

AUG 07 1985

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TITLE: THERMOELASTIC-PLASTIC FLOW AND DUCTILE FRACTURE IN SOLIDS

LA-UR--85-2529

DE85 015699

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SUBMITTED TO Lectures at:
 Istituto di Ingegneria Nucleare,
 CESNEF-Politecnico di Milano,
 Milano, Italy, August 12-30, 1985

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AND
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1985

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PREFACE

The main aim of these lectures is to develop the theory of elastic-plastic flow in solids, with application to a particular class of processes, namely those in which the dissipation of plastic work cannot be neglected. Examples of such processes are highspeed impact phenomena and cratering, shock compression, often shock release as well, and explosive deformation and welding. An important part of the development is to include the anisotropic elastic properties of a solid, and for this reason the theory applies to solids in general. On the other hand, it is possible that some of the results will have limited validity for nonmetals. The secondary aim of the lectures is to develop a general framework for ductile fracture, and describe the current practice in the field. Our theory is incremental in nature and suitable for integrating along a process in small timesteps as is done in "hydrodynamic" computer programs. A notational conflict arose, because in continuum mechanics the extensive quantities are taken per unit mass, while in thermodynamics they are usually per unit volume. The continuum mechanics normalization i.e. per unit mass, is used throughout, with the result that uncommon factors of density show up in the thermodynamic equations. For the first three lectures, I have heavily borrowed from a review document by Duane Wallace.¹

LECTURE I-MATERIAL RESPONSE

In this lecture, the basic material-response concepts underlying the entire theory are described in words. Some of the observations on time-rate effects, and in support of local thermodynamic equilibrium during plastic flow, were originally published in Physical Review.²

1. Equilibrium and Nonequilibrium

A system is a quantity of material whose behavior we want to study, as e.g. a cubic centimeter of gold, or a beaker of water. An isolated system is one for which nothing flows in or out of the system. If a system remains isolated, it will presumably reach a state which remains constant in all its macroscopic properties. This is an equilibrium state. Let us for the moment consider only states for which the forces applied to the system are isotropic, and make a partial list of the macroscopic properties of the equilibrium state: V = volume, P = pressure, U = internal energy, T = temperature, S = entropy.

Equilibrium thermodynamics is the study of processes by which a material (or several materials simultaneously) can pass from one equilibrium state to another, along paths which are constrained to pass only through equilibrium states. The primary physical laws invoked are conservation of energy, and the existence of an exact differential dS ; the rest of equilibrium thermodynamics is (almost entirely) mathematics. When the stress is isotropic

pressure, there are only two independent variables, which means two variables completely specify an equilibrium state of a given material. These variables can be any two from the above list, or two combinations of them, or other extensions. Take for example V and T as the independent variables. Then for all possible equilibrium states of a material, relations of the following form hold: $P = P(V,T)$, $U = U(V,T)$, $S = S(V,T)$. The term "equation of state" is sometimes used to denote the above equation for the pressure. A more general usage of the term, and that which will be followed in the present work, is to denote any or all of the information contained in the set of equations above.

To enter the realm of nonequilibrium states and processes, it is helpful to think in terms of statistical mechanics. Consider a monatomic nearly-ideal gas; mentally subdivide the space occupied by the gas into a large number of volume elements, each with the same volume. Each element contains a large number of atoms, and can be treated as a statistical subsystem. The important statistical measure of a subsystem is the distribution $f(\vec{p})$ of the atomic momenta \vec{p} . The equilibrium distribution is Maxwellian, with a temperature T : $f(\vec{p})$ is proportional to $\exp(-\beta \vec{p}^2/2M)$, where $\beta = 1/kT$, k is Boltzmann's constant, and M is the atomic mass. When the gas is in equilibrium, the mass and temperature are the same for each element. When the gas is not in equilibrium, the momentum distribution can be anything. If the distribution is not at least approximately Maxwellian, then the temperature cannot be defined. However, even when equilibrium-thermodynamic quantities such as temperature are not defined, mechanical quantities are always defined. The

most important macroscopic mechanical quantities representing each volume element, which in fact are just the zeroth, first, and second moments of $f(\vec{p})$, are the total mass, total linear momentum, and total energy. If the gas is thought of as a continuum, these quantities translate into local fields representing density, fluid velocity, and energy density.

