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TITLE TEMPERATURE AND HEAT-FLUX DISTRIBUTIONS IN A STRIP-HEATED COMPOSITE SLAB

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TEMPERATURE AND HEAT-FLUX DISTRIBUTIONS IN A STRIP-HEATED COMPOSITE SLAB*

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ABSTRACT

The steady temperature and heat-flux distributions for a composite slab consisting of a strip-heated, large-conductivity fin in intimate contact with a small-conductivity, convectively cooled substrate are obtained. Such a problem has application to the strip heating of process equipment and laboratory experiments where uniform thermal conditions are desired and where the conductivity of the substrate is small. Analytical methods are used to obtain closed-form solutions for the local temperature and heat flux for the full, two-dimensional problem and for the bounding case of no transverse conduction within the substrate. A design procedure to determine the strip-heater spacing necessary for a prescribed maximum variation in heat flux at the convectively cooled surface is presented. An application example is given and the results discussed. Expressions for the steady temperature and heat flux are also obtained for the limiting cases of infinite heat transfer coefficient and zero-thickness substrate.

INTRODUCTION

This paper concerns the problem of predicting the steady temperature and heat-flux distribution in a composite slab consisting of two dissimilar materials in intimate contact with each other. One material, of large conductivity, is heated on one side by equally spaced strips and the other material, which has a small conductivity, is cooled on the opposite side by convection to a fluid whose temperature is constant over the distance between the strips. The materials of large and small conductivity materials are referred to as the fin and the substrate, respectively. Such a problem has application to the strip-heating of process equipment and laboratory

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experiments where uniform or near-uniform temperatures or heat fluxes are required.* For the situation where the conductivity of the substrate is small (such as for glass or plastic materials), a large number of closely spaced strip heaters would be necessary to achieve uniformity. To reduce both the number and cost for such heaters, a thin sheet of material having a larger conductivity (a fin) is placed between the heat sources and the substrate. Accordingly, the flow of heat is distributed evenly between more coarsely spaced strips.

The problem of heat flow through a strip-heated single slab has been solved previously by Van Sant (Ref. 2) for the temperature distribution in a convectively cooled slab for constant heat flux and constant-temperature strips and by Schmitz (Ref. 3) for the heat-flux distribution in a slab having constant-temperature strips and cooled by a constant-temperature opposite side.

In this paper, a closed-form analytical solution is obtained from the simultaneous solution of the steady energy equation for both constituents of the composite medium. Although expressions for both local temperature and local heat flux are obtained, primary focus will be on the heat flux distribution at the convectively cooled side of the slab, this being the side over which uniformity is desired. A design procedure to determine the strip-heater spacing necessary for a prescribed maximum variation in heat flux, ϵ, at this surface is presented. Graphs and formulae for ϵ are developed for the upper-bounding case of no transverse conduction in the substrate (a quasi two-dimensional case) and for the case where two-dimensional heat conduction effects are considered. Expressions for the steady temperature and heat flux are also obtained for the limiting cases of infinite heat transfer coefficient (constant-temperature cooled surface) and zero-thickness substrate. Finally, an application example is presented and the results discussed.

ANALYSIS

The geometry for the problem at hand is shown in Fig. 1. The strip heater, which runs perpendicular to the plane of the figure, is positioned above the fin in region $0 \leq x \leq t$, where t is the half width of the strip. The heat-flow rate over this area per unit depth of the heater is Q'. In the

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*Processes involving natural convection, for instance, are sensitive to the imposed thermal-boundary conditions and may necessitate very uniform boundary heat fluxes or temperatures (Ref. 1).

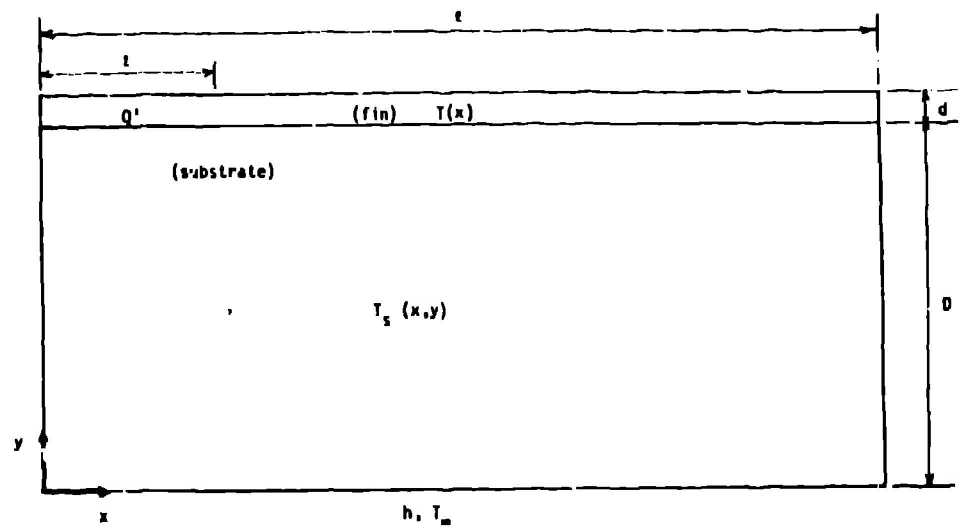

Fig. 1. Problem geometry.

