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STABILITY OF PLANAR MULTIFLUID PLASMA EQUILIBRIA BY ARNOLD'S METHOD

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STABILITY OF PLANAR MULTIFLUID PLASMA EQUILIBRIA BY ARNOLD'S METHOD

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ABSTRACT. A method developed by Arnold to prove nonlinear stability of certain steady states for ideal incompressible flow in two dimensions is extended to the case of barotropic, compressible, multifluid plasmas. This extension is accomplished by constructing conserved functionals derived from degeneracy of Poisson brackets. The results are applied to planar shear flows of the plasma.

I. INTRODUCTION. Arnold [1965a,1969] formulates a method for establishing sufficient conditions for stability of stationary (i.e., steady) motions of an ideal fluid against disturbances of small but finite amplitude. Stability is established by finding a priori estimates (expressed in a certain norm depending on the problem being considered) that place bounds on the subsequent size of the disturbances, as they develop in time. These estimates apply for as long as the solutions of the disturbed flow continue to exist. When such estimates have been established, the stationary motions are said to be "stable by Arnold's method."

Arnold's method is based on the construction of a conserved functional (a constant of the motion) that has a given stationary flow as its extremum (critical point). If this extremum is a true minimum or maximum relative to nearby flows within a neighborhood whose topology must be determined for each problem, then the corresponding stationary flow is stable in that topology. Such stability can be understood geometrically by a heuristic argument. Imagine the level surfaces of the conserved functional in function space, in a neighborhood of the point representing a given stationary flow. For a maximum or minimum, these level surfaces will be nested and closed, surrounding the equilibrium point. If the steady state flow is disturbed at some instant, the corresponding phase point in the function space will shift onto a nearby level

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surface and will remain on it throughout the subsequent time of motion, by conservation of the functional. If a priori estimates can establish that the distance in an appropriate norm from the equilibrium point to the nearby level surface upon which the disturbed motion takes place subsequently remains bounded, then the equilibrium point is stable by Arnold's method.

Bounded in a certain norm, motions stable by Arnold's method are also stable in the sense of Lyapunov: for each $\varepsilon > 0$ there exists a $\delta > 0$, such that if the initial values are disturbed by less than δ (in the norm determined by the a priori estimates) then the solution deviates from a specified solution (e.g., the stationary one) by less than ε during the entire subsequent motion. Having found by Arnold's method a norm $|\cdot|$ in which the perturbations δx_0 at time zero, and δx at time t satisfy $|\delta x| \leq K|\delta x_0|$, with $K \geq 1$ and for all time, one may choose $|\delta x_0| < \delta$; then $|\delta x| < \varepsilon = K\delta$. This is the type of stability result derived by Arnold's method.

Arnold [1965a,1969] studies incompressible planar fluid motion, where stability is established, among other examples, in the case of stationary flows satisfying Rayleigh's inflection point criterion. Dikii [1965] shows this type of stability for incompressible zonal circulation on a spherical surface, provided the stationary flows there satisfy a spherical analog of Rayleigh's criterion. Holm et al. [1983] establish conditions for stability by Arnold's method for compressible (barotropic) planar flows. Abarbanel et al. [1984] prove stability criteria by this method for two and three dimensional, stratified, incompressible flows, with buoyancy effects included. Holm et al. [1984] deal with additional examples of stability of stationary flows by this method: three dimensional adiabatic compressible hydrodynamics, magnetohydrodynamics, and multifluid plasma dynamics; two dimensional magnetohydrodynamics, both compressible and incompressible; Poisson-Vlasov, and Maxwell-Vlasov plasma equations; and multilayer quasigeostrophic systems. Wan et al. [1984] prove stability conditions for incompressible circular vortex patches in the plane by a method similar to Arnold's, but requiring more delicate analysis.

Arnold's stability method is assembled from several well known elements: extremal principles for conserved functionals, definiteness in sign of their second variations, and convexity arguments that establish a priori estimates. However, the success of this method in fluid dynamics derives from a less familiar element: degeneracy of Poisson brackets. Degeneracy of Poisson brackets for a given dynamical system means that certain quantities - the so-called "Casimirs" - are constants of the motion for *any* Hamiltonian. Thus, the Poisson bracket vanishes when taken between a Casimir and any other quantity depending upon the given dynamical variables. Casimirs are discribed from a geometrical viewpoint with finite-dimensional examples, in Weinstein [1984], in these proceedings. Construction of degenerate Poisson brackets for various fluid theories and their association to certain Lie algebras is treated in Marsden, Ratiu, and Weinstein [1983], Holm and Kupershmidt [1983], and Holm, Kupershmidt, and Levermore [1983]. Explicit derivation of Casimirs for Poisson brackets in fluid theories is discussed in Ratiu [1984], in these proceedings.

Arnold's stability method uses the Casimirs to construct conserved functionals. It imposes the Casimirs (as well as other constants of motion) essentially as Lagrange multiplier constraints for a variational principle that seeks conditional critical points of the energy. Denote by H this constrained energy, so that H is the sum of the energy and certain constants of motion. For stationary states, the first variation of H vanishes, i.e., H has a critical point, for appropriately chosen Lagrange multipliers. This critical point is locally a minimum, a maximum, or a saddle point, depending on whether the second variation of H at the critical point is, respectively, positive definite, negative definite, or indefinite.

Under certain conditions on the stationary states, the second variation at the critical point may be definite in sign. Under these conditions, the second variation defines a norm, which induces a weak type of stability, called "formal stability." Formal stability implies linearized stability against infinitesimal disturbances at the critical point, since the norm of the second variation is preserved by the linearized equations. This is only neutral stability, though, since the spectrum of the linearized ideal fluid equations lies on the imaginary axis. Formal stability in fluids and plasmas had been considered by a number of authors, even before Arnold [1965a]. For plasma theory, see, e.g., Kruskal and Oberman [1958], Newcomb (in Appendix I of Berstein, et al. [1964]), and Rosenbluth [1969]. For incompressible planar shear flows, formal stability is discussed in a geophysical context by Blumen [1971], and, more recently, for multilayer quasigeostropic flows, by Benzi et al. [1982].

Fortunately, the conditions on the stationary states that give formal stability via definiteness in sign of the second variation, can often be strengthened sufficiently to provide the desired a priori estimates; thereby expressing Lyapunov stability against disturbances of small but finite amplitude. These estimates are obtained via convexity arguments involving the constrained energy, H.

The present work establishes sufficient conditions for stability by Arnold's method for planar stationary plasma equilibria, as described by the ideal, compressible, multifluid plasma equations. This problem exemplifies the kind of results available for stability of fluids that are coupled self consistently with other fields, and displays the role in Arnold's method played by degenerate Poisson brackets possessing Casimirs. In the next section, after a brief introduction of energy principles in the context of potential flows, this role is reviewed for vortical incompressible flows in three dimensions (Beltrami flows) and in two dimensions (Arnold's case). In section 3, we study the multifluid plasma problem.

II. HOMOGENEOUS INCOMPRESSIBLE FLOWS

II.A. Potential Flows. The problem of establishing sufficient conditions for stability in ideal incompressible hydrodynamics can be introduced conveniently by recalling a result due to Lord Kelvin. Kelvin [1849] shows that ideal incompressible potential flows ($\underline{v} = \underline{\nabla}\phi$, div $\underline{v} = 0$, $\underline{v}$ the velocity, Φ its potential) satisfy a minimum energy principle among divergenceless flows in a simply connected domain $D \subset \mathbb{R}^3$ with prescribed normal flux at the surface.