We can now define nonequilibrium states which are "close to equilibrium." Any element of the gas is close to equilibrium if its momentum distribution is close to Maxwellian; specifically this means that $f(\vec{p}) = \exp(-\beta \vec{p}^2 / 2M) + \delta f(\vec{p})$, where the only restriction on $\delta f(\vec{p})$ is that it is small enough to be treated as a perturbation. But this will be the case only if the spatial and temporal variations of the mechanical quantities, the density, fluid velocity, and energy density, are sufficiently small. It is important always to differentiate between mechanical and thermodynamic quantities, and to remember that equilibrium thermodynamic quantities can be defined only for states which are close to equilibrium. A good exercise would be to explain in one's own words why this is so.

Irreversible thermodynamics is the study of processes which pass through nonequilibrium states, but only those which are close enough to equilibrium states so that the equilibrium-thermodynamic processes are characterized by the following properties.

(a) A "driving force" is present, which causes a flow that tries to cancel the driving force. The force-flow relation is called the constitutive relation, and the material properties which enter this relation are called constitutive properties. The constitutive relation couples to, and alters parts of, the equation of state.

(b) Dissipation is present (generation of entropy). The work done by the driving force against the flow is always at least partially dissipated, and usually it is totally dissipated. The equation for entropy increase acquires a dissipation term in addition to the heat flow term.

Consider a material in which there is a local temperature gradient; heat flows, in an attempt to cancel out the temperature gradient. A common constitutive relation sets the heat current proportional to the temperature gradient. The coefficient of proportionality, the thermal conductivity, is a constitutive property of the material. Consider a beaker of water, stirred gently with a stirring rod, and then isolated. Viscous stresses are present, working against the velocity gradients, and the macroscopic mechanical motion of the water gradually dies away, as it is turned into heat. The linear relation between viscous stresses and velocity gradients is the constitutive relation, and the viscosity coefficients are constitutive properties of the water.

It is important to remember that constitutive properties of a material are entirely separate from equilibrium properties; that equation-of-state data does not contain any information about constitutive properties of a material. In short, the constitutive relation, or constitutive behavior, of a material describes the dissipative force-flow part of an irreversible-thermodynamics process.

The present work is concerned only with processes in the irreversible-thermodynamic regime: equilibrium-thermodynamic quantities can always be defined, at least approximately. A point of usage needs to be addressed. It

seems that physicists are raised to think of adiabatic, when used in the thermodynamic sense, as meaning isentropic. Engineers generally use adiabatic to mean "without heat flow." We could simply abandon this word in the present work, except that we will have much need for the adiabatic elastic moduli, which of course are isentropic elastic moduli. The following definition will therefore be followed: adiabatic means isentropic.

2. Elastic Response and Plastic Response

The significant property of condensed matter, which makes it condensed, is the dominance of forces within the material. For an ordinary solid or fluid composed of atoms, or more accurately, composed of ions and electrons, the forces derive from effective potentials between the ions. These forces are elastic forces. Since the effective ion-ion potentials operate through the electrons, they are "instantaneous" potentials, as far as the present work is concerned. There are also forces arising from thermal energy in a material, e.g. the thermal excitations of electrons and phonons. Thermal forces can remain close to equilibrium as long as their spatial and temporal variations are slow compared to relaxation lengths and times within the electron-phonon system. Such relaxation lengths and times are quite short, being measured in terms of lattice spacings and picoseconds, respectively. The elastic forces and thermal forces taken together are called thermoelastic forces, and for solids we have the following conclusion: stresses are supported by the thermoelastic forces, and local thermoelastic equilibrium will be a valid approximation up to very high spatial and/or temporal rates of change.

When an anisotropic stress is applied to a solid material, a multitude of processes begin, all of them acting to reduce the stress. Some of these processes can be effective only on geological time scales; others are important in minutes or days. In the present work, we are interested in rather fast processes, say things which happen in one second or less. We will not be concerned with any stress relaxation mechanism which operates on a slower timescale. We will be concerned with the mechanisms of plastic flow and ductile fracture. Plastic flow does not have to be due to dislocations; however, it usually is, and we will often use dislocation concepts in examining the nature of plastic flow.

