PORTIONS
OF THIS
DOCUMENT
ARE
ILLEGIBLE

LA-UR-81-3324

LA-UR--81-3324

DE82 004336

MASTER

TITLE: ON DUAL SPACES OF DIFFERENTIAL LIE ALGEBRAS

AUTHOR(S): B. A. Kupershmidt

NOTICE
PORTIONS OF THIS REPORT ARE ILLEGIBLE. It has been reproduced from the best available copy to permit the broadest possible availability.

SUBMITTED TO: Proceedings of Lie Algebras Conference
University of Rutgers, New Jersey
May 29-31, 1981

DISCLAIMER

By acceptance of this article, the publisher recognizes that the U.S. Government retains a nonexclusive, royalty free license to publish or reproduce the published form of this contribution, or to allow others to do so, for U.S. Government purposes.

The Los Alamos Scientific Laboratory requests that the publisher identify this article as work performed under the auspices of the U.S. Department of Energy

University of California

LOS ALAMOS SCIENTIFIC LABORATORY
Post Office Box 1663 Los Alamos, New Mexico 87545
An Affirmative Action/Equal Opportunity Employer

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

UNITED STATES
DEPARTMENT OF ENERGY
CONTRACT W-7405-ENG. 36

# ON DUAL SPACES OF DIFFERENTIAL LIE ALGEBRAS

B. A. Kupershmidt*

0. Let $A$ be a commutative algebra over a field $\bar{k}$ of characteristic zero, and let $\partial_i : A \to A$, $i = 1, \ldots, m$, be $m$ mutually commuting derivations of $A$ over $\bar{k}$.

Suppose $L = A^N$, $N \leq \infty$, is a differential algebra over $\bar{k}$. That means that if $\bar{X}_i = (0, \ldots, X_i, \ldots, 0)$ (nonzero element on $i^{th}$-place), then a multiplication $*$ in $L$ is given by

$$(\bar{X}_i * \bar{Y}_j)_k = \sum_{\sigma,\nu \in \mathbb{Z}_+^m} c^k_{ij,\sigma,\nu}\, \partial^\sigma X_i \cdot \partial^\nu Y_j, \quad c^k_{ij,\sigma,\nu} \in A, \tag{0.1}$$

where $\mathbb{Z}_+ = \mathbb{N} \cup 0$, $\partial^\sigma := \partial_1^{\sigma_1} \cdots \partial_m^{\sigma_m}$ for $\sigma = (\sigma_1, \ldots$

If $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\bar{k}$, then $\ldots$ (understood as a function on $\mathfrak{g}^*$) one assigns a derivation $x_h$ of the ring $\ldots$ is map $\bar{\Gamma}.S(\mathfrak{g}) \to \mathrm{Deri}\, S(\mathfrak{g})$ is <u>Hamiltonian</u>, meaning (see [1] $\ldots$ Iv

$$x_{\{h,f\}} = [x_h, x_f], \quad \forall h, f \in S(\mathfrak{g}), \tag{0.2}$$

where the <u>Poisson bracket</u> is defined by

$$\{h, f\} = x_h(f) \tag{0.3}$$

Here I describe what is an analog of this well-known map in the differential case.

1. First we need an analog of functions on $L^*$. Let $u_1, \ldots, u_n$ be $n$ differentially independent variables. Consider a differential ring $A_u = A[u_1^{(\ )}]$

*Supported in part by NSF ...

with derivations $\partial$ extended on $A_u$ to act as $\partial_\ell(u_i^{(\sigma)}) = u_i^{(\sigma+1_\ell)}$. If $X \in L$, $X = (X_1, \ldots)$, we denote

$$\langle u, X\rangle := \Sigma u_i X_i \tag{1.1}$$

Denote $\mathrm{Im}\,\mathfrak{D} = \Sigma_\ell \mathrm{Im}\,\partial_\ell$ in $A_u$; we write $a \equiv b$ if $(a - b) \in \mathrm{Im}\,\mathfrak{D}$

Definition 1.2. A derivation $V: A_u \to A_u$ is called evolutionary if $[V, \partial_\ell] = 0$, $\ell = 1, \ldots, m$.