Euler's equations for an ideal incompressible fluid are

$$\begin{aligned} \partial_t \underline{v} &= -(\underline{v} \cdot \underline{\nabla})\underline{v} - \underline{\nabla} p \\ \operatorname{div} \underline{v} &= 0 \quad , \end{aligned} \tag{1}$$

where p is pressure, and the constant fluid density has been set equal to unity. These equations conserve the kinetic energy

$$E = \int_D \tfrac{1}{2}|\underline{v}|^2 \, d^3x \quad .$$

In Kelvin's minimum energy principle for potential flows, the kinetic energy is minimized subject to the two conditions that div $\underline{v} = 0$ in domain D and $\underline{v} \cdot \underline{n} = Q(\underline{x})$ on the boundary ∂D, where $\underline{n}$ is the unit vector normal to the boundary and Q is the prescribed normal flux consistent with conservation of energy. These two conditions will be imposed by choosing Lagrange multiplier functions, Φ, χ, respectively. Thus, one considers the functional

$$H_{\Phi,\chi}(\underline{v}) = \int_D [\tfrac{1}{2}|\underline{v}|^2 + \Phi(\underline{x}) \operatorname{div} \underline{v}] d^3x + \int_{\partial D} \chi(\underline{x})(\underline{v} \cdot \underline{n} - Q(\underline{x})) d^2x \quad .$$

The first variation of $H_{\Phi,\chi}$ is, for arbitrary variations $\delta\underline{v}$, $\delta\Phi$, $\delta\chi$,

$$\delta H_{\Phi,\chi} := DH_{\Phi,\chi}(\underline{v}) \cdot (\delta\underline{v}, \delta\Phi, \delta\chi) \quad ,$$

$$= \int_D [(\underline{v} - \underline{\nabla}\Phi) \cdot \delta\underline{v} + \delta\Phi \operatorname{div} \underline{v}]\, d^3x$$

$$+ \int_{\partial D} [(\Phi + \chi)\delta\underline{v} \cdot \underline{n} + \delta\chi(\underline{v} \cdot \underline{n} - Q)]\, d^2x \quad . \tag{2}$$

The first variation $\delta H_{\Phi,\chi}$ vanishes for an equilibrium velocity, $\underline{v}_e$, which is a stationary potential flow,

$$\underline{v}_e - \underline{\nabla}\Phi(\underline{x}) = 0 \quad ,$$

under the conditions imposed by the Lagrange multipliers,

$$\operatorname{div} \underline{v}_e = 0 \quad \text{in } D \quad ,$$

$$\underline{v}_e \cdot n - Q = 0 \quad \text{on } \partial D \quad ,$$

provided also $\Phi+\chi = 0$. Note that if $Q \equiv 0$, e.g., for a fixed, impermeable boundary, then the equations $\Delta\Phi = 0$ in D and $\underline{n} \cdot \underline{\nabla}\Phi = 0$ on ∂D imply that Φ will be constant, so $\underline{v}_e$ will vanish. Plainly, this static solution $\underline{v}_e \equiv 0$ would be a trivial minimum of $H_{\Phi,\chi}$. We seek nontrivial minima.

Taking the second variation of $H_{\Phi,\chi}$ leads to

$$2\delta^2 H_{\Phi,\chi} := D^2 H_{\Phi,\chi}(\underline{v}_e) \cdot (\delta\underline{v}, \delta\Phi, \delta\chi)^2$$

$$= \int_D (|\delta\underline{v}|^2 + 2\delta\Phi \operatorname{div} \delta\underline{v})\, d^3x + 2\int_{\partial D} \delta\chi\, \delta\underline{v} \cdot \underline{n}\, d^2x$$

which is positive definite in the class of divergenceless velocity variations ($\operatorname{div} \delta\underline{v} = 0$ in D) for the prescribed flux ($\delta\underline{v} \cdot \underline{n} = 0$ on ∂D). So the kinetic energy has a conditional minimum for potential flows. This is Kelvin's minimum energy principle.

Kelvin's minimum energy principle indicates how to establish stability of these stationary potential flows. Noting that $H_{\Phi,\chi}(\underline{v})$ is conserved, the following quantity is also conserved

$$\hat{H}_{\Phi,\chi}(\delta\underline{v}) = H_{\Phi,\chi}(\underline{v}_e + \delta\underline{v}) - H_{\Phi,\chi}(\underline{v}_e) - DH_{\Phi,\chi}(\underline{v}_e) \cdot (\delta\underline{v}, \delta\Phi, \delta\chi)$$

$$= \int_D |\delta\underline{v}|^2\, d^3x \quad . \tag{3}$$

Letting $\delta\underline{v}_0$ denote the initial value of a velocity perturbation that takes a value $\delta\underline{v}$ at a certain time t later, one has

$$\int_D |\delta\underline{v}|^2 \, d^3x = \int_D |\delta\underline{v}_0|^2 \, d^3x > 0 \quad . \tag{4}$$

Thus, Euler's equations conserve an energy norm (3), which is an L^2 norm in $\delta\underline{v}$. In this norm, ideal, stationary, potential flows are stable, according to the a priori estimate (4).

II.B Beltrami Flows: Introduction of Casimirs. For Beltrami flows, the velocity is an eigenfunction of the curl operator:

$$\text{curl } \underline{v} = \alpha \underline{v} \quad , \ \alpha = \text{const.} \tag{5}$$

Thus, expressing Euler's equations (1) as

$$\partial_t \underline{v} = \underline{v} \times \underline{\omega} - \underline{\nabla}(|\underline{v}|^2/2 + p) \tag{1'}$$

where $\underline{\omega} := \text{curl } \underline{v}$ is vorticity, one sees that Beltrami flows are stationary states of Euler's equations, when $\underline{\nabla}(|\underline{v}_e|^2/2 + p_e)$ vanishes. We shall show that Beltrami flows extremalize the kinetic energy, constrained by the "helicity", F_h, defined as

$$F_h = \int_D \underline{v} \cdot \underline{\omega} \, d^3x \quad , \ \underline{v} := -\Delta^{-1} \text{ curl } \underline{\omega} \quad , \tag{6}$$

in a finite domain $D \subset IR^3$, with vanishing normal flux at the fixed boundary, ∂D. However, we shall see that this helicity constraint will not be enough to establish the norm required to prove stability of Beltrami flows in three dimensions by Arnold's method. Nevertheless, stable Beltrami flows may still exist. We wish to use this apparently negative example to emphasize that even when successful, in most cases Arnold's method provides conditions that are only sufficient, not necessary and sufficient, for stability. This example also introduces the use of Casimirs, for a Poisson bracket in terms of the vorticity.

Euler's equations are Hamiltonian in terms of the vorticity. Namely, upon taking the curl of [illegible] identifying $\underline{v} = -\Delta^{-1}$ curl $\underline{\omega}$ in D, one finds

$$\partial_t \underline{\omega} = \{\underline{\omega}, E(\underline{\omega})\} = \text{curl } (\underline{v} \times \underline{\omega}) \tag{7}$$

with Hamiltonian

$$E(\underline{\omega}) = \int_D \underline{\omega} \cdot (-\Delta^{-1}\underline{\omega}) d^3x$$

and Poisson bracket $\{\cdot,\cdot\}$ defined by

$$\{F,G\} = \int_D \underline{\omega} \cdot (\text{curl } \frac{\delta F}{\delta \underline{\omega}} \times \text{curl } \frac{\delta G}{\delta \underline{\omega}}) d^3x \quad , \tag{8}$$

for functionals $F(\underline{\omega})$ and $G(\underline{\omega})$ of $\underline{\omega}$, where $\delta F/\delta\underline{\omega}$ and $\delta G/\delta\underline{\omega}$ are variational derivatives, defined by

$$[\frac{d}{d\varepsilon} F(\underline{\omega} + \varepsilon \underline{q})]_{\varepsilon=0} = \int_D \frac{\delta F}{\delta \underline{\omega}} \cdot \underline{q} \, d^3x$$

for an arbitrary function, $\underline{q}$. The time development of a functional $F(\underline{\omega})$ thus obeys

$$\partial_t F = \{F, E(\underline{\omega})\} \quad .$$

The helicity F_h in (6) is a Casimir for the Poisson bracket (8), i.e., the Poisson bracket $\{F_h,G\}$ vanishes for <u>every</u> Hamiltonian $G(\underline{\omega})$,

$$\{F_h, G\} = 0 \quad , \forall \; G(\underline{\omega}) \quad .$$

In particular, the bracket $\{F_h,E(\underline{\omega})\}$ vanishes, so the helicity is a constant of motion for Euler's equations (7) in D, with the boundary conditions of zero normal flux on ∂D. In addition, the Hamiltonian formed by the sum

$$H_\lambda = E + \lambda F_h \quad , \lambda = \text{const},$$

that is,

$$H_\lambda = \int_D [\tfrac{1}{2}\underline{\omega} \cdot (-\Delta^{-1}\underline{\omega}) + \lambda \underline{v} \cdot \underline{\omega}] d^3x \tag{9}$$

also generates the Euler equations (7) via Poisson bracket (8), with E replaced by H_λ for any value of λ, which we now regard as a Lagrange multiplier for the conserved constraint, F_h.

