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Generalized Helicity and Its Time Derivative

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I. Introduction

Spheromaks can be sustained against resistive decay by helicity injection because they tend to obey the minimum energy principle. This principle states that a plasma-laden magnetic configuration will relax to a state of minimum energy subject to the constraint that the magnetic helicity is conserved. Use of helicity as a constraint on the minimization of energy was first proposed by Voltjer[1] in connection with astrophysical phenomena. Use of the helicity constraint was first applied to the spheromak by Wells and Norwood.[2] The principle was later applied to the reversed-field pinch (RFP) by Taylor,[3] who was most responsible for the eventual acceptance and recognition of the principle as being important for RFP-type confinement devices. Of course, helicity does decay on the resistive diffusion time. However, if helicity is created and made to flow continuously into a confinement geometry, these additional linked fluxes can relax and sustain the configuration indefinitely against the resistive decay.

In this paper we will present an extension of the definition of helicity to include systems where $\vec{B}$ can penetrate the boundary and the penetration can be varying in time. We then discuss the sustainment of RFPs and spheromaks in terms of helicity injection.

II. Generalized Helicity

Magnetic helicity has traditionally been defined as a global quantity applicable to configurations of closed magnetic fields:

$$K = \int \vec{A}\cdot\vec{B}dV \tag{1}$$

with $\vec{B}\cdot\vec{n} = 0$ at the boundary of the volume in question. In this form, and in conjunction with the Taylor principle, this concept has had great utility in generating bounded equilibrium states. Difficulties can arise, however, when attempts are made to reduce the concept to more local terms, such as by defining a helicity density or a helicity flux. Analyzing helicity injection schemes for which $\vec{B}\cdot\vec{n} \neq 0$ can also present a problem. The central difficulty is that the artificial character of the vector potential, $\vec{A}$, makes manipulation of helicity sensitive to the choice of gauge. Indeed, if we apply the gauge transformation $\vec{A}' = \vec{A} + \nabla u$, it is readily shown from the above definition that

$$K' - K = \int u\vec{B}\cdot\vec{n}dS \tag{2}$$

so that for a generalized volume with $\vec{B}\cdot\vec{n} \neq 0$ over some region of its surface, the magnetic helicity is not well determined.

In the simple geometry of the CTX experiment, this rather technical theoretical issue is avoided by a direct approach. As illustrated by Moffatt [1], in the case of two separate, linked fluxes, ψ_1, and ψ_2, the above definition reduces to $K = 2\psi_1\psi_2$. In the CTX coaxial source, poloidal and toroidal fields are separate with the poloidal flux ψ_s being generated by a coil and the rate of change of the toroidal flux being given by the source voltage V_s. The result is the rule $dK/dt = 2\psi_s V_s$ for the helicity injection rate, which has been confirmed by CTX data.

The gauge problem that arises in less elementary cases has been studied independently by a number of workers.[4,5,6] The results of these studies indicate that the issue is now essentially resolved. Three approaches will be described briefly below: the work of Jensen and Chu at GA Technologies, the work of Berger and Field at the Harvard-Smithsonian Center for Astrophysics, and the results of our thinking on the subject. Jensen and Chu[5] treated the case of a boundary with $\vec{B}\cdot\vec{n} \neq 0$ but with $d(\vec{B}\cdot\vec{n})/dt = 0$. Though a more general view was implied, emphasis was on boundaries consisting primarily of conductors having gaps across which voltages were applied. In such cases, the vector potential is usefully represented as a superposition of eigenfunctions (Taylor states) plus an "inhomogeneous" part, $\vec{A}_I$, arising from $\vec{B}\cdot\vec{n}$ at the boundary. Jensen and Chu noted that a "total helicity", K_c, could be defined as

$$K_c = \int \vec{A}\cdot\vec{B}dV - \int \vec{A}_I\cdot(\nabla \times \vec{A}_I)dV \tag{3}$$

and that K_c can be shown to be gauge invariant. The second term on the right is referred to as the "vacuum helicity" and it vanishes when $\vec{B}\cdot\vec{n} = 0$. In this same study, a "Poynting vector" for helicity flux is proposed in the form