Consider a small region of a solid material, and suppose there is an anisotropic stress in the region, which results from forces applied by the surrounding material. There may be a dislocation in the region, or one might be generated; in any case the dislocation moves in such a way as to reduce the local anisotropic stress. Transforming this picture to the language of irreversible thermodynamics, we say that the stress drives plastic flow, and that for a given material in a given state, the material constitutive equation specifies the plastic response to any applied stress. We have thus arrived at an important point of logic in the present theoretical construction, stated as follows: For plastic flow within a solid material, at all times and locations, the stresses satisfy two separate conditions, viz. the equilibrium thermoelastic equations, and the plastic constitutive equation. If ductile fracture is also taking place, a constitutive relation for the fracture damage is also obeyed.

There is a subtle discord between the nature of plastic flow and the customary "textbook picture" of irreversible processes. This can be illustrated with a simple example. Suppose a shear stress τ is applied to a solid, and the solid deforms elastically for $\tau < \tau_0$, and at τ_0 the solid begins to flow plastically. The irreversible-thermodynamic driving force is τ_0 . This constitutive behavior cannot be represented by a linear phenomenological law, according to which the driving force is zero in equilibrium, and is a linear function of some measure of the departure from equilibrium. The essential nonlinearity of plastic constitutive behavior has led to the statement that "plastic flow cannot be treated by irreversible thermodynamics." This statement can safely be ignored. But let us continue with the example, and ask about the connection between time-rates and reversibility. The customary picture is that if you make a process slow enough, it will be arbitrarily close to equilibrium; that slow is reversible. Is this really true? In an ordinary solid, adiabatic elastic waves can be transmitted at very high frequencies, under near-equilibrium conditions. On the other hand, in driving plastic flow, the shear stress can be adjusted so that the plastic strainrate is arbitrarily small, yet the process is still irreversible: the driving force τ_0 is finite for an arbitrarily slow process. This result allows the following important conclusion to be made: The thermodynamic reversibility of a process is not determined by its rate.

So far, we have ignored the heterogeneous nature of plastic flow. In fact, plastic flow is intrinsically heterogeneous, and what is worse, it is heterogeneous on several different length and time scales. The finest scale of heterogeneity is that of a single dislocation. A dislocation is a line

defect, surrounded by a nonuniform elastic strain field. When a dislocation moves, it presumably acts as a source of mechanical excitations, and with the appropriate relaxation time, this mechanical energy becomes thermalized to heat. Larger scales of heterogeneity are associated with dislocation substructure, the networks and cells and so on. On this large scale, plastic instabilities may develop during the course of a process. For example, shear bands may form, within which the plastic flow may tend to become localized. The question arises, how is the heterogeneous nature of plastic flow to be addressed by irreversible thermodynamics?

First we will simply assume that the finest scale of heterogeneity can be ignored; that single-dislocation effects can be averaged for any mass element, whether the mass element contains only a few dislocations, or is located within an active shear band. There are two ways in which larger-scale inhomogeneities can be treated. The first way is simply to resolve the inhomogeneities which occur in a given flow process. After all, the continuum-mechanics and irreversible-thermodynamic theory is a local field theory, and applies in principle to spatially and temporally inhomogeneous processes.

An alternate procedure, which entails a sacrifice of resolution on a certain scale, is to replace a certain type of inhomogeneity by a homogeneous model. In this way, an additional field variable, or parameter, is introduced into the continuum theory, and the modeled inhomogeneity formally disappears. It should always be remembered, however, that the model has to be consistent with the basic thermoelastic properties of the solid. For example, consider a single crystal with a single straight dislocation; apply a shear stress to

the crystal and the dislocation moves. Now replace the whole slip plane of the dislocation by a shear band, and apply the same shear stress. The stress is supported elastically, with the same elastic stress-strain relation in both cases; the only difference in the two cases is the plastic constitutive relation. Hence, the appearance and growth of shear bands can be modeled homogeneously, by using a combination of plastic constitutive relations, together with a local field variable denoting what part of the total plastic flow is due to the shear bands. Another example is the growth of voids in a metal under tension. A single void contributes to the flow problem in the following way: plastic flow proceeds around the void's surface, as the void grows, and the cross section of the void does not support stress. These effects have been modeled in a manner consistent with the thermoelastic properties of a metal, in a calculation of necking and ductile fracture.³