region $t \le x \le \ell$, the fin is insulated on top. The side boundaries at $x = 0$ and $x = \ell$ are lines of symmetry, thus making the spacing between strip-heater centerlines a distance 2ℓ. In addition to the previously mentioned assumptions, we further impose the usual condition that $d/\ell \ll 1$ so that conduction in the fin in the y direction may be lumped. The steady, constant-property energy equation and boundary conditions for the fin and substrate become

$$\frac{Q'}{t} + kd\,\frac{d^2T}{dx^2} - k_s\,\frac{\partial T_s}{\partial y}\bigg|_{y=D} = 0 \qquad 0 \le x \le t\ , \qquad (1a)$$

$$kd\,\frac{d^2T}{dx^2} - k_s\,\frac{\partial T_s}{\partial y}\bigg|_{y=D} = 0 \qquad t \le x \le \ell\ , \qquad (1b)$$

$$\frac{dT}{dx}\bigg|_{x=0} = 0 = \frac{dT}{dx}\bigg|_{x=\ell}\ , \qquad (1c)$$

$$T(x = t_+) = T(x = t_-)\ ,$$

$$\left.\frac{dT}{dx}\right|_{x = t_+} = \left.\frac{dT}{dx}\right|_{x = t_-} , \tag{1d}$$

$$\frac{\partial^2 T_s}{\partial x^2} + \frac{\partial^2 T_s}{\partial y^2} = 0 . \tag{2a}$$

and $$\left.\frac{\partial T_s}{\partial x}\right|_{x = 0} = 0 = \left.\frac{\partial T_s}{\partial x}\right|_{x = \ell} , \tag{2b}$$

$$k_s \left.\frac{\partial T_s}{\partial y}\right|_{y = 0} = h \, [T_s(x, y = 0) - T_\infty] , \tag{2c}$$

$$T_s \, (x, y = D) = T(x) , \tag{2d}$$

where the left-hand side of Eqs. (1c) and (1d) refers to the temperature distribution over $0 \leq x \leq t$ and the right-hand side refers to that over $t \leq x \leq \ell$. By imposing Eqs. (1d) and (2d), temperature and heat flux continuity between the two fin regions and between the fin and substrate are ensured. Such conditions must always be satisfied in conjugate problems, of which this is one.

The governing equations and boundary conditions are nondimensionalized by

$$\begin{aligned} &\phi \equiv \frac{T - T_\infty}{Q'/k} , \quad \eta \equiv \frac{x}{\ell} , \quad \xi \equiv \frac{y}{D} , \quad \lambda \equiv \frac{k_s}{k} , \\ &\gamma \equiv \frac{t}{\ell} , \quad \nu \equiv \frac{d}{\ell} , \quad \mu \equiv \frac{D}{\ell} , \quad Bi \equiv \frac{hD}{k_s} , \end{aligned} \tag{3}$$

where Bi is the Biot number. Eqs. (1) and (2) become

$$\frac{d^2\phi}{d\eta^2} + \frac{1}{\nu\gamma} - \frac{\lambda}{\mu\nu} \left.\frac{\partial\phi_s}{\partial\xi}\right|_{\xi = 1} = 0 \qquad 0 \le \eta \le \gamma \quad , \tag{4a}$$

$$\frac{d^2\phi}{d\eta^2} - \frac{\lambda}{\mu\nu} \left.\frac{\partial\phi_s}{\partial\xi}\right|_{\xi = 1} = 0 \qquad \gamma \le \eta \le 1 \quad , \tag{4b}$$

$$\left.\frac{d\phi}{d\eta}\right|_{\eta = 0} = 0 = \left.\frac{d\phi}{d\eta}\right|_{\eta = 1} \tag{4c}$$

$$\begin{aligned} &\phi(\eta = \gamma) = \phi(\eta = \gamma) \quad , \\ &\left.\frac{d\phi}{d\eta}\right|_{\eta = \gamma} = \left.\frac{d\phi}{d\eta}\right|_{\eta = \gamma} \quad , \end{aligned} \tag{4d}$$

$$\mu^2 \frac{\partial^2\phi_s}{\partial\eta^2} + \frac{\partial^2\phi_s}{\partial\xi^2} = 0 \quad , \tag{5a}$$

$$\left.\frac{\partial\phi_s}{\partial\eta}\right|_{\eta = 0} = 0 = \left.\frac{\partial\phi_s}{\partial\eta}\right|_{\eta = 1} \quad , \tag{5b}$$

$$\left.\frac{\partial\phi_s}{\partial\xi}\right|_{\xi = 0} - Bi\ \phi_s\ (\eta, \xi = 0) = 0 \quad , \tag{5c}$$

$$\phi_s\ (\eta, \xi = 1) = \phi(\eta) \quad , \tag{5d}$$

where the same restrictions imposed on Eqs. (1c) and (1d) are applied to Eqs. (4c) and (4d).

The traditional separation-of-variables method of solution fails for the above system because the boundary condition at $\xi = 1$ [Eq. (5d)] is an as-yet undetermined function of ϕ and thus, ϕ_s, this being a direct consequence of the conjugate nature of the problem. In addition, because μ is small,* Eq. (5a) is "stiff" in the η direction, and attempts at conventional numerical methods of solution also fail because Eq. (5b) cannot be satisfied. It is because of the diminutive nature of μ that a perturbation method of solution (Ref. 3) is attempted here using μ^2 as the perturbation parameter.

