
h_4 &= \frac{1}{4}(1+r)(1-s) - \frac{1}{2}h_7 - \frac{1}{2}h_8 - \frac{1}{4}h_9 \quad , \\
h_5 &= \frac{1}{2}(1-r^2)(1+s) - \frac{1}{2}h_9 \text{ or } 0 \quad , \\
h_6 &= \frac{1}{2}(1-r)(1-s^2) - \frac{1}{2}h_9 \text{ or } 0 \quad , \\
h_7 &= \frac{1}{2}(1-r^2)(1-s) - \frac{1}{2}h_9 \text{ or } 0 \quad , \\
h_8 &= \frac{1}{2}(1+r)(1-s^2) - \frac{1}{2}h_9 \text{ , or } 0 \text{ , and} \\
h_9 &= (1-r^2)(1-s^2) \text{ or } 0 \quad .
\end{aligned} \tag{1}
$$

The numerical integration used for serendipity elements is discussed in Ref. [7].

The QUAD8s element referred to in this paper consists of the first eight nodal points shown in Fig. 3 ($h_9 = 0$ in Eq. (5)). The QUAD9s element includes all the nodal points shown in Fig. 3 and all the terms in Eq. (5). The location of nodal point 9 is at

$$x_9 = -\frac{x_1 + x_2 + x_3 + x_4}{4} + \frac{x_5 + x_6 + x_7 + x_8}{2} \tag{6}$$

$$y_9 = -\frac{y_1 + y_2 + y_3 + y_4}{4} + \frac{y_5 + y_6 + y_7 + y_8}{2} \quad .$$

This location is the location of $r = 0$, $s = 0$ using the first eight nodal points in Eq. (5) ($h_9 = 0$).

The midside nodal points are centered for both serendipity elements in the same manner as those discussed in the previous section. This region has many elements; some are in the interior of the region to be analyzed, and some have boundaries that are on the boundary of the region to be analyzed. Those midside nodal points that are in the interior of the region being analyzed are centered on straight lines, and the midside nodal points on the boundary of the region being analyzed are centered on the boundary, which may or may not be curved.

III. APPROXIMATION FUNCTIONS

An understanding of the approximation functions for triangular and serendipity elements will clarify the results of the two transversely loaded beam problems discussed in Sec. IV.

The shape functions defined in Sec. II are used to establish the transformations $x(r,s)$ and $y(r,s)$ from the natural coordinate system (r,s) to the global coordinate system (x,y) and also for the approximation functions for displacements (I will use $u(r,s)$ as a representative displacement). When the same shape functions are used for both transformations and approximation functions, the elements are isoparametric. The shape functions defined in Sec. II were derived to maintain a compatibility between elements for the geometry (x,y) and displacement u. But what limitations do the resulting approximation functions have?

To approximate a general function, I would recommend using a Taylor's series expansion, which suggests a polynomial series. The polynomial terms should be complete to assure the same degree of approximation for each possible deformation state (include all terms of the same degree). (See Ref. [8] for a discussion of complete polynomials for finite elements.) Thus for two dimensions, the ideal approximations are

$$u(x,y) = c_0 + c_1\, x + c_2\, y \text{ ...(linear) and}$$
$$u(x,y) = c_0 + c_1\, x + c_2\, y + c_3\, x^2 + c_4\, xy + c_5\, y^2 \text{...(quadratic)}$$

The approximation functions for triangular and serendipity elements will be compared with Eq. (7).

A. Triangular Element

Element 1 (shown in Fig. 4) has the following transformation for x(r,s):

$$x = h_1^1\, x_1 + h_2^1\, x_2 + h_4^1\, x_4 + h_5^1\, x_5 + h_8^1\, x_8 + h_9^1\, x_9 \quad . \tag{8}$$

In Fig. 4,

$$x_5 = \frac{x_1 + x_2}{2} + a_5 \quad , \quad x_8 = \frac{x_4 + x_1}{2} + a_8 \quad , \quad \text{and } x_9 = \frac{x_2 + x_4}{2} \quad . \tag{9}$$

Thus

$$x = h_1^1 + \left(\frac{h_5^1 + h_8^1}{2}\right) x_1 + \left(h_2^1 + \frac{h_5^1 + h_9^1}{2}\right) x_2 + \left(h_4^1 + \frac{h_8^1 + h_9^1}{2}\right) x_4$$
$$+ h_5^1\, a_5 + h_8^1\, a_8 \quad . \tag{10}$$

Substituting the shape functions (Eq. (1)) into Eq. (10) gives

$$x = \frac{x_2 + x_4}{2} + r\left(\frac{x_1 - x_2}{2} + a_5 + a_8\right) + s\left(\frac{x_1 - x_4}{2} + a_5 + a_8\right)$$

$$- r^2 a_5 - rs\,(a_5 + a_8) - s^2 a_8 \quad . \tag{11}$$

Similarly, $y(r,s)$ is

$$y = \frac{y_2 + y_4}{2} + r\left(\frac{y_1 - y_2}{2} + b_5 + b_8\right) + s\left(\frac{y_1 - y_4}{2} + b_5 + b_8\right)$$

$$- r^2 b_5 - rs\left(b_5 + b_8\right) - s^2 b_8 \quad . \tag{12}$$

Using the same six-nodal-point triangular shape functions (Eq. (1)) as approximation functions for displacements,

$$u(r,s) = h_1^1 u_1 + h_2^1 u_2 + h_4^1 u_4 + h_5^1 u_5 + h_8^1 u_8 + h_9^1 u_9 \quad ,$$

$$= u_9 + r\left(- \frac{u_1 + u_2}{2} + u_5 + u_8 - u_9\right) + s\left(- \frac{u_1 + u_4}{2} + u_5 + u_8 - u_9\right)$$

$$+ r^2\left(\frac{u_1 + u_2}{2} - u_5\right) + rs\left(u_1 - u_5 - u_8 + u_9\right) + s^2\left(\frac{u_1 + u_4}{2} - u_8\right) . \tag{13}$$

When the boundaries of the triangular element in Fig. 4 are straight and the midside nodal points are centered, the a's and b's are zero and $x(r,s)$, $y(r,s)$ in Eqs. (11) and (12) are linear functions of (r,s). Thus when Eqs. (11) and (12) are substituted into Eq. (7), the displacement function of Eq. (7) (quadratic) has the same polynomial terms as Eq. (13). Conversely, when the a's and b's are not zero, the boundaries are curved, or the midside nodal points are not centered, and $x(r,s)$ and $y(r,s)$ in Eqs. (11) and (12) are quadratic functions of (r,s). Thus when Eqs. (11) and (12) are substituted into Eq. (7), the displacement function (linear) has the same polynomial terms as Eq. (13).