3. Fluids

All fluids are presumably viscoelastic. This means the fluid response is viscous at low and moderate strainrates, but the response is elastic at high strainrates. For a simple monatomic fluid, the elastic regime begins at strainrates around the inverse mean-atomic-vibration time, or 10^{13} s^{-1} at ordinary temperatures and pressures. The elastic response in a fluid is due to the interatomic potentials, and does not occur in a gas, where all interactions are represented simply by two-particle elastic collisions. That a fluid and a gas are essentially different is demonstrated by the behavior of the viscosity, which decreases with temperature for a fluid, and increases with temperature for a gas. The onset of elastic response in a fluid at high

strainrates suggests that viscous stresses cannot be arbitrarily large; this is certainly the case in the shock process, as will be discussed in later.

When we speak of a viscous fluid, we mean a real fluid (a viscoelastic fluid) which is operating in the viscous regime. The most universally popular myth in high-strainrate materials response today is that an elastic-plastic solid is equivalent to, or approximately equivalent to, a viscous fluid. The difference in constitutive behavior between an elastic-plastic solid and a viscous fluid is not trivial, it is vital. Any experiment performed on a solid, and interpreted with viscous fluid theory, will indicate an enormous "viscosity," and one which depends intimately on the experiment itself. To add to the confusion, there is also the popular "viscoelastic solid." Just because this term is contained in a textbook, in a list of constitutive models, does not mean that any solid or earth behaves this way.

LECTURE II-CONTINUUM MECHANICS AND ANISOTROPIC THERMOELASTICITY

In this lecture we assume the student has a basic knowledge of the topics, either from prior study or other lectures. We will list equations to establish notation and will point out special features that are important to nonequilibrium thermodynamics. We treat in abbreviated form anisotropic and finite strain elasticity with which the student may be unfamiliar.

1. Mechanical Equations

The Eulerian form of the equation for conservation of mass is the following:

$$\frac{\partial \rho}{\partial t} \Big|_{\vec{x}} + \frac{\partial (v_i \rho)}{\partial x_i} \Big|_{\vec{x}} = 0, \quad (2.1)$$

where t is time, ρ is mass per unit volume, \vec{x} is the vector of Eulerian position coordinates, and v_i is the i th Cartesian component of the particle velocity. The term "particle" means a small piece of the material which continues smoothly into its neighbors and does not imply that the material is broken up into pieces. Double indices are to be summed over. The Eulerian position coordinates are simply the coordinates in a coordinate frame fixed in space through which the material moves. This equation describes how the material's mass compresses or expands as the material moves. The momentum conservation equation appears as:

$$\rho \partial v_j / \partial t \Big|_{\vec{X}} = \partial \tau_{ij} / \partial x_i \Big|_t, \quad (2.2)$$

where τ_{ij} is the Cauchy stress tensor (referred to real areas in the Eulerian coordinate system), and \vec{X} is the Cartesian Lagrangian coordinate which gives the position of a particle at some initial time. Thus, a particle is labeled uniquely by the \vec{X} it had at this initial time. The partial with respect to time holding \vec{X} fixed means a material derivative or one which follows the particle along its path. This equation is simply Newton's second law in continuum form: "force = time rate of change of momentum." This equation is given in the mixed Eulerian-Lagrangian form commonly used in computer calculations.