We seek a solution, ϕ_s, in the form

$$\phi_s(\eta, \xi) = \phi_{s0}(\eta, \xi) + \mu^2 \phi_{s2}(\eta, \xi) + O(\mu^4) \quad . \tag{6}$$

Combining Eqs. (6) and (5a) and equating the coefficients of the different powers of μ, we obtain

$$\frac{\partial^2 \phi_{s0}}{\partial \xi^2} = 0 \quad , \tag{7a}$$

and
$$\frac{\partial^2 \phi_{s0}}{\partial \eta^2} + \frac{\partial^2 \phi_{s2}}{\partial \xi^2} = 0 \quad . \tag{7b}$$

Integrating Eq. (7a) and applying Eqs. (5c) and (5d), we obtain the zeroth-order solution as

$$\phi_{s0}(\eta, \xi) = \phi(\eta) \left| \frac{1 + Bi\ \xi}{1 + Bi} \right| \quad . \tag{8}$$

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*In the physical problem, we attempt to make ℓ as large as possible by ensuring thermal uniformity by adding a large-conductivity fin at the top of a low-conductivity substrate. For all cases of practical concern, ℓ may be made large enough so that $D/\ell \equiv \mu << 1$.

Combining Eqs. (8) and (7b), integrating, applying Eqs. (5c) and (5d), and rearranging we produce the following from Eq. (6):

$$\phi_s(\eta,\xi) = \frac{1 + Bi\ \xi}{1 + Bi} \left\{ (1 + \mu^2)\ \phi(\eta) - \mu^2 \left[\frac{Bi}{6(1 + Bi)} \frac{\partial^2\phi}{\partial\eta^2} \left(\frac{\xi^3(1 + Bi)}{(1 + Bi\ \xi)} - 1 \right) + \frac{1}{2(1 + Bi)} \frac{\partial^2\phi}{\partial\eta^2} \left(\frac{\xi^2(1 + Bi)}{(1 + Bi\ \xi)} - 1 \right) \right] \right\} , \tag{9}$$

which satisfies Eq. (5b) because of Eq. (4c). In the limit as $\mu^2 \to 0$, Eq. (9) reduces to Eq. (8), as we expect.

The fin temperature $\phi(\eta)$ is obtained next by combining Eq. (9) and Eqs. (4a) and (4b) and applying Eq. (4d). First, the function $\partial\phi_s/\partial\xi \big|_{\xi = 1}$ is evaluated from Eq. (9) as

$$\frac{\lambda}{\mu\nu} \frac{\partial\phi_s}{\partial\xi} \bigg|_{\xi = 1} = A\ \phi(\eta) - B \frac{\partial^2\phi}{\partial\eta^2} , \tag{10a}$$

where

$$A = \frac{\lambda\ Bi\ (1 + \mu^2)}{\mu\nu\ (1 + Bi)} , \tag{10b}$$

$$B = \frac{\lambda\ Bi\ \mu\ \Lambda}{\nu\ (1 + Bi)} , \text{ and} \tag{10c}$$

$$\Lambda = \frac{Bi^2 + 3\ Bi + 3}{3\ Bi\ (1 + Bi)} . \tag{10d}$$

Combining Eqs. (4a) and (4b) and Eq. (10a) we obtain

$$\frac{\partial^2 \phi}{\partial \eta^2} - p^2 \phi = \frac{-1}{\gamma \nu \ (B + 1)} \qquad 0 \leq \eta \leq \gamma \ , \tag{11a}$$

$$\frac{\partial^2 \phi}{\partial \eta^2} - p^2 \phi = 0 \qquad \gamma \leq \eta \leq 1 \ , \tag{11b}$$

where

$$p^2 = \frac{A}{B + 1} = \frac{1 + \mu^2}{\Lambda\mu^2 + \gamma^2/m^2} = \frac{(t/\gamma D)^2 + 1}{\Lambda + (t/Dm)^2} \ , \tag{11c}$$

$$m^2 = \frac{ht^2/kd}{1 + hD/k_s} \ , \tag{11d}$$

and we further note that m is independent of the fin half-length, ℓ.

Eqs. (11a) and (11b) are now solved and the constants of integration evaluated by Eqs. (4c) and (4d). The resulting distribution is

$$\phi(\eta) = \frac{\gamma}{m^2 \nu \ (1 + \mu^2)} \left\{ 1 - \frac{\sinh p(1 - \gamma) \cosh p\eta}{\sinh p} \right\} \qquad 0 \leq \eta \leq \gamma \ , \tag{12a}$$

$$\phi(\eta) = \frac{\gamma \sinh p\gamma \cosh p(1 - \eta)}{m^2 \nu \ (1 + \mu^2) \sinh p} \qquad \gamma \leq \eta \leq 1 \ . \tag{12b}$$

The temperature distribution in the substrate is determined from Eq. (9), $\phi(\eta)$ and $\partial^2\phi/\partial\eta^2$ having been evaluated from Eqs. (12). Thus,

$$\phi_s(\eta, \zeta) = \frac{\gamma(1 + Bi \ \xi)}{n^2 \nu \ (1 + Bi)} \left\{ 1 - \frac{\sinh p(1 - \gamma) \cosh p\eta}{\sinh p} \right. \tag{13a}$$

$$\left[1 - (\Lambda\mu^2 + \gamma^2/m^2)^{-1} \left(\frac{Bi}{6(1 + Bi)} \left(\frac{\xi^3(1 + Bi)}{1 + Bi \ \xi} - 1 \right) + \right. \right.$$

$$\left.\left.\frac{1}{2(1 + Bi)} \left(\frac{\xi^2(1 + Bi)}{1 + Bi\ \xi} - 1\right)\right)\right]\Bigg\} \qquad 0 \le \eta \le \gamma \ ,$$

$$\phi_s(\eta, \xi) = \frac{\gamma(1 + Bi\ \xi)\sinh p\gamma \cosh p(1 - \eta)}{m^2\nu\ (1 + Bi)\sinh p} \Bigg\{ 1 -$$

$$(\Lambda\mu^2 + \gamma^2/m^2)^{-1} \left[\frac{Bi}{6(1 + Bi)} \left(\frac{\xi^3(1 + Bi)}{1 + Bi\ \xi} - 1\right) + \right. \tag{13b}$$

$$\left.\frac{1}{2(1 + Bi)} \left(\frac{\xi^2(1 + Bi)}{1 + Bi\ \xi} - 1\right)\right]\Bigg\} \qquad \gamma \le \eta \le 1 \ .$$

Temperature continuity at the surface ($\eta = \gamma$, ξ) in Eqs. (13) is guaranteed because Eqs. (13) satisfy Eq. (4d) at $\xi = 1$. Also, Eqs. (13) satisfy Eq. (5d) to each order in μ.