Summarizing, when the geometric boundaries are straight and the midside nodal points are centered, the displacement approximation functions are quadratic in (x,y), but when the geometric boundaries are curved or the midside nodal points are not centered, the displacement approximation functions are linear in (x,y).

Three-nodal-point triangular elements have linear shape functions (see Eq.(1)). Thus both the transformations x(r,s) and y(r,s) and the approximation functions u(r,s) are linear in (r,s). From Eq. (7) we can see that the approximation function is also linear.

B. Serendipity Element

Using the element in Fig. 5, the transformation for x(r,s) is

$$x = \sum_{i=1}^{9} h_i \, x_i \quad . \tag{14}$$

In Fig. 5

$$x_5 = \frac{x_1 + x_2}{2} + a_5 \; , \quad x_6 = \frac{x_2 + x_3}{2} + a_6 \; , \quad x_7 = \frac{x_3 + x_4}{2} + a_7 \; ,$$

$$x_8 = \frac{x_4 + x_1}{2} + a_8 \; , \text{ and } x_9 = \frac{x_1 + x_2 + x_3 + x_4}{4} + \frac{a_5 + a_6 + a_7 + a_8}{2} \quad . \tag{15}$$

Thus

$$x = x_1\left(h_1 + \frac{h_5 + h_8}{2} + \frac{h_9}{4}\right) + x_2\left(h_2 + \frac{h_5 + h_6}{2} + \frac{h_9}{4}\right) + x_3\left(h_3 + \frac{h_6 + h_7}{2} + \frac{h_9}{4}\right)$$

$$+ x_4\left(h_4 + \frac{h_7 + h_8}{2} + \frac{h_9}{4}\right) + a_5h_5 + a_6h_6 + a_7h_7 + a_8h_8$$

$$+ \frac{1}{2}\left(a_5 + a_6 + a_7 + a_8\right) h_9 \quad . \tag{16}$$

Substituting the shape function (Eq. (5)) into Eq. (16) gives

$$x = \frac{x_1 + x_2 + x_3 + x_4}{4} + \frac{a_5 + a_6 + a_7 + a_8}{2} + r\left(\frac{x_1 - x_2 - x_3 + x_4}{4} + \frac{a_8 - a_6}{2}\right)$$

$$+ s\left(\frac{x_1 + x_2 - x_3 - x_4}{4} + \frac{a_5 - a_7}{2}\right) - r^2\left(\frac{a_5 + a_7}{2}\right) + rs\left(\frac{x_1 - x_2 + x_3 - x_4}{4}\right)$$

$$- s^2\left(\frac{a_6 + a_8}{2}\right) + r^2 s\left(\frac{a_7 - a_5}{2}\right) + s^2 r\left(\frac{a_6 - a_8}{2}\right) . \qquad (17)$$

Similarly, $y(r,s)$ is

$$y = \frac{y_1 + y_2 + y_3 + y_4}{4} + \frac{b_5 + b_6 + b_7 + b_8}{2} + r\left(\frac{y_1 - y_2 - y_3 + y_4}{4} + \frac{b_8 - b_6}{2}\right)$$

$$+ s\left(\frac{y_1 + y_2 - y_3 - y_4}{4} + \frac{b_5 - b_7}{2}\right) - r^2\left(\frac{b_5 + b_7}{2}\right) + rs\left(\frac{y_1 - y_2 + y_3 - y_4}{4}\right)$$

$$- s^2\left(\frac{b_6 + b_8}{2}\right) + r^2 s\left(\frac{b_7 - b_5}{2}\right) + s^2 r\left(\frac{b_6 - b_8}{2}\right) . \qquad (18)$$

Using the nine-nodal-point serendipity shape functions (Eq. (5)) as approximation functions for displacements,

$$u = \sum_{i=1}^{9} h_i u_i \quad \text{and}$$

$$= u_9 + r\left(\frac{u_8 - u_6}{2}\right) + s\left(\frac{u_5 - u_7}{2}\right) + r^2\left(\frac{u_6 + u_8}{2} - u_9\right)$$

$$+ rs\left(\frac{u_1 - u_2 + u_3 - u_4}{4}\right) + s^2\left(\frac{u_5 + u_7}{2} - u_9\right)$$

$$+ r^2 s\left(\frac{u_1 + u_2 - u_3 - u_4}{4} + \frac{u_7 - u_5}{2}\right) + rs^2\left(\frac{u_1 - u_2 - u_3 + u_4}{4} + \frac{u_6 - u_8}{2}\right)$$

$$+ r^2 s^2\left(\frac{u_1 + u_2 + u_3 + u_4}{4} - \frac{u_5 + u_6 + u_7 + u_8}{2} + u_9\right) . \qquad (19)$$

When the boundaries of the serendipity element in Fig. 5 are straight and the midside nodal points are centered, the a's and b's are zero and $x(r,s)$, $y(r,s)$ in Eqs. (17) and (18) are

$$x(r,s) = p_0 + p_1\, r + p_2\, s + p_3\, rs \quad \text{and}$$

$$y(r,s) = q_0 + q_1\, r + q_2\, s + q_3\, rs \quad . \tag{20}$$

When Eq. (20) is substituted into Eq. (7), the displacement function of Eq. (7) (quadratic) has the same polynomial terms as Eq. (19). Conversely, when the a's and b's are not zero, the boundaries are curved or the midside nodal points are not centered, and $x(r,s)$, $y(r,s)$ in Eqs. (17) and (18) have all the quadratic terms and two third-degree polynomial terms. Thus when Eqs. (17) and (18) are substituted into Eq. (7), the displacement function (linear) has the same polynomial terms as Eq. (19) except that the fourth-degree term $(r^2 s^2)$ is not needed.