One can define a mean compressive stress \bar{P} as $-(1/3) \tau_{ii}$. \bar{P} is not to be confused with the "pressure", however; which we reserve to mean the "thermodynamic pressure." The pressure is defined only when the stress tensor is isotropic, $\tau_{ij} = -P \delta_{ij}$, and when the material is in local thermodynamical equilibrium. The minus sign is present because stresses are outward forces, while pressure is a force inward on a body. Hence, even when local thermodynamic equilibrium obtains, \bar{P} is not the "pressure" if τ_{ij} is anisotropic. The concept of thermodynamical pressure is not then general enough to fit the situation. For example, P is always a single valued function of V and T , but \bar{P} can be made to take on a continuous range of values for fixed V and T by adjusting the components of τ_{ij} .

The energy belonging to each mass element is divided into two parts, the translational kinetic energy, $(1/2)v^2$ per unit mass and the center-of-mass or

internal energy, E per unit mass. The increase of translational kinetic energy of the mass element is

$$d[(1/2)m\vec{v}^2] = mv_j \partial v_j / \partial t \Big|_{\vec{x}} dt \quad . \quad (2.3)$$

A change in the center-of-mass energy is divided into two parts, work done and heat transported. Before introducing the equation for internal energy conservation, we provide a description of heat. Heat is the excitation of a system, or a mass element, at fixed configuration. For an atomic solid or fluid, it is the motional energy of the atoms in the center-of-mass frame; in a metal, the excitation of conduction electrons is also included. Heat is defined without regard to thermodynamic equilibrium. Also, there is no (significant) transport of mass or momentum associated with heat transport. It should be recognized, however, that the real operational definition of heat is not contained in these words, but is contained in the constitutive equation for heat transport which one uses in any given calculation. Some common forms for this equation are discussed later.

The equation of internal energy conservation, for a single material particle moving in the solid, is the following:

$$\rho \partial E / \partial t \Big|_{\vec{x}} = \tau_{ij} \partial \epsilon_{ij} / \partial t \Big|_{\vec{x}} - \partial J_i / \partial x_i \Big|_t \quad , \quad (2.4)$$

where E is the internal energy per unit mass, and \vec{J} is the Eulerian heat flux vector. ϵ_{ij} is the infinitesimal strain tensor defined in terms of infinitesimal Eulerian displacements u_j from the current configuration as follows:

$$\epsilon_{ij} = 1/2 (\partial u_i / \partial x_j + \partial u_j / \partial x_i) . \quad (2.5)$$

We will define u_{ij} to be $\partial u_i / \partial x_j$. The term involving $\partial \epsilon_{ij}$ in equation (2.4) is the total internal mechanical work done on the particle per unit volume, which we call dW . The term involving J_1 is the total increase in heat per unit volume for the particle, which we call dQ . The above Lagrangian equation can be more simply expressed as: $dE = dW + dQ$, where dE is the internal energy increase of the particle.

We need to emphasize an aspect of the incremental work dW : this is work done on a mass element, by its surroundings, through the action of stresses. Nothing was said about dW being conservative, or reversible, and in fact it generally is not. The applied stresses can drive plastic flow in a solid, or viscous flow in a fluid, and these are both dissipative processes. The work term can be partially or totally dissipative. Because of this, it is not possible to keep track of the heat content of a given mass element by simply integrating the term dQ ; dQ represents the net heat transported into the mass element, but dW acts partially or totally as a source term for heat as well.

An important point is that the continuum-mechanic conservation equations (2.1) - (2.4) above hold for any material with any constitutive behavior; they apply to dissipative processes. They were derived from very basic physical principles which always hold. They don't need to be "corrected." The quantities they contain are general mechanical quantities which are defined whether or not thermodynamical equilibrium holds.

2. Thermoelasticity

We will be concerned with thermoelasticity for a solid of arbitrary crystal symmetry in the presence of anisotropic stresses. For irreversible thermodynamics, we need the assumption of local thermodynamic equilibrium, in which the thermodynamic variables are fields, but all the equilibrium relations hold locally. In other words, we assume that if the material is subdivided finely enough, the equilibrium thermodynamic relations will apply to the pieces individually with suitable terms added to account for irreversibility. Equilibrium thermodynamics assumes that the thermodynamic quantities are uniform throughout the solid, in irreversible thermodynamics we assume that quantities are uniform locally. The thermodynamic relations will hold in the rest frame of the material.