A dimensionless heat flux at the surface $\xi = 0$ is defined by

$$\hat{q}(\eta) = \frac{\ell}{Q'} q(x, y = 0) = \frac{\ell}{Q'} k_s \frac{\partial T_s}{\partial y}\Bigg|_{y = 0} = \frac{\lambda}{\mu} \frac{\partial \phi_s}{\partial \xi}\Bigg|_{\xi = 0} \ ,$$

and with Eqs. (13), this becomes

$$\hat{q}_{2D}(\eta) = \frac{1}{\gamma} \left\{ 1 - \frac{\sinh p(1 - \gamma) \cosh p\eta}{\sinh p} \left[1 + \frac{3 + Bi}{6(1 + Bi)[\Lambda + (t/Dm)^2]} \right] \right\} \tag{14a}$$

$$0 \le \eta \le \gamma \ ,$$

$$\hat{q}_{2D}(\eta) = \frac{\sinh p\gamma \cosh p(1 - \eta)}{\gamma \sinh p} \left[1 + \frac{3 + Bi}{6(1 + Bi)[\Lambda + (t/Dm)^2]} \right] \tag{14b}$$

$$\gamma \le \eta \le 1 \ .$$

The subscripts 2D in Eqs. (14) designate that the effect of η-direction conduction within the substrate was considered in the development of the expressions.

By inspecting Eqs. (11c) and (14), we note that the heat flux depends upon four parameters: Bi, t/D, m, and γ. However, in the limit as $\mu^2 \to 0$, $p^2 \to m^2/\gamma^2$ and the term within the large square brackets in Eqs. (14) becomes unity, eliminating the dependence upon Bi and t/D and simplifying the functional representation of the heat flux. In this limit, the effect of heat conduction in the η direction within the substrate is neglected and the analysis becomes quasi two dimensional. Because this component of conduction tends to make the temperature more uniform at the convectively cooled surface, the quasi two-dimensional case thus represents an upper bound on the nonuniformity of heat flux at that surface. This result is a direct consequence of the constant heat-transfer coefficient and coolant temperature assumed for this analysis.

For the quasi two-dimensional case, the heat flux at the surface, $\xi = 0$, becomes

$$\hat{q}_{qD}(\eta) = \frac{1}{\gamma} \left\{ 1 - \frac{\sinh m(\gamma^{-1} - 1) \cosh m\gamma^{-1}\eta}{\sinh m\gamma^{-1}} \right\} \qquad 0 \leq \eta \leq \gamma \quad , \qquad (15a)$$

$$\hat{q}_{qD}(\eta) = \frac{\sinh m \cosh m\gamma^{-1}(1 - \eta)}{\gamma \sinh m\gamma^{-1}} \qquad \gamma \leq \eta \leq 1 \quad , \qquad (15b)$$

where the subscript qD designates the quasi two-dimensional approximation.

To assess the degree of uniformity of heat flux over the surface $\xi = 0$, we define

$$\epsilon \equiv \frac{\hat{q}(\eta = 0, \xi = 0) - \hat{q}(\eta = 1, \xi = 0)}{\hat{q}(\eta = 0, \xi = 0)} \quad , \qquad (16)$$

which is the difference in heat flux between the locations $\eta = 0$ and $\eta = 1$ at $\xi = 0$, expressed as a fraction of that at $\eta = 0$ and $\xi = 0$. In particular, for the two-dimensional (2D) and quasi two-dimensional (qD) cases, Eq. (16) becomes

$$\epsilon_{2D} = 1 - \frac{G \sinh p\gamma}{\sinh p - G \sinh p(1 - \gamma)} \quad , \qquad (17a)$$

where

$$G = 1 + \frac{3 + Bi}{6(1 + Bi)[\Lambda + (t/Dm)^2]} \text{ , and}$$

$$\varepsilon_{qD} = 1 - \frac{\sinh m}{\sinh m\gamma^{-1} - \sinh m(\gamma^{-1} - 1)} \text{ ,} \qquad (17b)$$

respectively.