When the shape of the serendipity element is a parallelogram and the midside nodal points are centered, the a's and b's are zero and $x(r,s)$, $y(r,s)$ in Eqs. (17) and (18) are linear. This can be shown by realizing that the diagonals of a parallelogram intersect at their center points. Thus

$$\frac{x_1 + x_3}{2} = \frac{x_2 + x_4}{2} \quad \text{and} \quad \frac{y_1 + y_3}{2} = \frac{y_2 + y_4}{2} \quad , \tag{21}$$

or

$$x_1 - x_2 + x_3 - x_4 = 0 \quad \text{and} \quad y_1 - y_2 + y_3 - y_4 = 0 \quad , \tag{22}$$

which are the coefficients of the (rs) terms in Eqs. (17) and (18).

Therefore $x(r,s)$ and $y(r,s)$ are linear.

$$x(r,s) = p_0 + p_1 r + p_2 s \quad \text{and}$$

$$y(r,s) = q_0 + q_1 r + q_2 s \; . \tag{23}$$

When Eq. (23) is substituted into Eq. (7), the displacement approximation function of Eq. (7) (quadratic) has fewer terms than Eq. (19). Thus, for a parallelogram-shaped element, the displacement approximation function has three terms more than are needed for a quadratic approximation in (x,y). The (r^2s), (rs^2) and (r^2s^2) terms are not needed.

Summarizing, when the geometric shape of the serendipity element is a parallelogram and the midside nodal points are centered, the displacement approximation functions have three terms more than are needed for a quadratic approximation in (x,y). When the geometric boundaries are straight and the midside nodal points are centered, the displacement approximation functions are quadratic in (x,y). However, when the geometric boundaries are curved or the midside nodal points not centered, the displacement approximation functions have one term more than is needed for a linear approximation in (x,y).

When using the element in Fig. 5, if the ninth nodal point is ignored, the transformations for $x(r,s)$ and $y(r,s)$ are identical to Eqs. (17) and (18).

If the eight-nodal-point serendipity shape functions in Eq. (5) are considered as approximation functions for displacements, the ninth nodal point is again ignored, and

$$u = \sum_{i=1}^{8} h_i \, u_i \quad \text{and}$$

$$= -\frac{u_1 + u_2 + u_3 + u_4}{4} + \frac{u_5 + u_6 + u_7 + u_8}{2} + r\left(\frac{u_6 - u_6}{2}\right) + s\left(\frac{u_5 - u_7}{2}\right)$$

$$+ r^2\left(\frac{u_1 + u_2 + u_3 + u_4}{4} - \frac{u_5 + u_7}{2}\right) + rs\left(\frac{u_1 - u_2 + u_3 - u_4}{4}\right)$$

$$+ s^2\left(\frac{u_1 + u_2 + u_3 + u_4}{4} - \frac{u_6 + u_8}{2}\right) + r^2 s\left(\frac{u_1 + u_2 - u_3 - u_4}{4} + \frac{u_7 - u_5}{2}\right)$$

$$+ rs^2\left(\frac{u_1 - u_2 - u_3 + u_4}{4} + \frac{u_6 - u_8}{2}\right) \quad . \tag{24}$$

Again, the transformations $x(r,s)$ and $y(r,s)$ are the same for the eight- and nine-nodal-point serendipity elements; however, the displacement approximation functions are different. Thus when the boundaries of the serendipity element are straight and the midside nodal points are centered, the a's and b's are zero, and the transformations for $x(r,s)$ and $y(r,s)$ are Eq. (20). However, when Eq. (20) is substituted into Eq. (7), the displacement function of Eq. (7) (quadratic) has a fourth-degree term that Eq. (24) does not have. Conversely, when a's and b's are not zero, the boundaries are not straight or the midside nodal points are not centered, and $x(r,s)$ and $y(r,s)$ in Eqs. (17) and (18) have all the quadratic terms and two third-degree polynomials identical to the QUAD9S element. However, when Eqs. (17) and (18) are substituted into Eq. (7), the displacement function (linear) has the same polynomial terms as Eq. (24).

As shown for the QUAD9S element, the parallelogram-shaped element results in a linear transformation for $x(r,s)$, $y(r,s)$. Thus Eq. (24) has two terms more than are needed for a quadratic displacement approximation in (x,y) from Eq. (7). The $(r^2 s)$ and (rs^2) terms are not needed.

Summarizing, when the geometric shape of a serendipity element is a parallelogram and the midside nodal points are centered, the displacement approximation functions have two terms more than are needed for a quadratic approximation in (x,y). When the geometic boundaries are straight and the midside nodal points are centered, the displacement approximation functions lack the fourth-degree term (r^2s^2) to be quadratic in (x,y), but when the geometric boundaries are curved or the midside nodal points are not centered, the displacement approximation functions are linear in (x,y).

For a four-nodal-point element, the transformation equations $x(r,s)$ and $y(r,s)$ are identical to Eq. (20) for a skewed-shaped element and identical to Eq. (23) for a parallelogram-shaped element. Using the four-nodal-point serendipity shape functions in Eq. (5) for approximation functions for displacements $u(r,s)$,

$$
\begin{aligned}
u &= \sum_{i=1}^{4} h_i u_i \\
&= \frac{u_1 + u_2 + u_3 + u_4}{4} + r\left(\frac{u_1 - u_2 - u_3 + u_4}{4}\right) + s\left(\frac{u_1 + u_2 - u_3 - u_4}{4}\right) \\
&\quad + rs\left(\frac{u_1 - u_2 + u_3 - u_4}{4}\right) \quad .
\end{aligned}
\qquad ([illegible])
$$

Thus when Eq. (20) is substituted into Eq. (7) (linear), the displacement function of Eq. (7) (linear) has the same polynomial terms as Eq. (25). However, when Eq. (23) is substituted into Eq. (7), the displacement function of Eq. (7) (linear) does not have the (rs) term.

Summarizing, when the geometric shape of a four-nodal-point serendipity element is a parallelogram, the displacement approximation function is linear in (x,y) plus an (rs) term. When the shape in skewed, the approximation function is linear in (x,y). The element approximation functions are listed in Table II.