Taking the first and second variations of the constrained kinetic energy H_λ in (9) yields the formulas

$$\delta H_\lambda = \int_D (-\Delta^{-1}\underline{\omega} + 2\lambda\underline{v}) \cdot \delta\underline{\omega}\, d^3x = \int_D (\underline{v} + 2\lambda \operatorname{curl} \underline{v}) \cdot \delta\underline{v}\, d^3x, \tag{10}$$

$$2\delta^2 H_\lambda = \int_D [\delta\underline{\omega} \cdot (-\Delta^{-1}\delta\underline{\omega}) + 2\lambda\delta\underline{\omega} \cdot \delta\underline{v}]d^3x \quad , \tag{11}$$

with definitions

$$\delta H_\lambda = DH_\lambda(\underline{\omega}) \cdot \delta\underline{\omega} \quad ,$$

$$2\delta^2 H_\lambda = D^2 H_\lambda(\underline{\omega}) \cdot (\delta\underline{\omega})^2 \quad ,$$

$$\delta\underline{v} = -\Delta^{-1} \operatorname{curl} \delta\underline{\omega} \quad ,$$

and with surface terms having been set equal to zero whenever they appear due to integration by parts, according to the boundary conditions. From the first variation, δH_λ, which vanishes for equilibrium velocity $\underline{v}_e$ such that $\underline{v}_e + 2\lambda \operatorname{curl} \underline{v}_e = 0$, one sees that Beltrami flows do extremalize H_λ, and for a given Beltrami flow (5) with eigenvalue α, one has $\lambda = (2\alpha)^{-1}$ for the Lagrange multiplier, λ.

The second variation $\delta^2 H_\lambda$ is *indefinite* unless $\lambda = 0$, in which case the equilibrium flow is static. Indeed, $\delta^2 H_\lambda$ is equal to the following conserved quantity

$$\hat{H}_\lambda := H_\lambda(\underline{\omega}_e + \delta\underline{\omega}) - H_\lambda(\underline{\omega}_e) - DH_\lambda(\underline{\omega}_e) \cdot \delta\underline{\omega},$$

where $\underline{\omega}_e$ is the equilibrium vorticity distribution and $\delta\underline{\omega}$ can now be a *finite* perturbation. The quantity $\hat{H}_\lambda$ is conserved, since $H_\lambda(\underline{\omega}_e + \delta\underline{\omega})$ is conserved for any $\delta\underline{\omega}$, $H_\lambda(\underline{\omega}_e)$ is merely a constant real number, and $DH_\lambda(\underline{\omega}_e) \cdot \delta\underline{\omega}$ vanishes. With the quantity $\hat{H}_\lambda$ indefinite, no norm is established and constancy of $\hat{H}_\lambda$ does not restrict the growth of perturbations.

Besides introducing Casimirs into the construction of the conserved quantity $\hat{H}_\lambda$, this example illustrates the following point: when $\hat{H}_\lambda$ is indefinite, no conclusion is indicated by Arnold's method about either stability, or instability. In particular, one cannot draw the conclusion now that all Beltrami flows with $\alpha \neq 0$ are unstable, cf. Arnold [1965b]. Such indefiniteness, though, does suggest employing a complementary technique. For example, one could seek sufficient conditions for linear instability of Beltrami flows, using, say, normal mode analysis.

II.C. Arnold's Theorem. Arnold's theorem uses an extremal energy principle to obtain stability criteria for stationary, planar, vortical flow of an ideal, incompressible fluid. Arnold [1965a,1965b,1969] considers incompressible fluid motion in a fixed domain $D \subset \mathbb{R}^2$, in the (x,y) plane, with velocity tangent to the boundary, ∂D. In this case, vorticity is defined by a scalar function ω, as

$$\text{curl } \underline{v} = \hat{z}\, \omega(x,y,t) \quad ,$$

where $\hat{z}$ is the unit vector normal to the (x,y) plane. The Poisson bracket (8) becomes

$$\{F,G\} = \int_D \omega\left[\frac{\delta F}{\delta \omega},\frac{\delta G}{\delta \omega}\right]dx\, dy \tag{12}$$

for functionals F,G of ω, with $[\cdot,\cdot]$ being the jacobian (or the canonical Poisson bracket), defined by

$$[f,g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y} \tag{13}$$

for functions $f(x,y)$, $g(x,y)$. Also, the energy E in (7) becomes

$$E(\omega) = \tfrac{1}{2}\int_D \omega(-\Delta^{-1}\omega)dx\, dy \quad ,$$

whereby the equation of motion results,

$$\partial_t \omega = \{\omega,E\} = [-\Delta^{-1}\omega,\omega] \tag{14}$$

using (12). Defining the stream function ψ such that $\omega = -\Delta\psi$ leads to the standard formula,

$$\partial_t \omega = [\psi,\omega] \quad .$$

Consequently, a certain functional dependence exists for stationary flows ψ_e, ω_e, expressible as

$$\psi_e = \tilde{\psi}(\omega_e) \quad , \tag{15}$$

since the jacobian $[\psi_e,\omega_e]$ vanishes for stationary flows.

A Casimir for the Poisson bracket (12) is, with an arbitrary function $\Phi(\omega)$,

$$F_\Phi(\omega) = \int_D \Phi(\omega)dxdy \quad . \tag{16}$$

By direct computation, one shows that F_Φ satisfies $\{F_\Phi,G\} = 0$ for every Hamiltonian, G,

$$\{F_\Phi(\omega),G(\omega)\} = \int_D \omega\left[\frac{\delta F}{\delta\omega},\frac{\delta G}{\delta\omega}\right]dx\,dy$$

$$= \int_D \frac{\delta G}{\delta\omega}\,[\omega,\Phi'(\omega)]dx\,dy$$

$$= 0 \quad , \quad \forall\, G(\omega),\ \Phi(\omega) \quad ,$$

upon using the properties of the jacobian (13) and integrating by parts. In particular, the bracket $\{F_\Phi,E\}$ vanishes so that F_Φ in (16) is a family of constants of motion for the two dimensional Euler equations.