Note that $V(\mathrm{Im}\,\mathfrak{D}) \in \mathrm{Im}\,\mathfrak{D}$ and $V$ is defined by its action on $u_i$'s only: $V(u_i^{(\sigma)}) = \partial^\sigma V(u_i)$.

Define the matrix $B \in A_u[[\partial_1, \ldots, \partial_m]]$ by

$$\langle u, X * Y\rangle \equiv X^t B Y, \forall X, Y \in L, \tag{1.3}$$

where "t" means "transpose" and elements of L are understood as vectors. From (1.3), (0.1) we get

$$B_{ij} = \sum_{k,\sigma,\nu} (-1)^{|\sigma|} \partial^\sigma \cdot u_k C^k_{ij,\sigma,\nu} \cdot \partial^\nu \quad , \tag{1.4}$$

where $|\sigma| = \sigma_1 + \ldots + \sigma_m$.

Finally, we need "functional derivatives." For $P \in A_u$, the vector $\frac{\delta P}{\delta u}$ is defined by $\left(\frac{\delta P}{\delta \bar{u}_i}\right) = \frac{\delta P}{\delta u_i}$, where

$$\frac{\delta P}{\delta u_i} = \sum_\sigma (-\partial)^\sigma \left(\frac{\partial P}{\partial u_i^{(\sigma)}}\right) \tag{1.5}$$

We set $\frac{\delta P}{\delta \bar{u}^t} = \left(\frac{\delta P}{\delta \bar{u}}\right)^t$.

Proposition 1.6. The matrix B is skew-adjoint iff multiplication in L is skew-commutative.

2. Thereafter we assume B (and L) to be skew. The matrix B generates a map $\Gamma = B\delta$ from $A_u$ into $D^{ev}(A_u)$ (= evolution derivations of $A_u$) by

$$X_H(u_i) = \sum_j B_{ij}\left(\frac{\delta H}{\delta u_j}\right), \ \forall H \in A_u \tag{2.1}$$

We will write $X_H = B\,\frac{\delta H}{\delta \bar{u}}$.

Lemma 2.2. For any $P, Q \in A_u$,

$$\frac{\delta}{\delta \bar{u}}\left(\frac{\delta P}{\delta \bar{u}}^t B \frac{\delta Q}{\delta \bar{u}}\right) = D\left(\frac{\delta P}{\delta \bar{u}}\right)^{\sim} B \frac{\delta Q}{\delta \bar{u}} - D\left(\frac{\delta Q}{\delta \bar{u}}\right) B \frac{\delta P}{\delta \bar{u}} + \frac{\delta P}{\delta \bar{u}} * \frac{\delta Q}{\delta \bar{u}} ,$$

where the Frechet derivative $D(\bar{R})$, $\forall \bar{R} \in (A_u)^N$, is a matrix (differential operator) defined by

$$D(\bar{R})V\bar{u} = V(\bar{R}), \forall V \in D^{ev}(A_u) \ , \tag{2.3}$$

see e.g., [2], Ch. 1.

The proof is a straightforward analog of the computations in the lemma 7.14, Ch. I of [2].

Remark 2.4. One can make $A_u$ into an algebra using the Poisson bracket: $\{P,Q\} = X_P(Q)$. It is easy to see that, $\forall P,Q,R \in A_u$, $\{P,\{Q,R\}\} + c.p. \equiv 0$ iff $\Gamma$ is Hamiltonian, that is, $X_{\{P,Q\}} = [X_P, X_Q]$, $\forall P,Q \in A_u$.

Now I can formulate the main result.

Theorem 2.5. $\Gamma$ is Hamiltonian iff L is a Lie algebra.