IV. APPLICATIONS

In this section I will present three beam problems: a longitudinal beam, a cantilevered beam, and a circular beam. Each problem has been analyzed with the three elements defined in Sec. II.

The longitudinal beam problem has linear displacements and was solved exactly by all three elements independent of whether their shapes were rectangular or skewed. However, my original intent was to use the same integration for QUAD9t as I did for the serendipity elements. This was satisfactory for the other two beam problems, but the calculation using the QUAD9t elements gave meaningless answers for this problem until a triangular integration scheme was developed for this element. This integration scheme is documented in Sec. II.

The cantilevered beam problem is identical to the problem used by Professor Strickland, et al., in Ref. [4] except that the material properties, loads, and sizes have been given metric values. This problem is illustrated in Fig. 6, and the theoretical and calculated results are tabulated in Table III.

From the derivations in Sec. III, we can interpret these results. For the case where the quadrilateral elements are rectangular, the approximation functions are quadratic plus additional terms for the serendipity elements with QUAD9s having one more term than QUAD8s. The QUAD9t has identical quadratic approximations. Thus QUAD9s gives the best results, QUAD8s the second best results, and QUAD9t gives the third best results; however, these results have less than 3.5% error. Independent of what problem is solved, if the quadrilaterals are parallelograms, the QUAD9s element will approximate the solution better than or as well as the QUAD8s element; and the QUAD8s element will approximate the solution better than or as well as the QUAD9t element.

For the case where the quadrilateral elements are skewed, the QUAD8s approximation functions are lacking a term to be able to approximate quadratic functions. Also, the region for approximation is large; thus the calculated results are very stiff. The QUAD9s element does have quadratic approximation capability, but its region for approximation is also large compared with the QUAD9t element, which breaks each quadrilateral region into two triangular regions each with quadratic approximation capability. Thus, when the quadrilateral elements have skewed shapes, the QUAD9t elements are good, the QUAD9s elements are poor, and the QUAD8s elements are very poor. Independent of what problem is being solved, if the quadrilaterals are skewed geometry, the QUAD9t element will approximate the solution better than or as well as the QUAD9s element, and the QUAD9s element will approximate the solution better than or as well as the QUAD8s element.

The circular beam problem is illustrated in Fig. 7, and the theoretical and calculated results are tabulated in Table IV. The horizontal displacement is used as a gauge of the solution because it is proportional to the potential energy of the problem. This problem was chosen because it is similar in thickness, length, material properties, and loads to the cantilevered beam problem. I studied this problem to better understand curved boundaries. A check of the theoretical solution of this problem and the previous problem shows that the displacements to be approximated are similar [9]. However, it required six quadrilateral elements to get as much accuracy for this problem as that obtained with the previous problem with only three elements. This confirms the lower degree approximation functions for displacements when the boundaries are curved; this was discussed in Sec. III. Also, for this problem, the QUAD9s elements are not as much better than the QUAD8s elements as they were in the previous problem. This is because the QUAD8s element has a linear approximation function for curved boundaries and the QUAD9s element had the same, with one additional term. Again, the QUAD9t element is much better than the other two elements because the region it approximates with its linear approximation function is much smaller than that of the other two elements.

V. CONCLUSIONS AND RECOMMENDATIONS

From the definitions in Sec. II, the derivations in Sec. III, and the problems in Sec. IV, I conclude the following.

(1) Whenever elements are rectangular-shaped or uniformly curved (diagonals are equal), the QUAD8s element should be used. The QUAD9s element is more accurate for these element shapes; however, the ninth nodal point is of minimal importance for these elements, and the efficiency gained by dropping a nodal point is worth it.

(2) All other element shapes should use the QUAD9t element.

Ideally, these elements could be designated with a mesh generator. It should mesh as much of the region to be analyzed as possible using rectangular elements with eight nodal points for QUAD8s elements. The rest of the region should be meshed with nine-nodal-point quadrilaterals for QUAD9t elements. These QUAD9t elements should have straight boundaries unless straight boundaries significantly change the geometry of the region; then they should be curved. When curved boundaries are used, the analyst should remember that the accuracy of the approximation functions is reduced.

I would like to emphasize that when these higher-degree elements are used, it is very important to center the midside nodal points. The results calculated for the beam problems in Sec. IV would be much less accurate if the midside nodal points were moved even a small amount. This implies a special handling of the midside nodal points for Eulerian-(updated Lagrangian) type finite element nonlinear geometric programs.

ACKNOWLEDGMENT

This work was supported by the Department of Energy, Division of Military Applications, and was performed by the Los Alamos National Laboratory, University of California.

REFERENCES

[1] K. J. Bathe, "ADINA - A Finite Element Program for Automatic Dynamic Incremental Nonlinear Analysis," Acoustics and Vibration Laboratory report 82446-1, Dept. Mech. Eng., M.I.T., September 1975 (December 1, 1978).

[2] J. Ergatoudis, B. M. Irons, and O. C. Zienkiewicz, "Curved, Isoparametric, 'Quadrilateral' Elements for Finite Element Analysis," Int. J. Solids Structures 4, 31-42 (1968).

[3] O. C. Zienkiewicz, B. M. Irons, J. Ergatoudis, and F. Scott, "Iso-parametric and Associated Element Families for Two and Three-Dimensional Analysis," in Finite Element Methods in Stress Analysis, I. Holand and K. Bell, Eds. (Tapir, Tech. Univ. of Norway, Trondheim, Norway 1969) Chap. 13.

[4] J. A. Strickland, W. S. Ho, E. Q. Richardson, and W. E. Haisler, "On Isoparametric vs Linear Strain Triangular Elements," Int. J. Num. Meth. Eng. 11, 1041-1043 (1977).

[5] L. N. Gifford, "More on Distorted Isoparametric Elements," Int. J. Num. Meth. Eng. 14, 290-291, (1979).

[6] J. Backlund, "On Isoparametric Elements," Int. J. Num. Meth. Eng. 12, 731-732 (1978).

[7] K. J. Bathe and E. L. Wilson, Numerical Methods in Finite Element Analysis (Prentice-Hall, Englewood Cliffs, New Jersey, 1976).