Following Arnold [1969], one defines the sum

$$H_\Phi(\omega) = E(\omega) + F_\Phi(\omega) \quad , \tag{17}$$

which is a conserved functional. Taking the first and second variations of $H_\Phi(\omega)$ yields

$$\delta H_\Phi := DH_\Phi(\omega)\cdot\delta\omega = \int_D [-\Delta^{-1}\omega + \Phi'(\omega)]\delta\omega\, dxdy \quad , \tag{18}$$

$$2\delta^2 H_\Phi := D^2 H_\Phi(\omega)\cdot(\delta\omega)^2 = \int_D [\delta\omega(-\Delta^{-1}\delta\omega) + \Phi''(\omega)(\delta\omega)^2]dxdy \quad . \tag{19}$$

The first variation δH_Φ vanishes, provided ω takes equilibrium values, ω_e, satisfying

$$-\Delta^{-1}\omega_e + \Phi'(\omega_e) = 0 \quad . \tag{20}$$

That is, for stationary flows δH_Φ vanishes, and $\Phi(\omega_e)$ is determined for a given stationary flow satisfying (15), by

$$\Phi'(\omega_e) = -\bar{\psi}(\omega_e) \quad . \tag{21}$$

As mentioned in the introduction, either negative, or positive definiteness of the second variation $\delta^2 H_\Phi$ suggests that Lyapunov stability can be established. Both cases are shown to be possible in Arnold [1969]. In each case, a convexity argument for the function $\Phi(\omega)$ is used in combination with the conserved quantity $\hat{H}_\Phi$,

$$\hat{H}_\Phi := H_\Phi(\omega_e + \delta\omega) - H_\Phi(\omega_e) - DH_\Phi(\omega_e) \cdot \delta\omega \tag{22}$$

to establish Lyapunov stability in a certain norm. Here, $\delta\omega$ is considered to be a vorticity disturbance at a certain time, t, which has the value $\delta\omega_0$ at time zero. The quantity $\hat{H}_\Phi$ is conserved, since $H_\Phi(\omega_e + \delta\omega)$ is conserved for any $\delta\omega$, $H_\Phi(\omega_e)$ is merely a constant real number, and $DH_\Phi(\omega_e) \cdot \delta\omega$ vanishes, by (20).

<u>Case 1</u>. According to (19), the second variation $\delta^2 H_\Phi$ will be positive definite, provided

$$\Phi''(\omega_e) > 0 \quad ,$$

since $(-\Delta^{-1})$ is a positive operator. By using (15) and (21), this condition can be expressed as

$$\Phi''(\omega_e) = -\tilde{\psi}'(\omega_e) = \frac{\underline{\nabla}\psi_e}{\underline{\nabla}\Delta\psi_e} > 0 \quad . \tag{23}$$

For example, flows parallel to the x-axis in the strip $\{Y_1 \le y \le Y_2\}$ and periodic in x have

$$\psi_e = \psi_e(y) \quad , \underline{\nabla}\psi_e = v(y)\hat{\underline{y}} \quad , \Delta\psi_e = v'(y) \quad , \underline{\nabla}\Delta\psi_e = v''(y)\hat{\underline{y}} \quad .$$

Consequently, for such flows (23) becomes

$$\Phi''(\omega_e(y)) = \frac{v(y)}{v''(y)} > 0 \quad , \tag{24}$$

provided an inertial frame can be chosen so that the sign of v is everywhere the same as the sign of v''. Thus, all flows having no inflection points have $\delta^2 H_\Phi$ positive definite.

Positive definiteness of $\delta^2 H_\Phi$, by itself, does not imply Lyapunov stability. Arnold [1969] supplies a convexity argument which does prove

Lyapunov stability criteria in this case. Strengthening the condition (23) to

$$0 < a \le \Phi''(\zeta) \le A < \infty \tag{25}$$

and extending the definition of $\Phi(\zeta)$ over the entire ζ axis subject to inequality (25), implies that, for any h,

$$\frac{a}{2} h^2 \le \Phi(\zeta+h) - \Phi(\zeta) - \Phi'(\zeta)h \le \frac{A}{2} h^2 \quad . \tag{26}$$

Hence, according to the definition (22)

$$\begin{aligned} 2\hat{H}_\Phi(t) &\ge \int_D [\delta\omega(-\Delta^{-1}\delta\omega) + a(\delta\omega)^2]dxdy > 0 \\ 2\hat{H}_\Phi(0) &\le \int_D [\delta\omega_0(-\Delta^{-1}\delta\omega_0) + A(\delta\omega_0)^2]dxdy \end{aligned} \tag{27}$$

and $\hat{H}_\Phi(t) = \hat{H}_\Phi(0)$, so that the growth of a disturbance $\delta\omega$ is bounded in terms of its initial value $\delta\omega_0$. The estimate (27) implies Lyapunov stability of stationary flows with $\underline{\nabla}\psi_e/\underline{\nabla}\Delta\psi_e \ge a > 0$.

<u>Case 2</u>. Consider stationary flows with $\underline{\nabla}\psi_e/\underline{\nabla}\Delta\psi_e < 0$. Let a stationary flow be such that

$$0 < a \le -\Phi''(\zeta) \le A < \infty \tag{28}$$

and extend the definition of $\Phi(\zeta)$ over the entire ζ axis, subject to (28). Then one bounds $-2\hat{H}_\Phi$, to find

$$\begin{aligned} -2\hat{H}_\Phi(t) &\ge \int_D [-\delta\omega(-\Delta^{-1}\delta\omega) + a(\delta\omega)^2]dxdy \ge \int_D (-k_{min}^{-2} + a)(\delta\omega)^2 dxdy \\ -2\hat{H}_\Phi(0) &\le \int_D [-\delta\omega_0(-\Delta^{-1}\delta\omega_0) + A(\delta\omega_0)^2]dxdy \le \int_D A(\delta\omega_0)^2 dxdy \end{aligned} \tag{29}$$

where k_{min}^2 is the minimum eigenvalue of minus the Laplacian $(-\Delta)$ in domain D. Consequently, if

$$a = \min |\Phi''(\zeta)| > k_{min}^{-2}$$

then perturbation growth is bounded, since again $\hat{H}_\Phi(t) = \hat{H}_\Phi(0)$. The estimate (29) establishes Lyapunov stability of stationary flows with $-\underline{\nabla}\psi_e/\underline{\nabla}\Delta\psi_e \ge a > k_{min}^{-2}$.

III. COMPRESSIBLE MULTIFLUID PLASMA STABILITY IN TWO DIMENSIONS

The multifluid plasma (MFP) equations describe motion of a system of ideal, charged fluids interacting together via selfconsistent electromagnetic forces. The fluid species are labeled by superscript s (Note: no summation convention is imposed on the superscript s in this section.); each species is composed of particles of mass m^s and charge q^s, with charge to mass ratio $a^s = q^s/m^s$. Dynamical fluid variables are: fluid velocity $\underline{v}^s$; mass density ρ^s (with barotropic partial pressure $p^s = p^s(\rho^s)$ and specific internal energy $e^s = e^s(\rho^s)$, each depending *only* on ρ^s); electric field $\underline{E}$; and magnetic field $\underline{B}$.

The MFP equations consist of dynamical Maxwell equations for the electromagnetic fields; a continuity equation for each species; and the MFP motion equations:

$$\begin{aligned}
\partial_t\underline{B} &= -\mathrm{curl}\ \underline{E} \\
\partial_t\underline{E} &= \mathrm{curl}\ \underline{B} - \sum_s a^s\rho^s\underline{v}^s \\
\partial_t\rho^s &= -\mathrm{div}\ \rho^s\underline{v}^s \\
\partial_t\underline{v}^s &= \underline{v}^s \times (\underline{\omega}^s + a^s\underline{B}) - \frac{1}{\rho^s}\underline{\nabla}p^s - \tfrac{1}{2}\underline{\nabla}|\underline{v}^s|^2 + a^s\underline{E}\quad .
\end{aligned}\tag{30}$$

The static Maxwell equations

$$\mathrm{div}\ \underline{B} = 0\quad ,\ \mathrm{div}\ \underline{E} - \sum_s a^s\rho^s = 0\quad ,\tag{31}$$

although nondynamical, are compatible with the flow, i.e., if true initially (31) will remain true under MFP dynamics.