RESULTS

The heat-flux distribution from Eqs. (15) are shown in Fig. 2 for several m and γ combinations. We note that the heat flux becomes more

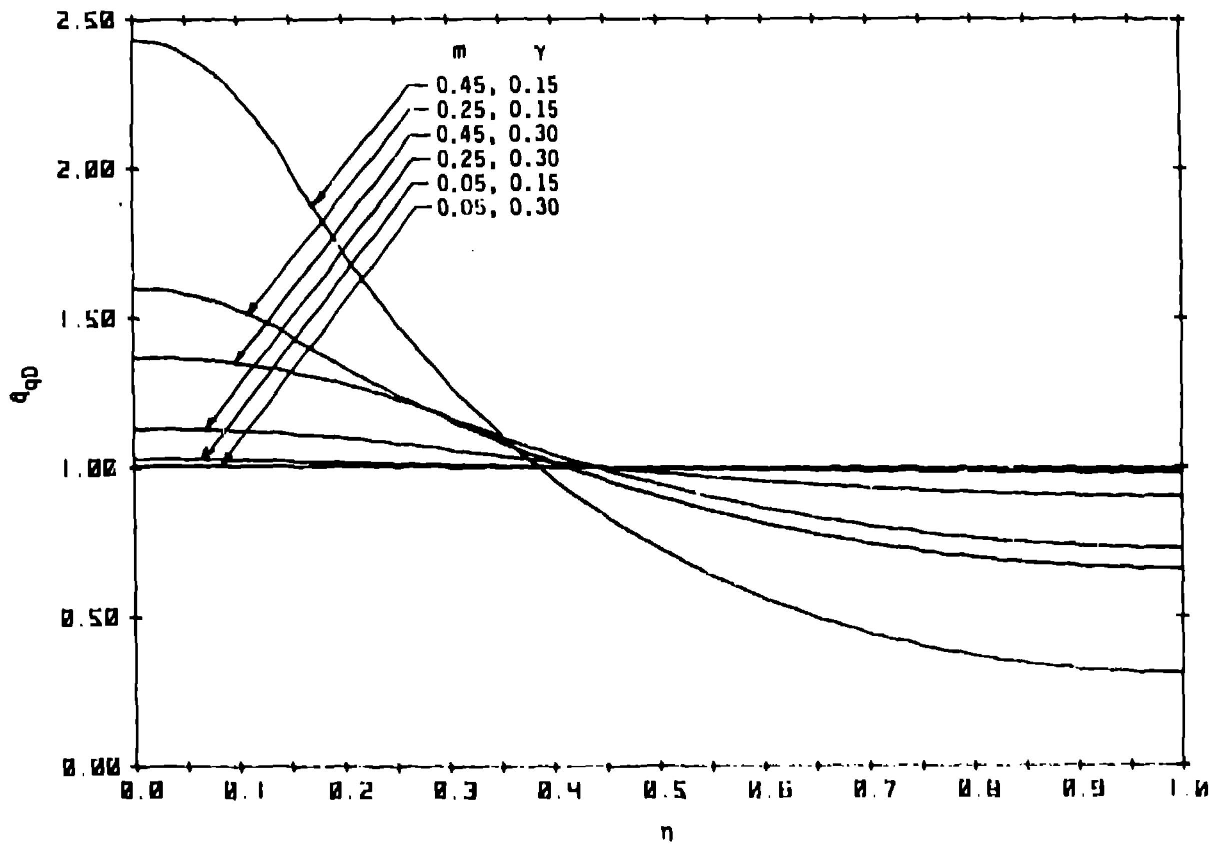

Fig. 2. Dimensionless quasi two-dimensional heat-flux distribution at ξ = 0 for various combinations of m and γ parameters.

uniform with increased γ and decreased m. For a fixed Bi and t, this implies an increased fin conductance, kd, and a decreased fin half-length, ℓ, respectively. Recalling fundamental fin theory (Ref. 5), both these effects are seen to cause an increased uniformity in the heat-flux distribution within the fin, thus verifying the trends indicated in Fig. 2.

The figure-of-merit variable ϵ_{qD} from Eq. (17b) is shown in Fig. 3 for a wide range of m values. Just as in Fig. 2, a decrease in the uniformity of heat flux at the surface $\xi = 0$ for increased m values is evident for fixed values of γ. This figure serves as the basis for the strip-heater-design procedure described in the next section.

The variable ϵ_{2D} from Eq. (17a) is shown in Fig. 4 for several D/t and $\Lambda^2 + (t/Dm)^2$ combinations and for a fixed value of Bi = 15. The latter value corresponds approximately to natural convective flow of water over a