It is worth mentioning that the equilibrium thermodynamic equations are independent of time. They involve only differences between equilibrium states in which time is not involved.

With anisotropic stresses, an equilibrium thermodynamic state can be specified by the elastic configuration, or by the stress tensor, plus one more variable such as entropy or temperature. We will take the independent variables to be the elastic configuration and the entropy. Since it will be necessary eventually to distinguish between elastic and plastic strains, we will use a superscript e to denote elastic strains. The incremental elastic strain gradients du_{ij}^e carry the current configuration \vec{x} into the next configuration $\vec{x} + d\vec{x}$. We do, in fact, include finite strains but introduce only infinitesimal strain changes in the theory. The incremental entropy dS represents entropy from all sources. It includes that from heat flow here and will be generalized in the nonequilibrium case to include dissipation. Thus, the

independent incremental variables are the set $\{du_{ij}^e, dS\}$. The symmetric elastic strains $d\epsilon_{ij}^e$ are $d\epsilon_{ij}^e = 1/2 (du_{ij}^e + du_{ji}^e)$. The rotations are to be thought of as rigid. Elastic strain does not contribute to rotation, nor does plastic strain. Hence, rotation increments are denoted simply $d\omega_{ij}$, where $d\omega_{ij} = 1/2 (du_{ij}^e - du_{ji}^e)$. Note this implies that the displacement gradients are not entirely elastic: the antisymmetric part of du_{ij}^e is rigid rotation. Using the elastic and rotation tensors, the complete set of independent variables becomes $d\epsilon_{ij}^e$, $d\omega_{ij}$, and dS .

In order to simplify notation, we will omit the indication, in partial derivatives, of which variables are to be held constant. This can be done because we use a single set of independent variables; any partial derivative is carried out with the remaining independent variables held fixed. In our algebra, we will take all ϵ_{ij}^e and ω_{ij} to be independent variables with the constraint that ϵ_{ij}^e is symmetric in ij and ω_{ij} is anti-symmetric.

In thermoelasticity, all strains are elastic and we can write the energy increment equation as:

$$\rho dU = \tau_{ij} d\epsilon_{ij}^e + \rho T dS \quad . \quad (2.6)$$

This equation does not contain $d\omega_{ij}$, which indicates that dU is invariant under rigid rotations. The customary expressions for stresses and temperature, as energy derivatives, follow directly from equation (2.6):

$$\tau_{ij} = \rho \partial U / \partial \epsilon_{ij}^e \quad , \quad (2.7)$$

$$T = \partial U / \partial S \quad . \quad (2.8)$$

We will express increments in τ_{ij} and T in terms of those of the independent variables, and break the thermodynamic hierarchy there. We will consider the coefficients in the resulting equations as known, from experiment or theoretically. These equations, then, are the ones to be used in actual calculations.

A subtle but very important point arises. Both sides of a thermodynamic equation are evaluated at the same state and both sides are true for any thermodynamic state. Hence, in principle the equation can be differentiated, since differentiation is just a process of taking differences between neighboring states. However, equation (2.7) for τ_{ij} is not a general relation; the left hand side is τ_{ij} only when the strains ϵ_{ij} are zero. Hence it cannot be simply differentiated with respect to this quantity. The wrong answer results. A proper general equation is given in reference (4) where the proper way to differentiate this equation is described. The algebra is not too hard to follow, but too lengthy to be included here. The result, along with the result for stress changes with rotation, is written schematically as

$$\partial \tau_{ij} / \partial \epsilon_{kl}^0 = B_{ijkl} \quad , \quad (2.9)$$

$$\partial \tau_{ij} / \partial \omega_{kl} = 1/2 (\tau_{il} \delta_{jk} - \tau_{ik} \delta_{jl} + \tau_{jl} \delta_{ik} - \tau_{jk} \delta_{il}) \quad , \quad (2.10)$$

where B_{ijkl} is an adiabatic stress-strain coefficient. It tells how the stress tensor changes when the strains change infinitesimally. It is symmetric in ij and kl but not with respect to interchanging ij and kl . This equation does not include rotations of the mass element. If the mass element has in fact rotated, then the stress has rotated with it. Hence, components of stress in the laboratory system are changed, and it is this change which equation (2.10) gives. To give added insight into the B_{ijkl} coefficients, we write them in terms of the special second order elastic constants, C_{ijkl} :

$$B_{ijkl} = C_{ijkl} + 1/2(\tau_{ik}\delta_{jl} + \tau_{il}\delta_{jk} + \tau_{jk}\delta_{il} + \tau_{jl}\delta_{ik} - 2\tau_{ij}\delta_{kl}) . \quad (2.11)$$