The MFP equations are shown to be Hamiltonian in Spencer [1982] with Poisson bracket {F,G} defined in terms of $\{\rho^s, \underline{M}^s{:=}\rho^s\underline{v}^s, \underline{E}, \underline{B}\}$ by

$$\begin{aligned}
\{F,G\} = \sum_s \int d^3x\ [\rho^s(&\frac{\delta G}{\delta \underline{M}^s}\cdot\underline{\nabla}\frac{\delta F}{\delta\rho^s} - \frac{\delta F}{\delta\underline{M}^s}\cdot\underline{\nabla}\frac{\delta G}{\delta\rho^s}) \\
&+ M_i^s(\frac{\delta G}{\delta\underline{M}^s}\cdot\underline{\nabla}\frac{\delta F}{\delta M_i^s} - \frac{\delta F}{\delta\underline{M}^s}\cdot\underline{\nabla}\frac{\delta G}{\delta M_i^s}) \\
&+ a^s\rho^s(\frac{\delta F}{\delta\underline{M}^s}\cdot\frac{\delta G}{\delta\underline{E}} - \frac{\delta G}{\delta\underline{M}^s}\cdot\frac{\delta F}{\delta\underline{E}} + \underline{B}\cdot\frac{\delta F}{\delta\underline{M}^s}\times\frac{\delta G}{\delta\underline{M}^s})] \\
&+ \int d^3x\ (\frac{\delta F}{\delta\underline{E}}\cdot\mathrm{curl}\ \frac{\delta G}{\delta\underline{B}} - \frac{\delta G}{\delta\underline{E}}\cdot\mathrm{curl}\ \frac{\delta F}{\delta\underline{B}})
\end{aligned}\tag{32}$$

and Hamiltonian energy function

$$E = \int\{\sum_s [\frac{1}{2\rho^s} |\underline{M}^s|^2 + \rho^s e^s(\rho^s)] + \tfrac{1}{2}|\underline{E}|^2 + \tfrac{1}{2}|\underline{B}|^2\}d^3x \quad . \tag{33}$$

The time development of any functional, F, of the MFP dynamical variables obeys

$$\partial_t F = \{F,E\} \quad .$$

Moreover, one readily shows that the static Maxwell equations (31) correspond to the following Casimirs,

$$G_{\underline{E}} = \int \phi(\underline{x})(-\mathrm{div}\ \underline{E} + \sum_s a^s \rho^s)d^3x$$

$$G_{\underline{B}} = \int \tilde{\Phi}(\underline{x})(-\mathrm{div}\ \underline{B})d^3x \quad .$$

Each of the quantities $G_{\underline{E}}$, $G_{\underline{B}}$, for arbitrary functions $\phi, \tilde{\phi}$, Poisson commutes using (32) with every Hamiltonian $H[\rho^s, \underline{M}^s, \underline{E}, \underline{B}]$. Thus, not only the equations of motion, but the Poisson bracket (32) itself preserves the static Maxwell equations.

III.A <u>Planar MFP Flows</u>. We consider now planar MFP motion in some domain $D \subset \mathbb{R}^2$ in the (x,y) plane. In order that such motion remain planar, each of the dependent variables $\{\rho^s, \underline{v}^s, \underline{E}, \underline{B}\}$ must be functions only of (x,y t); $\underline{v}^s$ and $\underline{E}$ must lie in the (x,y) plane; and $\underline{\omega}^s$ and $\underline{B}$ must be directed normally to the plane, along $\hat{z}$,

$$\begin{aligned} \underline{\omega}^s &= \hat{z}\ \omega^s(x,y,t) \\ \underline{B} &= \hat{z}\ B(x,y,t) \quad . \end{aligned} \tag{34}$$

The planar MFP equations are

$$\begin{aligned} \partial_t B &= -\hat{z}\cdot \mathrm{curl}\ \underline{E} = E_{1,2} - E_{2,1} \quad , \\ \partial_t \underline{E} &= \underline{\nabla} B \times \hat{z} - \sum_s a^s \rho^s \underline{v}^s \\ \partial_t \rho^s &= -\mathrm{div}\ \rho^s \underline{v}^s \\ \partial_t \underline{v}^s &= -(\omega^s + a^s B)\ \hat{z} \times \underline{v}^s - \underline{\nabla}(\tfrac{1}{2}|\underline{v}^s|^2 + h^s(\rho^s)) + a^s \underline{E} \quad , \end{aligned} \tag{35}$$

where $h^s(\rho^s)$ is specific enthalpy, related to pressure p^s and specific internal energy e^s by

$$h^s = e^s + p^s/\rho^s$$
$$dh^s = (\rho^s)^{-1} dp^s \quad . \tag{36}$$

For a single fluid species and when $|\underline{E}|$ and B are absent, these equations reduce to the equations for planar motion of a barotropic fluid, whose stability criteria are proven by Arnold's method in Holm et al. [1983]. Taking the curl of the planar motion equation and using the continuity equation leads to the advected quantitites Ω^s, the so-called "modified vorticities",

$$\frac{d}{dt^s}\Omega^s = 0 \quad , \Omega^s := (\omega^s + a^s B)/\rho^s \quad , \tag{36}$$

with species material derivative

$$\frac{d}{dt^s} := \frac{\partial}{\partial t} + \underline{v}^s \cdot \underline{\nabla} \tag{37}$$

along the flow lines of each species. In view of (36) and the continuity equation for each s, for every real valued function of a real variable $\Phi^s(\zeta)$, each functional

$$F_{\Phi^s}(\Omega^s) := \iint \rho^s \Phi^s(\Omega^s) dxdy \tag{38}$$

is conserved by the planar HFP equations (provided the integral exists and the solutions are smooth; Ω^s would be created at a discontinuity). Another conserved quantity is the energy (33) expressed in two dimensions,

$$E := \iint_D \{\sum_s [\tfrac{1}{2}\rho^s|\underline{v}^s|^2 + \rho^s e^s(\rho^s)] + \tfrac{1}{2}|\underline{E}|^2 + \tfrac{1}{2}B^2\} dxdy \quad .$$

Either by direct computation from the Poisson bracket (32) specialized to planar motion, or by showing invariance under the coadjoint action of the semidirect product group whose Lie-Poisson bracket is a key ingredient of (32), one may readily show that each functional $F_{\Phi^s}(\Omega^s)$ in (38) is a

Casimir,

$$\{F_{\phi^s}, G\} = 0 \quad , \quad \forall\, G[\rho^s, \underline{M}^s, \underline{E}, B] \quad .$$

Likewise, Gauss's Law in (31) corresponds to the following Casimir,

$$G_{\underline{E}} = \int_D \phi(\underline{x})(-\mathrm{div}\ \underline{E} + \sum_s a^s\rho^s)\ dxdy \quad . \tag{39}$$

The Casimir $G_{\underline{B}}$ mentioned earlier is identically zero in two dimensions with $\underline{B}$ normal to the plane.

Equilibrium States. The equilibrium states ρ_e^s, $\underline{v}_e^s$, $\underline{E}_e$, B_e, of the system (35) in the (x,y) plane are the stationary, two-dimensional, barotropic MFP flows. For such stationary flows, one has the relations

$$\begin{aligned}
&\underline{E}_e = -\underline{\nabla}\phi_e \\
&\underline{\nabla}B_e \times \hat{z} = \sum_s a^s\rho_e^s\underline{v}_e^s \\
&\mathrm{div}\ \rho_e^s\underline{v}_e^s = 0 \\
&\underline{v}^s_e \cdot \underline{\nabla}[\tfrac{1}{2}|\underline{v}_e^s|^2 + h^s(\rho_e^s) + a^s\phi_e] = 0 \\
&\underline{v}_e^s \cdot \underline{\nabla}\,\Omega_e^s = 0 \quad .
\end{aligned} \tag{40}$$