[8] P. C. Dunne, "Complete Polynomial Displacement Fields for Finite Element Method," The Aeronautical J. of the Royal Aeronatucial Society 72, 245-246 (March 1968).

[9] S. P. Timoshenko and J. N. Goodier, Theory of Elasticity (McGraw Hill, New York, 1951).

TABLE I

INTEGRATION FOR TRIANGULAR ELEMENTS

Triangle	Integration Element	r	s	Weighting Factor
1	1	1/3	1/3	2
1	3	2/3	2/3	2/3
1	3	2/3	-1/3	2/3
1	3	-1/3	2/3	2/3
2	1	-1/3	-1/3	2
2	3	-2/3	-2/3	2/3
2	3	-2/3	1/3	2/3
2	3	1/3	-2/3	2/3
3	1	-1/3	1/3	2
3	3	-2/3	2/3	2/3
3	3	-2/3	-1/3	2/3
3	3	1/3	2/3	2/3
4	1	1/3	-1/3	2
4	3	2/3	-2/3	2/3
4	3	2/3	1/3	2/3
4	3	-1/3	-2/3	2/3

TABLE II

ELEMENT APPROXIMATION FUNCTIONS

Element and Number of Nodal Points	Shape[a]	Approximation Function[b]
Triangular		
3	Triangle	Linear
6	Triangle	Quadratic
6	Curved boundaries	Linear
Serendipity		
4	Parallelogram	Linear + 1
4	Skewed	Linear
8	Parallelogram	Quadratic + 2
8	Skewed	Linear + 4
8	Curved boundaries	Linear
9	Parallelogram	Quadratic + 3
9	Skewed	Quadratic
9	Curved boundaries	Linear + 1

[a] These shapes are assumed to have all midside nodal points centered.

[b] + indicates the number of additional terms.

TABLE III

PLANE STRESS CANTILEVERED BEAM WITH TRANSVERSE LOAD[a]

Element Type	Element Shapes	Vertical Tip Deflection % Error	Vertical Tip Deflection Meters
QUAD8s	Rectangular	3.10	.03876
QUAD9s	Rectangular	1.58	.03937
QUAD9t	Rectangular	3.38	.03865
QUAD8s	Skewed	56.03	.01759
QUAD9s	Skewed	16.90	.03324
QUAD9s	Skewed	4.38	.03825
Theoretical		0	.0400

[a]See Fig. 6

TABLE IV

PLANE STRESS CIRCULAR BEAM WITH TRANSVERSE LOAD[a]

Element Type	Element Shapes	Horizontal Tip Deflection % Error	Meters
QUAD8s	Uniform	2.48	.02380
QUAD9s	Uniform	1.99	.02392
QUAD9t	Uniform	1.42	.02406
QUAD8s	Skewed	22.56	.01890
QUAD9s	Skewed	15.39	.02065
QUAD9t	Skewed	3.14	.02364
Theoretical		0	.02441

[a]See Fig. 7

FIGURES

Fig. 1. A quadrilateral element composed of two triangular elements (diagonal between nodal points 2 and 4).

Fig. 2. A quadrilateral element composed of two triangular elements (diagonal between nodal points 1 and 3).

Fig. 3. A quadrilateral serendipity element.

Fig. 4. Element I with two curved boundaries.

Fig. 5. Nine-nodal point serendipity element with curved boundaries.

Fig. 6. Plane stress cantilevered beam problem (see Ref. [4]).

Fig. 7. Plane stress circular beam problem.

NOMENCLATURE

a - x-coordinate distance of midside nodal point from the center point (see Figs. 4 and 5)

b - y-coordinate distance of midside nodal point from the center point (see Figs. 4 and 5)

c - constant coefficients used in Eq. (7)

E - Young's modulus (material property)

h - element shape functions

p - constant coefficients used in Eqs. (20) and (23)

P - pressure load

q - constant coefficients used in Eqs. (20) and (23)

r - natural coordinates used in element description (see Figs. 1 and 2)

s - natural coordinates used in element description (see Figs. 1 and 2)

u - a displacement; u represents the displacements in the x and y coordinate directions

x - cartesian coordinate (see Figs. 1 and 2)

y - cartesian coordinate (see Figs. 1 and 2)

ν - Poisson's ratio (material property)