The proof follows from lemma 2.2.

Remark 2.6. In the case L is an ordinary Lie algebra, $B_{ij} = \sum_k c^k_{ij} u_k$ and the equations of trajectories of the field $X_H$, $H \in S(L) \otimes C^\infty(L^*)$, are familiar

$$\dot{u}_i = \Sigma u_k c^k_{ij} \frac{\partial H}{\partial u_j} \tag{2.7}$$

Thus $X_H$ is tangent to the orbits of the coadjoint representation of a "group G" (whose Lie algebra is L). In the general differential situation, the notion of "orbits" is meaningless. That is why one needs to work algebraically. (The alternative would be either tight restrictions of the "compactness" type or pyramids of functional analysis.)

As a corollary, from the theorem 2.5 one gets

<u>Proposition 2.8.</u> The map $\Theta: L \to D^{ev}(A_u)$, $L \ni Y \mapsto X_{-\langle u,Y\rangle}$, is a Lie algebra homomorphism (assuming L is a Lie algebra).

3. Denote by $L_{(s)} = \{X \in L \mid X_i = 0,\ i \le s\}$. Consider the matrix $B^s$ defined by

$$B^s_{ij} = \sum_{k>s;\sigma,\nu} (-1)^{|\sigma|} \partial^\sigma \cdot u_k c^k_{ij,\sigma,\nu} \cdot \partial^\nu, \quad i,j > s. \tag{3.1}$$

<u>Proposition 3.2.</u> The matrix $B^s$ defines a Hamiltonian structure (i.e., if H,F depend upon $u_i$ with $i > s$ then the same is true for $\{H,F\}$) iff $L_{(s)}$ is a Lie subalgebra in L.

4. Suppose $L_1 = A^M$, $M \le \infty$, is another algebra over $\bar{k}$, and let $\phi: L \to L_1$ is a map given by a linear differential operator. Denote by $A_v = A[v_j^{(\nu)}]$, $j \le M$, the differential algebra which plays for $L_1$ the same role as $A_u$ plays for L. We can define the map $\phi^*: A_v \to A_u$ by

$$\begin{cases} 1)\ \ \phi^* \partial_\ell = \partial_\ell \phi^*, \ \ell \le m; \\ 2)\ \langle \phi^* u, X\rangle \equiv \langle u, \phi(X)\rangle, \ \forall X \in L. \end{cases} \tag{4.1}$$

Suppose L and $L_1$ are Lie algebras, thus corresponding matrices $B_u$ and $B_v$ define Hamiltonian structures. We say that $\phi^*$ is <u>canonical</u> if $\Gamma_u(\phi^* F)$ is $\phi^*$-compatible with $\Gamma_v(F)$, $\forall F \in A_v$ (see motivations in [1], Ch. IV).

Proposition 4.2. $\phi^*$ is canonical iff $\phi$ is a Lie algebra homomorphism.

5. Consider some examples.

5.1. Let $A[\xi] = \left\{ \sum_{|\sigma| \in M'} a_\sigma \xi^\sigma \,|\, a_\sigma \in A,\ \xi^\sigma := \xi_1^{\sigma_1} \dots \xi_m^{\sigma_m} \right\}$ be a ring of differential operators and $A[\xi, \xi^{-1}]$ the corresponging ring of pseudo-differential operators. For the Lie algebra generated by elements of $A[\xi, \xi^{-1}]$ of negative order, corresponding matrix B (for m = 1) was introduced by Manin ([2], Ch. I) as a formal limit of Gel'fand-Dikii operators. The Lie-algebraic interpretation of that matrix (for m = 1) was given in [3].