$$\vec{Q} = \phi\vec{B} + \vec{E} \times \vec{A} \tag{4}$$

where ϕ is the electrostatic potential. Berger and Field[6] devote considerable attention to the topological basis of flux linkage and view the gauge problem as arising from topological indeterminacy: when magnetic field lines leave the volume of interest ($\vec{B}\cdot\vec{n} \neq 0$) their external linkage is unknown. The resolution of this uncertainty is to compare the actual field $\vec{B}$ to a reference field, $\vec{P}$, obtained from a magnetic potential ($\nabla \times \vec{P} = 0$) but having the same values of $\vec{B}\cdot\vec{n}$ and $d(\vec{B}\cdot\vec{n})/dt$ at the boundary. By subtracting the reference contribution, a "relative helicity" is obtained which is gauge invariant and can be shown to satisfy a plausible sum rule. The subtraction step is similar to that of Jensen and Chu, but the treatment of the problem is more general in that $d(\vec{B}\cdot\vec{n})/dt$ need not be zero. Also the transfer of helicity between adjacent volumes is treated with more care. The rate of change of "relative helicity" within a particular volume is given (in their units) by

$$dH_R/dt = -2c\int \vec{E}\cdot\vec{E}dV + 2c\int (\vec{A}_p \times \vec{E} - \frac{1}{c}\frac{d\chi}{dt}\vec{A}_p)\cdot\vec{n}dS \tag{5}$$

where χ and $\vec{A}_p$ are the magnetic scalar and Coulomb gauge vector potentials of the reference case. The work of Berger and Field provides a lucid and rigorous discussion of the proper description of magnetic helicity and its transport. There is every reason to expect that, as it becomes better known, it will provide the standard approach to these topics.

For experimentalists, the concepts of field and flux often have more appeal than that of vector potential. In terms of these quantities, helicity can be thought of as a linkage of flux with flux and for a closed volume can be defined as

$$K = \int_0^{\psi_t} \psi_{\psi 1} d\psi \tag{6}$$

where ψ is the amount of flux linking the incremental closed flux tube $d\psi$ and ψ_t indicates that integral is performed on all flux tubes within the closed volume of interest.

We will now discuss changes in the helicity of a system and then write down an equation for the time derivative of K. If a flux tube ψ is added to a magnetic configuration the helicity is changed by two effects. The ψ links some flux ψ' giving a $K = \psi'\psi$ and the flux which links the flux ψ' is increased by ψ giving another $K = \psi\psi'$. Thus the increase in helicity is twice that computed for either case. This is a very general result following from the fact that if flux tube c links tube d then d must link c. The rate at which flux increases inside a closed flux tube $d\psi$ is $-\oint\vec{E}\cdot d\vec{l}$ along the closed tube. Thus the rate of change of helicity due to the change of linkage in $d\psi$ is $-2\oint\vec{E}\cdot d\vec{l}d\psi$. The factor of 2 is needed because $\oint E\cdot dl d\psi$ is only one of the two equal effects. Thus for the whole system

$$K = -2\int\oint \vec{E}\cdot d\vec{l}d\psi \tag{7}$$

where the line integral is taken along the flux tube $d\psi$ in the direction of the $\vec{B}$ field. In the case of an arbitrary boundary intersected by these tubes, the path of integration is to be taken across the boundary rather than following the external field. This process defines a unique dK/dt only if $d(\vec{B}\cdot\vec{n})/dt = 0$. The difficulty [when $d(\vec{B}\cdot\vec{n})/dt \neq 0$] is again resolved by subtracting a reference case: the fields produced in vacuum by the same boundary values of $\vec{B}\cdot\vec{n}$, $\vec{E}\times\vec{n}$ and the same flux linking the boundary for multiply connected systems. Thus

$$K = \int_0^{\psi_t} \psi d\psi - \int_0^{\psi_{t_v}} \psi_{,v} d\psi_v$$

$$\dot{K} = -2\int\!\!\int\!\oint\vec{E}\cdot d\vec{l}\,d\psi + 2\int\!\!\int\!\oint\vec{E}_v\cdot d\vec{l}_v d\psi_v \tag{8}$$

where the subscript v refers to the vacuum fields and fluxes obtained using the boundary values of the plasma case. The first term on the right gives the plasma dissipation of K which for ideal MHD is zero. Thus it is the second term that usually determines the injection rate although the first term may cause helicity generation within the plasma. Departing from the restriction of closed field lines, $d\vec{l}d\psi$ can be generalized as $\vec{B}dV$ giving

$$dK/dt = -2\int\vec{E}\cdot\vec{B}dV + 2\int\vec{E}_v\cdot\vec{B}_v dV \tag{9}$$