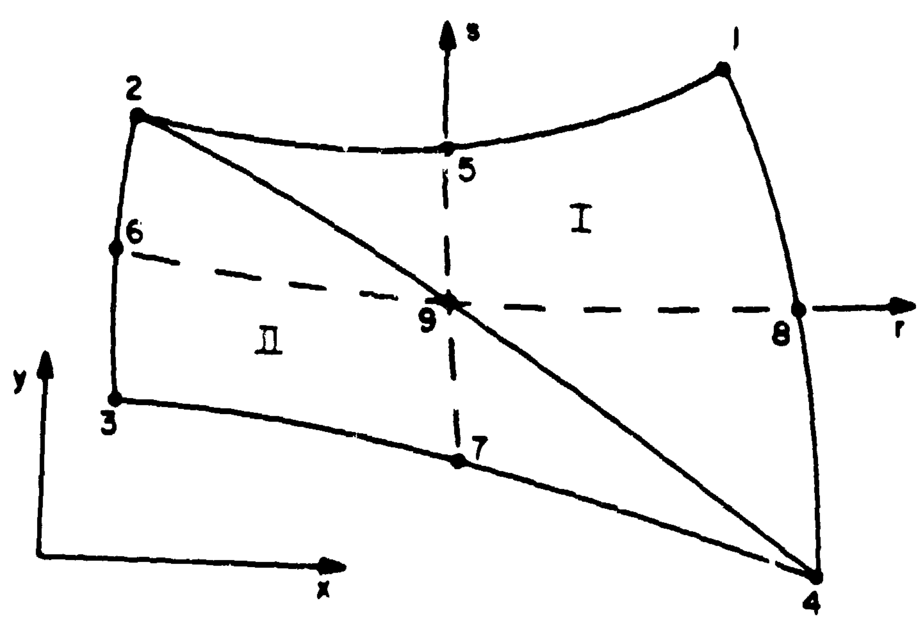

Fig. 1

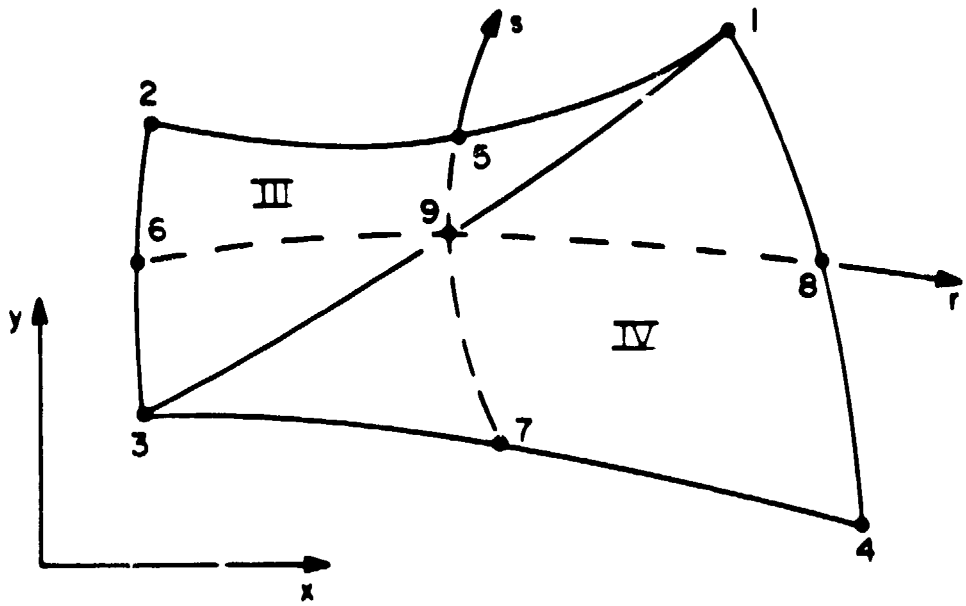

Fig. 2