According to the last two equations in (40), the gradient vectors $\underline{\nabla}\Omega_e^s$ and $\underline{\nabla}(\frac{1}{2}|\underline{v}_e^s|^2 + h^s(\rho_e^s) + a^s\phi_e)$ are orthogonal to the equilibrium species velocity $\underline{v}_e^s$. Consequently, these two gradient vectors are collinear, provided they or the velocity do not vanish. A sufficient condition for such collinearity in the plane is the functional relationship

$$\tfrac{1}{2}|\underline{v}_e^s|^2 + h^s(\rho_e^s) + a^s\phi_e = k^s(\Omega_e^s) \quad , \tag{41}$$

for certain functions $k^s(\zeta)$, $\zeta \in \mathbb{R}$; these are called the Bernoulli functions and (41) represents Bernoulli's Law for each species. Either applying the operator $(\Omega_e^s)^{-1}\ \hat{z} \times \underline{\nabla}$ to (41), or simply vector multiplying by $\hat{z}$ the stationary motion equation,

$$(\omega_e^s + a^sB_e)\ \hat{z} \times \underline{v}_e^s = -\underline{\nabla}(\tfrac{1}{2}|\underline{v}_e^s|^2 + h^s(\rho_e^s) + a^s\phi_e) \quad , \tag{42}$$

gives the relation

$$\rho_e^s \underline{v}_e^s = \frac{k^{s\prime}(\Omega_e^s)}{\Omega_e^s} \hat{z} \times \underline{\nabla}\Omega_e^s = \frac{1}{\Omega_e^s} \hat{z} \times \underline{\nabla} k^s(\Omega_e^s) \quad , \tag{43}$$

where prime ' denotes derivative of a function with respect to its stated argument. Substitution of (43) into the second equation in (40) (i.e., Ampere's Law) leads to another relation for stationary flows,

$$\underline{\nabla} B_e = -\sum_s \frac{a^s}{\Omega_e^s} \underline{\nabla}\, k^s(\Omega_e^s) \quad . \tag{44}$$

Relations (43) and (44) will be useful in establishing the following proposition.

Proposition. For smooth solutions with velocity fields parallel to the boundary and fixed circulation on the boundary, a stationary solution ($\underline{v}_e^s$, ρ_e^s, $\underline{E}_e$, B_e) *of the ideal planar MFP equations is a conditional extremum of the total energy* E *for fixed Casimirs* F_{Φ^s} *and* $G_{\underline{E}}$, *and an absolute extremum of* $H_F = E + F_{\Phi^s} + G_{\underline{E}}$, *where* $\phi = \phi_e$ *and*

$$\Phi^s(\zeta) = \zeta\left(\int^{\zeta} \frac{k^s(t)}{t^2}\, dt + \text{const}\right) \quad , \tag{45}$$

k^s being the Bernoulli function of species s.

The functional H_F in the Proposition is, explicitly,

$$H_F(\underline{v}^s, \rho^s, \underline{E}, B) = \iint_D \{\sum_s [\tfrac{1}{2}\rho^s|\underline{v}^s|^2 + \rho^s e^s(\rho^s) + \rho^s\Phi^s(\Omega^s)]$$

$$+ \tfrac{1}{2}|\underline{E}|^2 + \tfrac{1}{2}B^2 + \phi(\underline{x})(-\text{div}\,\underline{E} + \sum_s a^s\rho^s)\}\, dxdy \ . \tag{46}$$

After integration by parts, the variational derivative δH_F in the direction

$(\delta\underline{v}^s, \delta\rho^s, \delta\underline{E}, \delta B)$ becomes

$$\delta H_F = DH_F(\underline{v}^s, \rho^s, \underline{E}, B) \cdot (\delta\underline{v}^s, \delta\rho^s, \delta\underline{E}, \delta B)$$

$$= \iint_D \{ \sum_s [\tfrac{1}{2}|\underline{v}^s|^2 + h^s(\rho^s) + a^s\phi + \Phi^s(\Omega^s) - \Omega^s\Phi^{s\prime}(\Omega^s)]\, \delta\rho^s$$

$$+ \sum_s [\rho^s\underline{v}^s - \hat{z} \times \underline{\nabla}\, \Phi^{s\prime}(\Omega^s)] \cdot \delta\underline{v}^s + [B + \sum_s a^s\Phi^{s\prime}(\Omega^s)]\, \delta B \qquad (47)$$

$$+ (\underline{E} + \underline{\nabla}\phi) \cdot \delta\underline{E} + (-\mathrm{div}\, \underline{E} + \sum_s a^s\rho^s)\delta\phi\}\, dxdy$$

$$+ \int_{\partial D} \sum_s \Phi^{s\prime}(\Omega^s)\, \delta\underline{v}^s \cdot \underline{d\ell} + \int_{\partial D} \phi\delta\underline{E} \cdot (\hat{z} \times \underline{d\ell})$$

where $\underline{d\ell}$ is the line element along the boundary ∂D. For a stationary solution, the connected components of the boundary ∂D are both streamlines and equipotential lines. Thus, Ω_e^s and ϕ_e are constants on ∂D and the boundary integrals become

$$\sum_s \Phi^{s\prime}(\Omega_e^s)\Big|_{\partial D} \int_{\partial D} \delta\underline{v}^s \cdot \underline{d\ell} + \phi_e\Big|_{\partial D} \int_{\partial D} \delta\underline{E} \cdot (\hat{z} \times \underline{d\ell}) \quad . \qquad (48)$$

Let the variations $\delta\underline{v}^s$ and $\delta\underline{E}$ satisfy $\int_{\partial D} \delta\underline{v}^s \cdot \underline{d\ell} = 0$ and $\int_{\partial D} \delta\underline{E} \cdot \hat{z} \times \underline{d\ell} = 0$, respectively. Then the boundary integrals in (48) each vanish. In equation (47), the $\delta\rho^s$ coefficient vanishes for a stationary flow obeying (41), provided that Φ^s is related to the Bernoulli function k^s by

$$k^s(\zeta) + \Phi^s(\zeta) - \zeta\Phi^{s\prime}(\zeta) = 0 \quad ,$$

from which equation (45) in the Proposition follows. Differentiating with respect to ζ implies $\zeta^{-1} k^{s\prime}(\zeta) - \Phi^{s\prime\prime}(\zeta) = 0$. Then the $\delta\underline{v}^s$ and δB coefficients in (47) each vanish, by (43) and (44), respectively, since $\underline{\nabla}\Phi^{s\prime}(\Omega_e) = (\Omega_e^s)^{-1}\, \underline{\nabla}\, k^s(\Omega_e^s)$. If $\phi = \phi_e$, the $\delta\underline{E}$ coefficient vanishes. Finally, the $\delta\phi$ coefficient in (47) vanishes, by Gauss's Law in (31). □

The quadratic form defined by the second derivative of H_F at the stationary solution is

$$D^2H_F(\underline{v}_e^s, \rho_e^s, \underline{E}_e, B_e) \cdot (\delta\underline{v}^s, \delta\rho^s, \delta\underline{E}, \delta B)^2 = \int_D \{\sum_s [\rho_e^s|\delta\underline{v}^s + \underline{v}_e^s\delta\rho^s/\rho_e^s|^2$$

$$+ (h^{s\prime}(\rho_e^s) - |\underline{v}_e^s|^2/\rho_e^s)(\delta\rho^s)^2 + \rho_e^s\, \Phi^{s\prime\prime}(\Omega_e^s)(\delta\Omega^s)^2]$$

$$+ (\delta B)^2 + |\delta\underline{E}|^2 + (-\mathrm{div}\, \delta\underline{E} + \sum_s a^s\delta\rho^s)\delta\phi\}\, dxdy \quad .$$

(49)

The last term vanishes for variations that satisfy Gauss's Law. Sufficient conditions for this quadratic form to be positive definite are:

$$\text{(i)}\quad h^{s\prime}(\rho_e^s) - |v_e^s|^2/\rho_e^s = ((c_e^s)^2 - |\underline{v}_e^s|^2)/\rho_e^s > 0 \quad , \tag{50}$$

where c_e^s is the sound speed of species s for the stationary solution, defined by $\rho_e^s h^{s\prime}(\rho_e^s) = (c_e^s)^2$, i.e., the stationary flow is everywhere subsonic; and

$$\text{(ii)}\quad (\Omega_e^s)^{-1}\, k^{s\prime}(\Omega_e^s) = \Phi^{s\prime\prime}(\Omega_e^s) > 0 \quad , \tag{51}$$

i.e., the two collinear gradient vectors $\underline{\nabla}(\frac{1}{2}|\underline{v}_e^s|^2 + h^s(\rho_e^s) + s^s\phi_e)$ and $\underline{\nabla}(\Omega_e^s)^2$ point in the same direction throughout the flow. For a single, incompressible fluid without charge ($s = 1$, $\rho_e^s = 1$, $\delta\rho^s = 0$, $\delta B = 0$, $\delta\underline{E} = 0$), formula (49) reduces to Arnold's formula, equation (19) in Section III.C, discussed earlier.