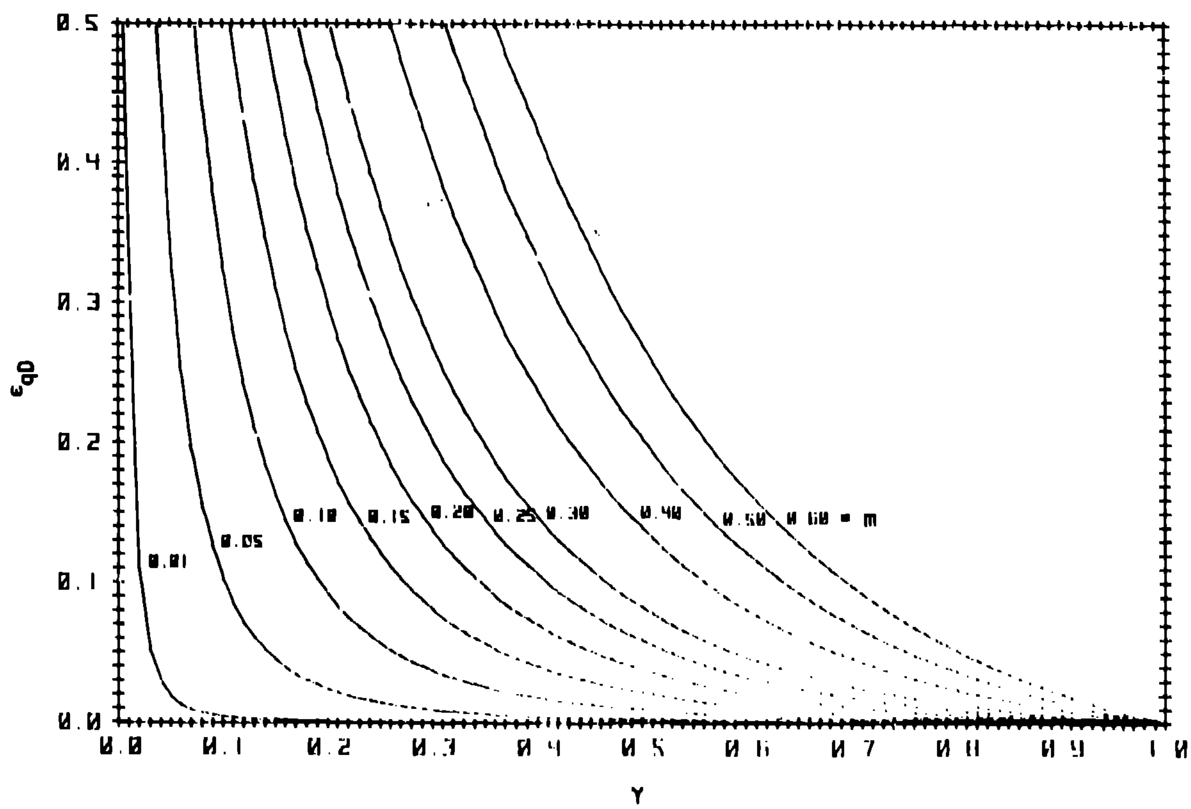

Fig. 3. Figure-of-merit variable ϵ_{qD} from the quasi two-dimensional model for various m parameter values.

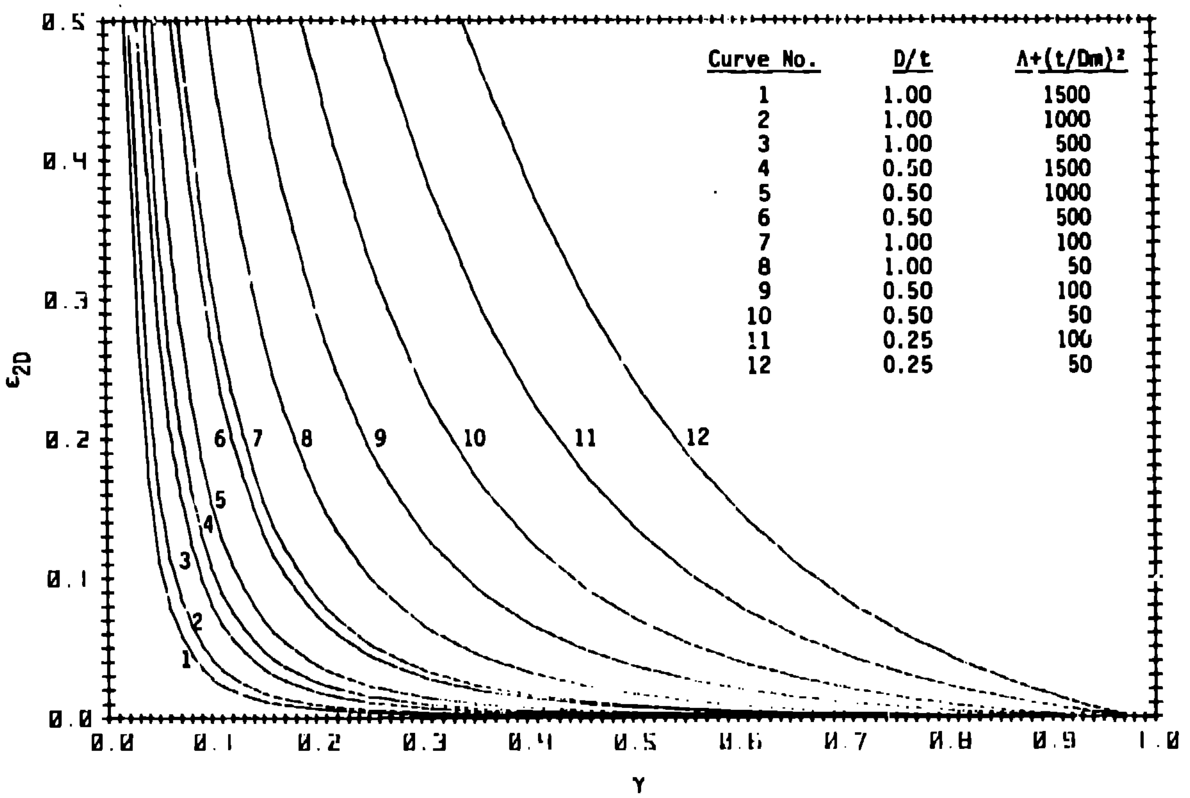

Fig. 4. Figure-of-merit variable ϵ_{2D} from two-dimensional model for various D/t and $\Lambda + (t/Dm)^2$ parameter values. Bi = 15.

moderately heated 2.54-cm- (1-in.) thick acrylic plate. For values of $(t/Dm)^2 \gtrsim 15$ and Bi $\gtrsim$ 15, ϵ_{2D} is fairly insensitive to Bi, which we note by inspecting Eqs. (10d) and (14). For all other cases, ϵ_{2D} should be calculated from Eq. (17a).