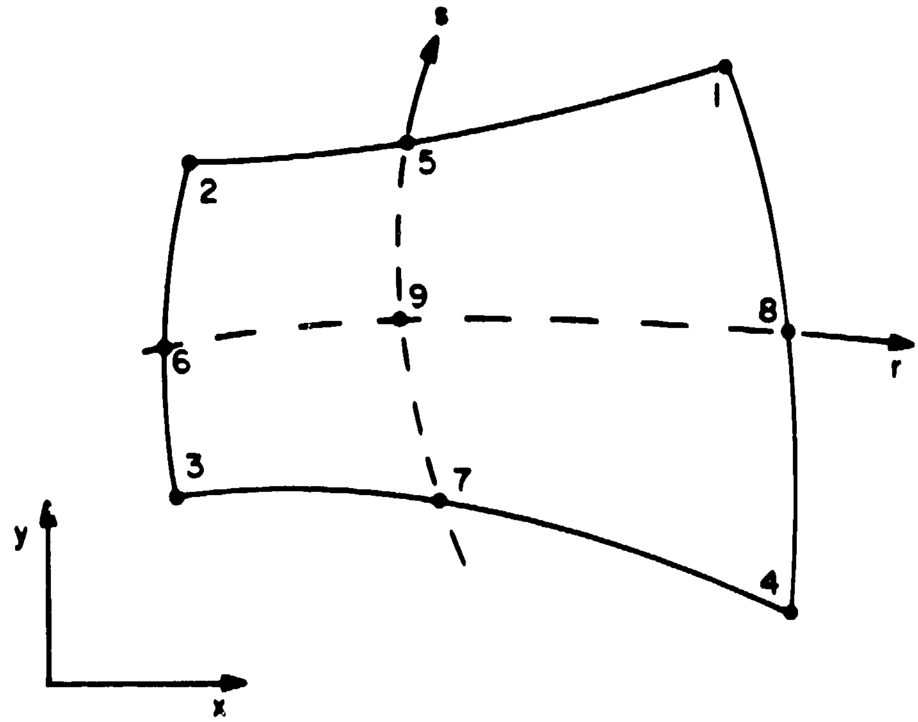

Fig. 3

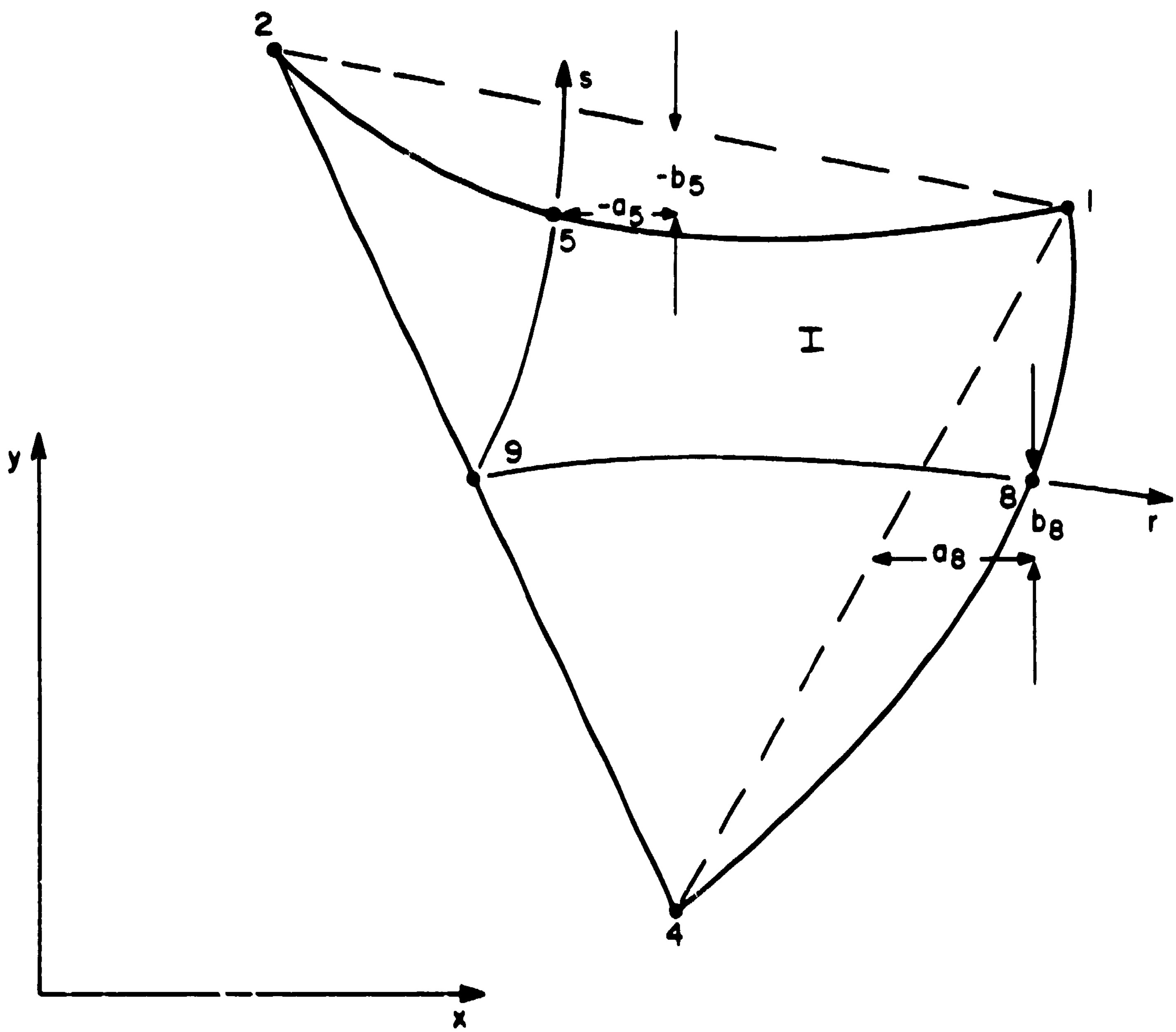

Fig. 4

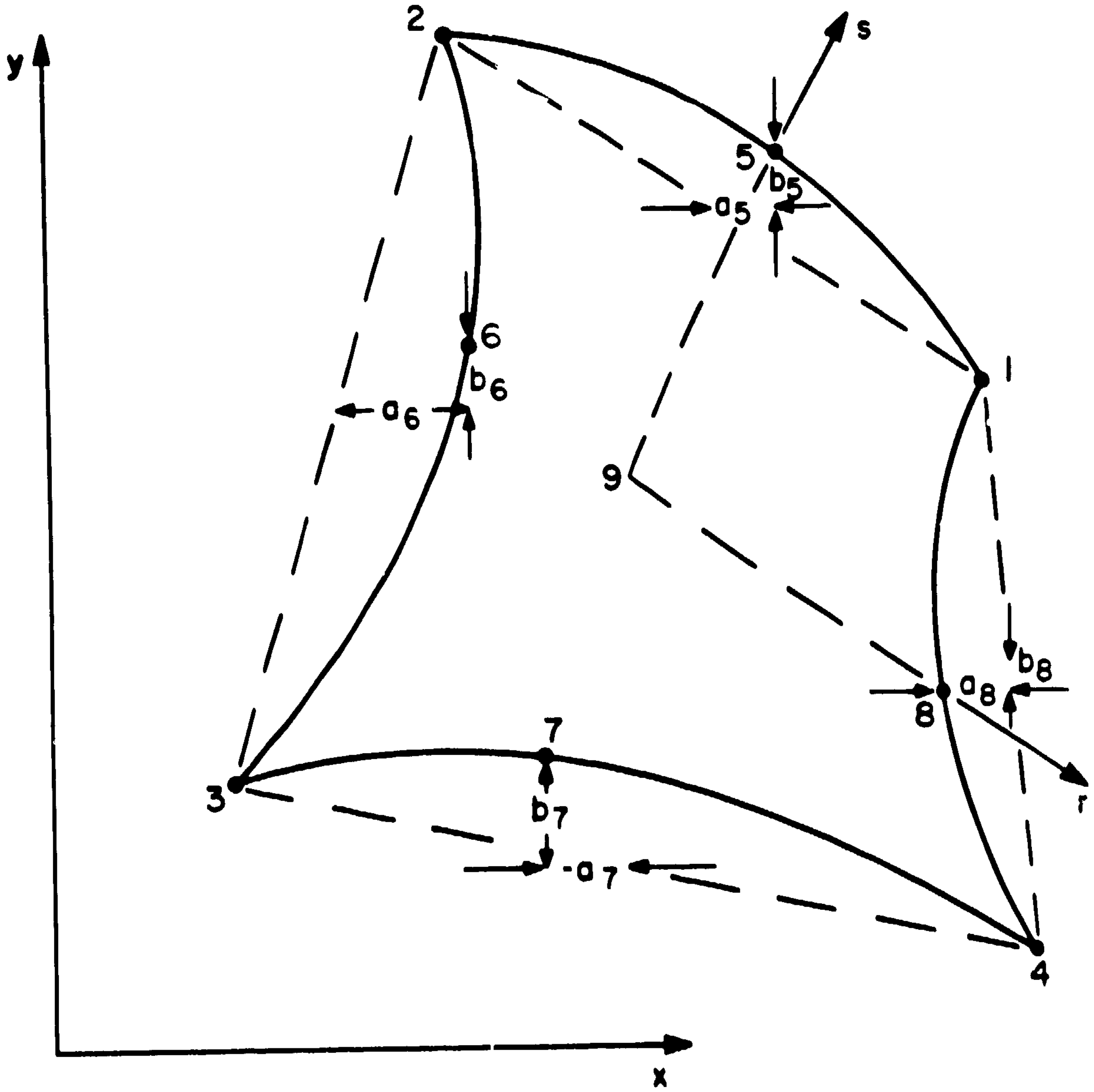