This result is equivalent to that of Berger and Field and is therefore not new. However, this representation does not explicitly contain the vector potential and is attractive in providing a quick intuitive guide for assessing the prospects of experimental helicity injection schemes. The dissipation is zero in ideal MHD ($\vec{E} = \vec{v}\times\vec{B}$ is the ohms law) because $E\cdot B = 0$ everywhere. However, the principle includes resistive MHD and the generalization should state that the volume integral is zero (even though $E\cdot B$ is non-zero locally) for all time scales shorter than the resistive decay time. Only on a resistive decay time is helicity dissipated. For example, the volume integral of the dissipation of helicity for the tearing modes should be zero. Thus the plasma term can be replaced by $-K/\tau_K$. The form of Eq. 8 is convenient especially in the simple geometries when only one voltage ($\int\vec{E}_v\cdot d\vec{l}_v = V$) is involved and when the "vacuum" flux to which this voltage is applied is obvious. For the simple geometries such as the RFP, S-1, coaxial source, and bumpy Z-pinch the contribution from the vacuum term is

$$\dot{K}_v = 2V\psi \tag{10}$$

where V is the voltage and ψ is the flux. For the RFP the only vacuum flux is the toroidal flux so ψ is the toroidal flux and V is $\int E\cdot dl$ along toroidal flux tube, i.e. the toroidal voltage. For the S-1 there is only vacuum poloidal flux so ψ is the poloidal flux which links the flux core and V is the poloidal voltage equal to $\dot{\psi}_t$. For the electrode type sources (where negligible flux penetrates the insulator such as the coaxial source and the bumpy Z-pinch) V is the voltage that is applied to the electrodes and ψ is the net flux into each electrode. Since $\vec{B}\cdot\vec{n} = 0$ everywhere in these cases these results can also be obtained from Jensen and Chu's equations.

The full generalized helicity injection rate (Eq. 8) is needed for the PS-1 device and conical Θ-pinch. They will be discussed together because the voltage ($\int E_v\cdot dl_v$) which dominates the helicity injection is applied to flux which penetrates the boundary through an insulator. This insulator penetration makes the helicity injection calculation more complicated because, in general, the $\int E_v\cdot dl_v$ along the field lines between boundary penetration will give a different voltage for each field line, and also $\vec{B}\cdot n \neq 0$. In order to calculate the helicity injection rate into these devices, one would need to measure $\vec{E} \times \vec{n}$ and $\vec{B} \cdot \vec{n}$. This could be done by measuring voltages between floating hoops just inside the insulating wall to determine $\vec{E} \times \vec{n}$ and by using flux loops just outside the wall to determine $\vec{B} \cdot \vec{n}$. If axial symmetry is assumed, these measurements would be quite possible on the present experiments. The helicity generated in these devices may come from the plasma term (which is forbidden by the minimum energy principle) and the measurement proposed here could help to determine the source of helicity generated in those objects by measuring the contribution from the vacuum term.

III. Steady State Methods

In order to achieve steady state sustainment $\dot{K}$ must equal zero so the contribution from the vacuum term must cancel the losses associated with the plasma term in Eq. 8. With electrode geometries a positive contribution of $2V\psi$ can be supplied indefinitely by simply maintaining a fixed voltage on and fixed flux into the electrodes.[7,8] On the inductive driven schemes such as S-1 or RFPs the time averaged V must be zero because the applied voltage is due to a change in external flux which cannot change in the same direction idefinitely. In this case, steady state time average net injection is achieved by oscillating both V and a fraction of ψ in phase so that there is a net contribution to the product while maintaining a zero time averaged voltage.[9]

Most RFPs have been able to maintain the magnetic configuration for several magnetic decay times of the fields. So far only one sign of V has been applied so the sustainment is limited by the volt seconds available. Such experiments are important in establishing the operation of the relaxation processes that are necessary for sustainment by helicity injection but are not directly extrapolatable to steady state operation. True steady state by oscillating fields is being proposed in the Los Alamos RFP program.[10] In the S-1 spheromak the ratio of the toroidal to poloidal flux relaxes toward the Taylor value and by injecting excess toroidal flux the poloidal flux of the spheromak can be increased. This result indicates that the minimum energy principle is operating and steady state drive may be possible by again oscillating both the voltage and a fraction of the flux.[11]

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Figure Caption

The voltage and flux of the coaxial source on CTX as a function of time and the spheromak current as a function of time. The smoother curve with the current curve is computed using Eq. 8.