The ratio ϵ_{2D} to ϵ_{qD} as a function of γ is presented in Fig. 5 for several D/t and $\Lambda + (t/Dm)^2$ combinations and for Bi = 15. Ordinate values smaller than or equal to unity verify that ϵ_{2D} is bounded by ϵ_{qD} from above as discussed before the development of Eqs. (14). In addition, the effect of η-direction conduction on the heat flux at the surface $\xi = 0$ may be assessed from this figure. In particular, for Bi $\geq$ 15, D/t $\leq$ 0.50, $\gamma \leq 0.40$, and m $\leq$ 0.23, the value of ϵ_{qD} represents less than an 8% overestimate in heat-flux variation at the surface $\xi = 0$. Thus, in many cases, only the quasi two-dimensional results shown in Fig. 2 need be used to obtain reasonably good estimates of heat-flux variations in strip-heated, composite-slab systems.

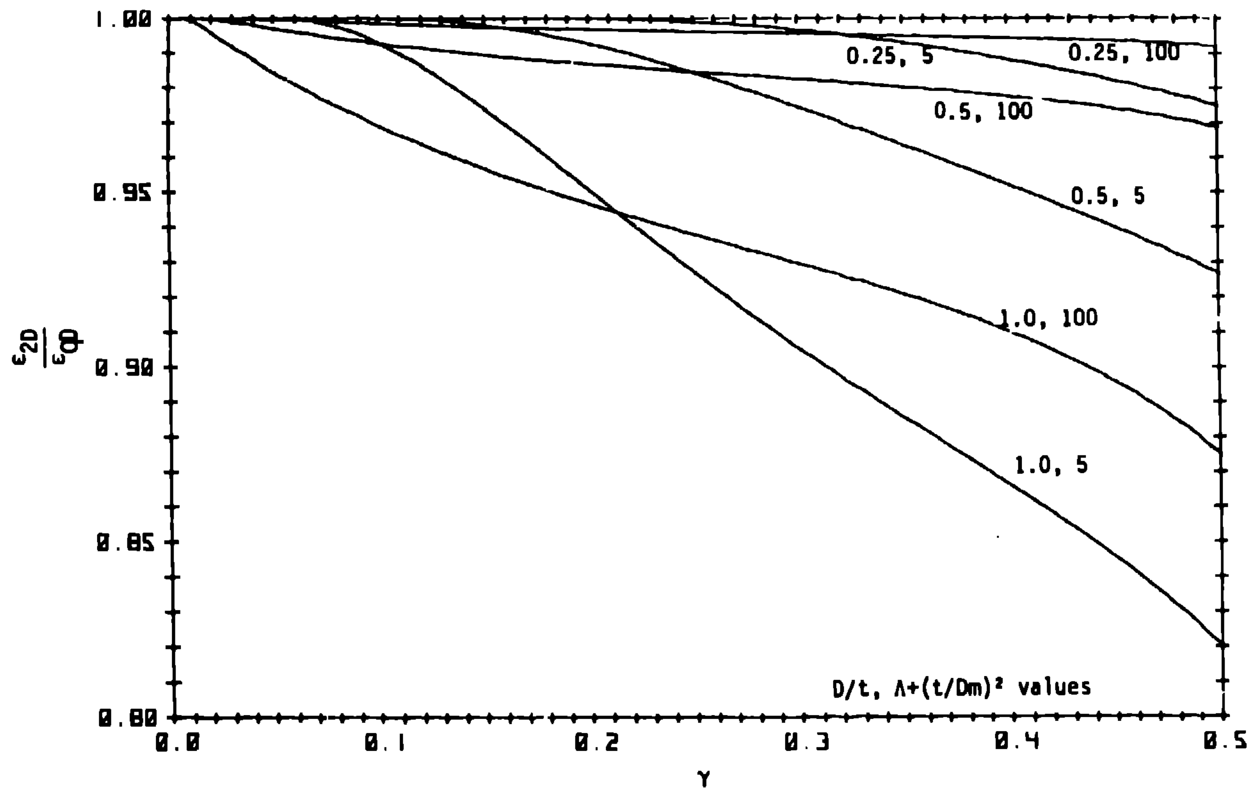

Fig. 5. Ratio of two-dimensional to quasi two-dimensional figures-of-merit for various D/t and $\Lambda + (t/Dm)^2$ parameter values. Bi = 15.

DESIGN METHODOLOGY

We present the following method to estimate strip-heater spacing for a prescribed maximum variation in heat flux over the convectively cooled side of a strip-heated composite slab.

1. From the problem specifications, obtain values for h, D, k_s, t, d, and k.

2. Calculate the following variables from the equation numbers shown after them in parentheses: Bi (3), Λ (10d), m (11d), and G (17a).

3. Choose a maximum allowable value of ε_{qD} (a value of 0.05 or smaller may be a good value in most cases).

4. Enter the value of m and ε_{qD} in Fig. 3 and estimate γ. Alternatively, Eq. (17b) may be used in place of Fig. 3.

5. Estimate half-spacing, ℓ, between two adjacent strip heaters by $\ell = t/\gamma$.

6. Estimate p from γ and Eq. (11c).

7. From Eq. (17a), estimate ϵ_{2D} [or alternatively, if, for the problem under consideration, Bi = 15 and the combination of t/D and $\Lambda + (t/Dm)^2$ appear on Fig. 4, the figure may be used in place of Eq. (17a)]. If ϵ_{2D} is much smaller than ϵ_{qD} because of significant two-dimensional effects, recalculate ϵ_{qD} using a slightly smaller value of γ than that obtained in step 4. Steps 5 through 7 may be repeated until the prescribed value of ϵ_{2D} is obtained.

If desired, this procedure may be repeated choosing different values of t, d, and k to optimize the strip-heated system with respect to cost, weight, material availability, etc. In addition, because ϵ parameters are obtained from closed-form functions (and not tables of numbers or graphs from numerical simulations), they, along with the other formulae needed, may be easily computer programmed and the design and optimization carried out in this manner.