A priori estimates expressing stability. Lyapunov stability criteria for planar stationary flows of MFP in the smooth regime can be proved by establishing sufficient conditions that imply certain a priori estimates bounding perturbation growth in terms of the Bernoulli functions k^s. These estimates can be obtained readily by following the same convexity argument as in Theorem 1 of Holm et al. [1983] for planar barotropic flows. Thus, one obtains the following result.

THEOREM. Assume that each Bernoulli function k^s in (41) and internal energy density function $\varepsilon^s := \rho^s e^s(\rho^s)$ satisfies

$$0 < q^s \le \zeta^{-1} k^{s\prime}(\zeta) \le Q^s < \infty \tag{52}$$

where q^s and Q^s are positive constants and similarly,

$$0 < r^s \le \varepsilon^{s\prime\prime}(\tau^s) \le R^s < \infty \tag{53}$$

with constants r^s, R^s, and for all values of the arguments. Let ($\delta\underline{v}^s$, $\delta\rho^s$, $\delta\underline{E}$, δB) be a small, but finite, smooth perturbation of a stationary solution ($\underline{v}_e^s$, ρ_e^s, $\underline{E}_e$, B_e) and denote its value at $t = 0$ by ($\delta\underline{v}_0^s$, $\delta\rho_0^s$, $\delta\underline{E}_0$, δB_0). Let the circulation $\int_{\partial D} \delta\underline{v}_0^s \cdot \underline{d\ell}$ and integral $\int_{\partial D} \delta\underline{E}_0 \cdot \hat{z} \times \underline{d\ell}$ each vanish. Then the perturbation ($\delta\underline{v}^s$, $\delta\rho^s$, $\delta\underline{E}$, δB) of the stationary solution ($\underline{v}_e^s$, ρ_e^s, $\underline{E}_e$, B_e) at *any time t is estimated in terms of* ($\delta\underline{v}_0^s$, $\delta\rho_0^s$, $\delta\underline{E}_0$, δB_0) *by*

$$\int_D \{\sum_s [\frac{|\delta(\rho^s\underline{v}^s)|}{\rho_e^s + \delta\rho^s} + (r^s - \frac{|\underline{v}_e^s|^2}{\rho_e^s+\delta\rho^s})(\delta\rho^s)^2 + q^s(\rho_e^s + \delta\rho^s)(\delta\Omega^s)^2]$$

$$+ (\delta B)^2 + |\delta\underline{E}|^2\}dxdy$$

$$\leq \int_D \{\sum_s [\frac{|\delta(\rho^s\underline{v}^s)_0|^2}{\rho_e^s + \delta\rho_0^s} + (R^s - \frac{|\underline{v}_e^s|^2}{\rho_e^s+\delta\rho_0^s})(\delta\rho_0^s)^2 \tag{54}$$

$$+ Q^s(\rho_e^s + \delta\rho_0^s)(\delta\Omega_0^s)^2] + (\delta B_0)^2 + |\delta\underline{E}_0|^2\}dxdy \quad ,$$

where $\delta(\rho^s\underline{v}^s)$ *and* $\delta\Omega^s$ *are defined by*

$$\delta(\rho^s\underline{v}^s) = (\rho_e^s + \delta\rho^s)(\underline{v}_e^s + \delta\underline{v}^s) - \rho_e^s\underline{v}_e^s$$

$$\delta\Omega^s = (\omega_e^s + a^sB_e + \delta\omega^s + a^s\delta B)/(\rho_e^s + \delta\rho^s) - (\omega_e^s + a^sB_e)/\rho_e^s \quad .$$

Just as in Holm et al. [1983], the proof of the Theorem proceeds by showing that a conserved functional

$$\hat{H}_F(\delta\underline{v}^s,\ \delta\rho^s,\ \delta\underline{E},\ \delta B) = H_F(\underline{v}_e^s + \delta\underline{v}^s,\ \rho_e^s + \delta\rho^s,\ \underline{E}_e + \delta\underline{E},\ B_e + \delta B)$$

$$- H_F(v_e^s,\ \rho_e^s,\ \underline{E}_e,\ B_e) - DH_F(\underline{v}_e^s,\ \rho_e^s,\ \underline{E}_e,\ B_e) \cdot (\delta\underline{v}^s,\ \delta\rho^s,\ \delta\underline{E},\ \delta B) \tag{55}$$

is bounded from below (above) by the left (right) hand side of (54). The a priori estimate (54) then implies Lyapunov stability for smooth solutions, provided $\rho_e^s + \delta\rho^s$ remains finite and bounded away from zero. Under such an additional hypothesis on the density, one has the following result.

Corollary 1. Let a stationary solution satisfy (41) for smooth functions $k^s(\zeta)$. *Assume that*

$$0 < q^s \leq \zeta^{-1} k^{s\prime}(\zeta) \leq Q^s < \infty \tag{56}$$

and

$$0 < \frac{(C^s_{min})^2}{\rho^s_{max}} \leq \varepsilon^{s}{}''(\tau^s) \leq \frac{(C^s_{max})^2}{\rho^s_{min}} < \infty \tag{57}$$

for all $\zeta \in \mathbb{R}$ *and* τ^s *such that* $0 < \rho^s_{min} \leq \tau^s \leq \rho^s_{max} < \infty$ *where* q^s, Q^s, $(C^s_{min})^2$, $(C^s_{max})^2$, ρ^s_{min}, ρ^s_{max}, *are positive constants. Also assume that*

$$0 < \lambda^s := (C^s_{min})^2/\rho^s_{max} - |\underline{v}^s_e|^2/\rho^s_{min} < \Lambda^s < \infty \tag{58}$$

for some other positive constants λ^s, Λ^s. *Then with the same definitions as in the Theorem, the following estimates obtain,*

$$\int_D \{\sum_s [\frac{|\delta(\rho^s \underline{v}^s)|^2}{\rho^s_{max}} + \lambda^s(\delta\rho^s)^2 + q^s\rho^s_{min}(\delta\Omega^s)^2] + (\delta B)^2 + |\delta\underline{E}|^2\}dxdy$$

$$\leq \int_D \{\sum_s [\frac{|\delta(\rho^s \underline{v}^s)_0|^2}{\rho^s_{min}} + \Lambda^s(\delta\rho^s_0)^2 + Q^s\rho^s_{max}(\delta\Omega^s_0)^2]$$

$$+ (\delta B_0)^2 + |\delta\underline{E}_0|^2\}dxdy \quad , \tag{59}$$

for solutions with densities satisfying $\rho^s_{min} \leq \rho^s \leq \rho^s_{max}$.

Corollary 1 follows immediately from the Theorem by replacing (53) by (57), imposing (58), and bounding ρ^s.

Remark. The a priori estimate (59) in Corollary 1 implies stability for smooth, planar, MFP solutions, in the sense of a norm estimate of small, but finite, circulation-preserving perturbations obeying Gauss's Law, that develop from a perturbed, initially steady flow. Because of the method of proof for the Theorem, the right hand side of the inequality (59) in Corollary 1 can be minimized by replacing it with $\hat{H}_F(\delta\underline{v}^s_0, \delta\rho^s_0, \delta\underline{E}_0, \delta B_0)$. Thus, we have shown another corollary.