5.2. Consider $C^\infty(\mathbb{R}^{2m}) = C^\infty(x, \xi)$ and its localization around subspace $\xi_1 = \dots = \xi_m = 0$. The usual Poisson bracket on $\mathbb{R}^{2m} = T^*(\mathbb{R}^m)$ leads to the following algebraic set up: $L = \{ \sum_{\mu \in \mathbb{Z}_+^m} X_\mu \xi^\mu \,|\, X_\mu \in A \} = ((X_\mu) \,|\, X_\mu \in A)$.

The bracket on L is given by

$$(X * Y)_\mu = \sum_{\sigma + \tau = \mu + 1_\ell} [\sigma_\ell X_\sigma \partial_\ell (Y_\tau) - \tau_\ell Y_\tau \partial_\ell (X_\sigma)] , \tag{5.3}$$

thus the corresponding Hamiltonian matrix B is

$$B_{\sigma, \tau} = \sum_{\ell = 1}^{m} [\sigma_\ell u_{\sigma + \tau - 1_\ell} \partial_\ell + \tau_\ell u_{\sigma + \tau - 1_\ell}] . \tag{5.4}$$

This matrix appears in m-D hydrodynamics. For m = 1 it was introduced in [4]; its Lie-algebraic interpretation is given in [5] (for m = 1).

5.5. In $\mathbb{R}^{r+1} = ((x, \xi_1, \dots, \xi_r)$ consider the Lie algebra of functions on $\mathbb{R}^{r+1}$, with the bracket

$$[f, g] = \sum_{\gamma = 1}^{r} \left( d_\gamma \frac{\partial}{\partial \xi_\gamma} f \cdot \partial g - \partial f \cdot d_\gamma \frac{\partial}{\partial \xi_\gamma} g \right), \quad \partial = \partial_1 = \partial / \partial x . \tag{5.5}$$

Localization to the line $\xi = 0$ leads to the algebra $L = \left\{ \sum_{\psi \in \mathbb{Z}_+^r} X_\psi \xi^\psi \,|\, X_\psi \in A, \partial = \partial_1 = \partial / \partial x \right\}$

with Lie bracket

$$(X*Y)_\psi = \sum_{a+b=\psi+1_\gamma} (a_\gamma d_\gamma X_a \partial Y_b - b_\gamma d_\gamma Y_a \partial X_b),\ a,b\in\mathbb{Z}_+^r \tag{5.7}$$

The corresponding matrix $B$ is given by

$$B_{\phi\psi} = \sum_{\gamma=1}^{r} (\phi_\gamma d_\gamma u_{\phi+\psi-1_\gamma}\partial + \partial\psi_\gamma d_\gamma u_{\phi+\psi-1_\gamma}) \ , \tag{5.8}$$

which is the multicomponent (for $r > 1$) generalization of the $m = 1$-case (5.4).

5.9. In $\mathbb{R}^{n+p}$ with coordinates $(y_1, \ldots, y_p; x_1, \ldots, x_n)$ consider Lie algebra $\bar{L}$ consisting of those vector fields $X\in\mathfrak{D}(\mathbb{R}^{n+p})$ which commute with $\frac{\partial}{\partial y_q}$, $q = 1, \ldots, p$.

The algebraic version of $\bar{L}$ would be the Lie algebra $\mathfrak{D}(A) \bigoplus_{q=1}^{p} A \frac{\partial}{\partial y_q}$ acting on $A[y_1, \ldots, y_p]$ as derivations. The corresponding matrix $B$ now yields Poisson bracket

$$(F,G) = \sum_{q=1}^{p} v_q \sum_{\ell=1}^{n} \left(\frac{\delta F}{\delta u_\ell}\, \partial_\ell \frac{\delta G}{\delta v_q} - \frac{\delta G}{\delta u_\ell}\, \partial_\ell \frac{\delta F}{\delta v_q}\right) +$$

$$+ \sum_{\ell=1}^{n} u_\ell \sum_{i=1}^{n} \left(\frac{\delta F}{\delta u_i}\, \partial_i \frac{\delta G}{\delta u_\ell} - \frac{\delta G}{\delta u_i}\, \partial_i \frac{\delta F}{\delta u_\ell}\right) \ , \tag{5.10}$$

which appeared in [6] (for $n=1$, $p=2$) in the context of hydrodynamics of compressible fluid.