Fig. 5

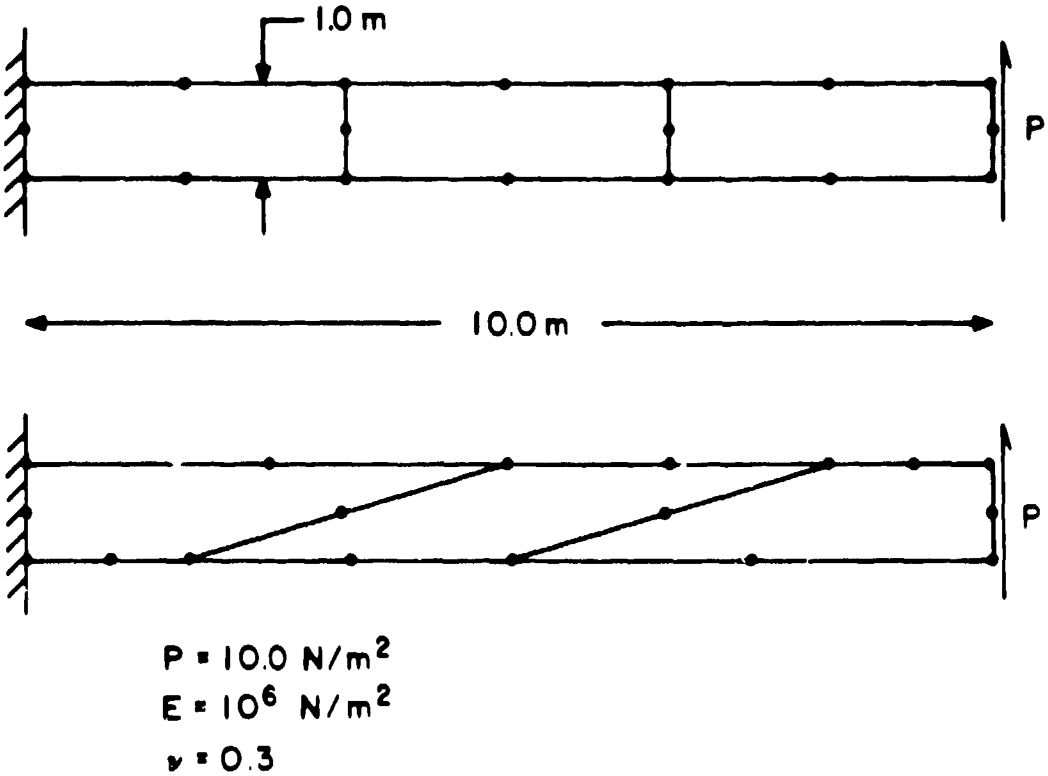

Fig. 6

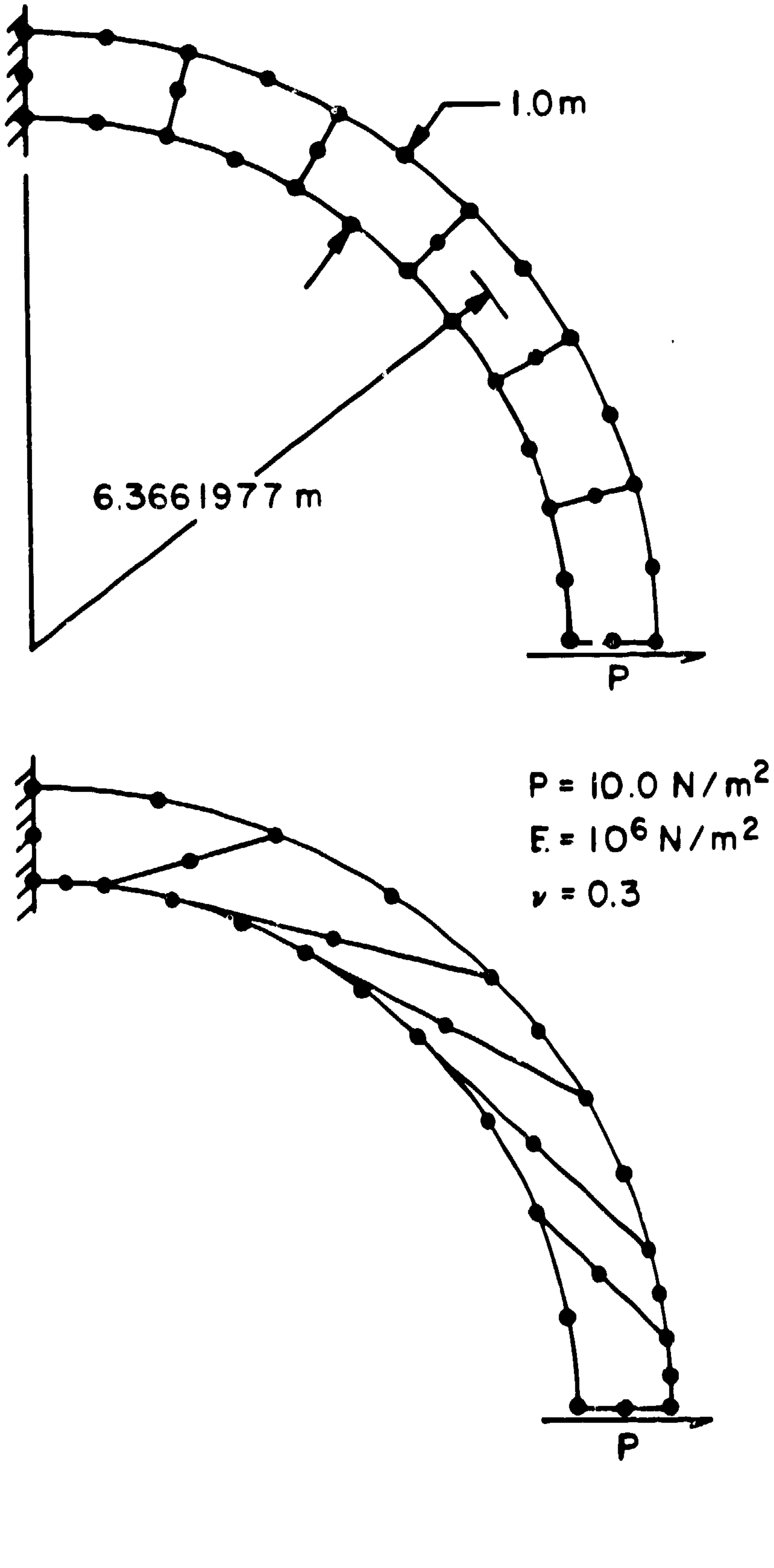

Fig. 7