Corollary 2. *Under the assumptions of Corollary 1 and the Theorem, the following a priori estimates obtain,*

$$\hat{H}_F(\delta\underline{v}^s_0,\ \delta\rho^s_0,\ \delta\underline{E}_0,\ \delta B_0) \geq \int_D \{\sum_s [\frac{|\delta(\rho^s\underline{v}^s)|^2}{\rho^s_{max}} + \lambda^s(\delta\rho^s)^2 + q^s\rho^s_{min}(\delta\Omega^s)^2]$$

$$+ (\delta B)^2 + |\delta\underline{E}|^2\}dxdy > 0 \quad . \tag{60}$$

When there is only a single fluid species and electromagnetic fields are absent, the result of the Theorem reduces to the estimate in Holm et al. [1983] for planar barotropic flow. These estimates can break down when smooth solutions cease to exist; for example, upon occurence of cavitation, and/or the formation of shocks from an initially-smooth, steady flow. When these phenomena occur, however, it is questionable whether the barotropic approximation should still be used. One could exclude cavitation by replacing (54) by an estimate as in Holm et al. [1983], modeling an elastic fluid. None of the estimates in this section apply to three-dimensional phenomena. That topic is discussed in Holm et al. [1984].

<u>Example. Subsonic Shear Flows</u>. A stationary solution of the MFP equations (35) in the strip $\{(x,y) \subset \mathbb{R}^2 \mid Y_1 \leq y \leq Y_2\}$ is a plane parallel flow along x, admitting arbitrary velocity profile $\underline{v}^s_e(x,y) = (\bar{v}^s(y),0)$, electrostatic potential $\phi_e(x,y) = \bar{\phi}(y)$, and density $\rho^s_e(x,y) = \bar{\rho}^s(y)$. The density profile is subject only to the subsonic condition (50), expressible as

$$\frac{d\bar{p}^s}{d\bar{\rho}^s}(y) - (\bar{v}^s(y))^2 > 0 \tag{61}$$

and depending on the barotropic relation $\bar{p}^s = p^s(\bar{\rho}^s)$. In this domain, the independent variable x can be either unrestricted on the entire real line, or periodic. The former case requires that initial perturbations be sufficiently integrable for $\hat{H}_F(\delta\underline{v}^s_0,\ \delta\rho^s_0,\ \delta\underline{E}_0,\ \delta B_0)$ to be finite and, thus, give a meaningful upper bound in (60).

To determine the limits of stability for subsonic stationary planar MFP flows, we proceed as follows. (i) Choose profiles $\bar{v}^s(y)$, $\bar{\phi}(y)$, and $\bar{\rho}^s(y)$, satisfying the subsonic condition (61). Relations (43) and (44) then imply y-dependence only, for magnetic field and modified vorticity: $B_e(x,y) = \bar{B}(y)$, $\Omega^s_e(x,y) = \bar{\Omega}^s(y)$. (ii) Use Ampere's Law in the form (44) to determine $\bar{B}(y)$

from $\bar{\rho}^s(y)$ and $\bar{v}^s(y)$, then compute $\bar{\Omega}^s(y)$ from its definition (36) in terms of $\bar{\rho}^s$, $\bar{v}^s$, $\bar{B}$. (iii) Solve for an expression for the quantity $(\bar{\Omega}^s)^{-1}k^{s\prime}(\bar{\Omega}^s)$ appearing in condition (52) of the stability theorem and consider its sign, thereby determining the limits of stability in terms of the profiles $\bar{\rho}^s(y)$, $\bar{v}^s(y)$, $\bar{B}(y)$.

Given the profiles $\bar{v}^s(y)$, $\bar{\rho}^s(y)$, and $\bar{\phi}(y)$, one finds $\bar{\omega}^s(y)$ and $\bar{\Omega}^s(y)$ from their definitions

$$\omega_e^s = \hat{z} \cdot \text{curl } \underline{v}_e^s = - \bar{v}^{s\prime}(y) =: \bar{\omega}^s(y) \tag{62}$$

and

$$\Omega_e^s = (\rho_e^s)^{-1}(\omega_e^s + a^s B_e) = (\bar{\rho}^s(y))^{-1}(-\bar{v}^{s\prime}(y) + a^s\bar{B}(y)) =: \bar{\Omega}^s(y) \quad . \tag{63}$$

Equations (43) and (44) give the relations

$$\bar{\rho}^s(y)\ \bar{v}^s(y) = - \frac{1}{\bar{\Omega}^s}\ k^{s\prime}(\bar{\Omega}^s)\ \bar{\Omega}^{s\prime}(y) \tag{64}$$

and

$$\bar{B}'(y) = \sum_s a^s\bar{\rho}^s(y)\bar{v}^s(y) \quad , \tag{65}$$

which determine $\bar{B}(y)$ and $(\bar{\Omega}^s)^{-1}k^{s\prime}(\bar{\Omega}^s)$. Solving (64) gives the formula

$$(\bar{\Omega}^s)^{-1}\ \frac{dk^s(\bar{\Omega}^s)}{d\bar{\Omega}^s} = - \frac{\bar{\rho}^s(y)\bar{v}^s(y)}{d\bar{\Omega}^s/dy}$$

$$= \frac{(\bar{\rho}^s)^2\ \bar{v}^s}{\bar{v}^{s\prime\prime} - (\bar{\rho}^{s\prime}/\bar{\rho}^s)\bar{v}^{s\prime} + a^s\bar{B}(\bar{\rho}^{s\prime}/\bar{\rho}^s - \bar{B}'/\bar{B})} \quad . \tag{66}$$

where, e.g., $\bar{v}^{s\prime\prime} = d^2\bar{v}^s(y)/dy^2$, $\bar{B}' = d\bar{B}(y)/dy$, etc. Thus, control of positivity of $(\bar{\Omega}^s)^{-1}k^{s\prime}(\bar{\Omega}^s)$ in (56) and, hence, of stability for MFP involves an interplay among velocity, density and magnetic field profiles, through the positivity condition,

$$\frac{\bar{v}^s(y)}{-\bar{\Omega}^{s\prime}(y)} > 0 \quad . \tag{67}$$

Given that an inertial frame can be chosen so that condition (67) holds, planar MFP flows will be stable, provided

$$\bar{\Omega}^{s\,\prime}(y) \neq 0 \quad . \tag{68}$$

We consider several cases.

Case A. In the case of neutral fluids ($a^s = 0$) and stationary flows with constant density ($\bar{\rho}^{s\,\prime}(y) = 0$), positivity of $(\bar{\Omega}^s)^{-1}k^{s\,\prime}(\bar{\Omega}^s)$ (67) reduces to

$$\bar{v}^s(y)/\bar{v}^{s\,\prime\prime}(y) > 0 \quad . \tag{69}$$

Provided an inertial frame can be chosen so that (69) holds throughout domain D, one recovers Rayleigh's criterion (24) for stability of shear flows: all flows in this case with no inflection points in their velocity profile are stable.

Case B. For the case of charged fluids ($a^s \neq 0$) at constant density ($\bar{\rho}^{s\,\prime}(y) = 0$), positivity in (67) reduces to

$$\frac{(\bar{\rho}^s)^2\bar{v}^s}{\bar{v}^{s\,\prime\prime} - a^s\bar{B}^{\prime}} > 0 \tag{70}$$

Provided an inertial frame can be chosen in which (70) holds throughout D, one obtains the following criterion for stability in this case,

$$\bar{v}^{s\,\prime\prime}(y) \neq a^s\bar{B}^{\prime}(y) \quad . \tag{71}$$

Case C. In the general MFP case, with charged, compressible fluids, ($a^s \neq 0$, $\bar{\rho}^{s\,\prime}(y) \neq 0$), when an inertial frame exists in which (67) holds, the stability condition (68) becomes

$$\bar{v}^{s\,\prime\prime} \neq (\bar{\rho}^{s\,\prime}/\bar{\rho}^s)(\bar{v}^{s\,\prime} - a^s\bar{B}) + a^s\bar{B}^{\prime} \quad , \tag{72}$$

which involves all three stationary profiles.

Note that the conditions obtained here by Arnold's method are *sufficient* for stability. Thus, violation of these conditions would be *necessary* for the onset of instability but *not necessary and sufficient*, except in the fortunate event where they coincide with instability conditions found by linear analysis.

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