5.11. On $C^\infty(\mathbb{R}^2)$ consider Lie algebra structure $[f,g] = (\alpha\xi\partial_\xi + \xi)f \cdot \partial g - (\alpha\xi\partial_\xi + \xi)g \cdot \partial f$, $\partial = \partial/\partial x$, $\alpha\in\mathbb{R}$. Localization on the line $\xi = 0$ leads to the following algebraic set up: $L = A^\infty = \{(X_i)\} \leftrightarrow \{X_i\}^i$, $m = 1$, with multiplication

$$(X*Y)_k = d_k \sum_{n+m=k} (X_n \partial Y_m - Y_m \partial X_n),\ \partial = \partial/\partial x = \partial_1 \ , \tag{5.12}$$

where $d_k = \alpha k + \beta$, and $\alpha,\beta \in \bar{k}$ are arbitrary. The corresponding matrix B[7]

$$B_{ij} = d_i u_{i+j} \partial + \partial d_j u_{i+j} \tag{5.13}$$

is the main source of integrable systems in two-space dimensions. Classification table of such systems (see, e.g. [7]) ; smoothly upon $(\alpha:\beta) \in \bar{k}P^1$. Despite of this, one can prove the following

Theorem 5.14. The Lie algebra L with multiplication (5.12) admits a nonzero ad-invariant (modulo Im $\partial$), bilinear (differential) form (not necessarily symmetric) iff $\frac{3\beta}{\alpha} \in \mathbf{Z}$:

$$(X,Y) = \sum_{i+j+3\beta\alpha^{-1}=0} X_i Y_j \tag{5.15}$$

6. Remark. The existence of Hamiltonian structures on dual spaces of Lie algebras is not the privilege of the differential case only. Analogous results can be found: a) when one substitutes commuting automorphisms $\Delta_1, \ldots, \Delta_m : A \to A$ instead of derivations, as in [8]; b) and moreover, when both derivations and automorphisms are present. The corresponding matrices B behave naturally with respect to "continuum limits" $\Delta_\ell \to \exp(\lambda \partial_\ell)$[8]. Details will be given elsewhere.

REFERENCES


[1] B. A. Kupershmidt, "Geometry of Jet Bundles and the Structure of Lagrangian and Hamiltonian Formalisms," in Lect. Notes Math. #775 (1980), 162-218, Springer-Verlag.

[2] Yu. I. Manin, "Algebraic Aspects of Nonlinear Differential Equations," J. Sov. Math 11, (1979), 1-122.

[3] D. R. Lebedev and Yu. I. Manin, "Hamiltonian Operator of Gel'fand-Dikii and Coadjoint Representation of Volterra Group," Funk. Anal. Appl. 13:4 (1979), 40-46.

[4] B. A. Kupershmidt and Yu. I. Manin, "Equations of Long Waves with a Free Surface II. Hamiltonian Structure and Higher Equations," Funk. Anal. Appl. 12:1 (1978), 25-32.

[5] D. R. Lebedev, "Benney's Long Waves Equations: Hamiltonian Formalism," Lett. Math. Phys. 3 (1979), 481-488.

[6] P. J. Morrison and J. M. Greene, "Nancanonical Hamiltonian Density Formulation of Hydrodynamics and Ideal Magnetohydrodynamics," Phys. Rev. Lett. 45:10 (1980), 790-794.

[7] B. A. Kupershmidt, "Deformations of Integrable Systems," Proc. Roy. Irish Acad.

[8] B. A. Kupershmidt, "On Algebraic Models of Dynamical Systems," Lett. Math. Phys.

Department of Mathematics
The University of Michigan
Ann Arbor, MI 48109