APPLICATION EXAMPLE

Consider the following example problem. The numbers on the far left correspond to the steps from the design procedure described in the last section.

1.

h	$= 85.17$ W/m^2 K	(15 Btu/h ft^2 °F)
D	$= 1.27$ cm	(0.5 in.)
t	$= 1.27$ cm	(0.5 in.)
d	$= 0.318$ cm	(0.125 in.)
k	$= 382.55$ W/m K	(221 Btu/h ft °F)
k_s	$= 0.346$ W/m K	(0.2 Btu/h ft °F)

2.

$$Bi = \frac{hD}{k_s} = 3.125$$

$$\Lambda = \frac{Bi^2 + 3Bi + 3}{3\,Bi(1 + Bi)} = 0.573$$

$$m = \left[\frac{ht^2/k_1}{1 + Bi} \right]^{1/2} = 0.052$$

$$G = 1 + \frac{3 + Bi}{6(1 + Bi)[\Lambda + (t/Dm)]^2} = 1.001$$

$$\Lambda + (t/Dm)^2 = 370.39 \quad .$$

3. Choose $\epsilon_{qD} = 0.05$.

4. From Fig. 3, $\gamma = 0.15$ (for this value of γ, Eq. (17b) gives $\epsilon_{qD} = 0.049$).

5. $\ell = t/\gamma = 1.27\text{cm}/0.15 = 8.467$ cm.

6. $p = \left[\frac{(t/D\gamma)^2 + 1}{\Lambda + (t/Dm)^2} \right]^{1/2} = 0.350$.

7. $\epsilon_{2D} = 1 - \frac{G \sinh p\gamma}{\sinh p - G \sinh p(1 - \gamma)} = 0.046$.

We note that two-dimensional effects reduce the upper-bound estimate of ϵ by about 7%. If the above results are acceptable, a 4.6% variation in heat flux across the surface at $\xi = 0$ should be expected for this problem for a centerline spacing between two adjacent strip heaters of 2 x 8.467 cm = 16.93 cm. The values of μ and μ^2 here are 0.150 and 0.022, respectively.

LIMITING CASES OF INFINITE Bi AND ZERO-THICKNESS SUBSTRATE

In this section we consider the two limiting cases of infinite Bi and zero-thickness substrate. The former corresponds to a constant temperature at the cooled surface, whereas the latter is for a single material in which ξ-direction conduction may be lumped.

For Bi $\to \infty$, Eqs. (10) and (11) become

$$A = \lambda(1 + \mu^2)/\mu\nu \quad , \tag{18a}$$

$$B = \lambda\mu/3\nu \quad , \tag{18b}$$

$$\Lambda = 1/3 \quad , \tag{18c}$$

$$p^2 = 2\lambda(1 + \mu^2)/\mu(\lambda\mu + 3\nu) \quad , \tag{19a}$$

$$m^2 = t^2 k_s/Dkd \quad . \tag{19b}$$

The temperature distribution in the fin [Eqs. (12)] remains unchanged except for the change in m and p given by Eqs. (19). The temperature distribution in the substrate becomes

$$\phi_s(\eta, \xi) = \frac{\gamma\xi}{m^2\nu}\left\{1 - \frac{\sinh p(1 - \gamma)\cosh p\eta}{\sinh p}\right\} \tag{20a}$$

$$\left[1 - \frac{(\xi^2 - 1)}{6(\mu^2/3 + \gamma^2/m^2)}\right] \qquad 0 \le \eta \le \gamma \quad ,$$

$$\phi_s(\eta, \xi) = \frac{\gamma\xi \sinh p\gamma \cosh p(1 - \eta)}{m^2\nu \sinh p} \tag{20b}$$

$$\left\{1 - \frac{(\xi^2 - 1)}{6(\mu^2/3 + \gamma^2/m^2)}\right\} \qquad \gamma \le \eta \le 1 \quad .$$

All other equations remain the same except that the term within the large brackets in Eqs. (14) becomes

$$1 + [2 + 6d/\lambda D]^{-1} \quad ,$$

which is also the new expression for G in Eq. (17a).

For the case of a zero-thickness substrate, $\mu = 0$ and Eqs. (11) become

$$m^2 = ht^2/kd \quad , \tag{21a}$$

$$p = m/\gamma \quad . \tag{21b}$$

The temperature distribution within the fin [Eqs. (12)] simplifies to

$$\phi(\eta) = \frac{\gamma}{m^2 \nu} \left\{ 1 - \frac{\sinh m(\gamma^{-1} - 1) \cosh m\gamma^{-1}\eta}{\sinh m\gamma^{-1}} \right\} \qquad 0 \leq \eta \leq \gamma \quad , \tag{22a}$$

$$\phi(\eta) = \frac{\gamma \sinh m \cosh m\gamma^{-1}(1 - \eta)}{m^2 \nu \sinh m \gamma^{-1}} \qquad 0 \leq \eta \leq 1 \quad . \tag{22b}$$

The term within the large brackets in Eqs. (14) reduces to unity for this case as does the expression for G in Eq. (17a). Thus, we note that Eqs. (14) and (15) and Eqs. (17a) and (17b) become the same for the limiting case of $\mu = 0$ as we expect because of the absence of any two-dimensional conduction effects.

The design procedure described above and the graphical results may be applied to both of these cases, when the above expressions for Λ, m, and p are used. Obviously, Figs. 4 and 5 (and steps 6 and 7 in the design procedure), which accounts for multidimensional heat conduction effects within the composite slab, need not be considered for the case of a zero-thickness substrate because no such effects occur within the fin.

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