The C_{ijkl} are defined by $\rho \partial U^2 / \partial \eta_{ij} \partial \eta_{kl}$ evaluated at the initial condition of zero η_{ij} , where η_{ij} are the symmetric finite strain parameters of Murnaghan defined by

$$\eta_{ij} = 1/2 (u_{ij} + u_{ji} + u_{ki}u_{kj}) . \quad (2.12)$$

u_{ij} is the displacement gradient $\partial u_i / \partial x_j$ and displacements are here not limited to infinitesimal displacements. When the stresses are zero, the C_{ijkl} correspond physically to what one is used to thinking of as elastic constants in the infinitesimal strain case. But when stresses are non-zero, they can be different. Equation (2.11) is derived in reference (4). Some idea of why the stress dependence should occur in B_{ijkl} is given by the following simple example. Consider a long rectangular bar of unit cross sectional area and of initial length L_0 pointing along the x -axis. We stretch it quasistatically to

length L , keeping the cross-sectional area constant. Incidentally, the bar is in the same condition it would be if it were part of a solid of infinite cross sectional area that experienced a uniform x -displacement to length L . We call this condition of the bar state 1. The total force on the bar is the same as the stress σ_{11} normal to bar's end and can be simply related to the stretch of the bar, $L - L_0$, by the equation $\sigma_{11} = k (L - L_0)$ where k is a "spring constant." We have used Voigt notation in σ_{11} . We now stretch the bar further by the infinitesimal length ΔL which increases the stress by $\Delta\sigma_{1j} = k\Delta L$. The infinitesimal strain experienced by the bar in this last stretch is $\epsilon_{11} = \Delta L/L$, with all other components being zero. Hence, we now can get the B_{11} coefficient for the bar of length L (state 1) by dividing $\Delta\sigma_{1j}$ by ϵ_{11} to get kL . The C'_{11} coefficient is defined as the usual elastic coefficient when the bar is in its initial unstressed condition. One can take two partials of the potential energy per unit volume $1/2 k (L'-L_0)^2/L_0$ with respect to the infinitesimal strain ϵ'_{11} referred to the initial condition to obtain C'_{11} . This strain is in fact $(L'-L_0)/L_0$, which enables one to easily obtain $C'_{11} = k L_0$. Finally, we rewrite $B_{11} = k L$ as $k L_0 + k (L - L_0)$, which is easily seen to be $C'_{11} + \sigma_{11}$. This completes the example. The result can be seen to follow because the infinitesimal strains used to define B_{11} and C'_{11} are referred to different configurations. We note that in equation (2.11) above which relates B_{ijkl} to C_{ijkl} and τ_{ij} , the C_{ijkl} is referred to the current positions or to state 1, while C'_{11} above was referred to state the initial condition. The C'_{11} nevertheless fits into equation (2.11) when specialized to this simple case. The resolution of the puzzle is that C'_{11} equals C_{11} for this simple example.

We continue the program of calculating derivatives required to construct equations for $d\tau_{ij}$ and dT . The anisotropic Gruneisen parameters γ_{ij} are defined by

$$\partial\tau_{ij}/\partial S = - \rho T \gamma_{ij} \quad . \quad (2.13)$$

The γ_{ij} represent the thermal energy contribution, or heat contribution, to the stresses, as can be seen by rewriting equation (2.13) in the form

$$\rho \gamma_{ij} = - \partial\tau_{ij}/\partial U]_{\eta} \quad . \quad (2.14)$$

The subscript η means at constant elastic configuration. The heat capacity at constant configuration is C_{η} , defined by $\partial T/\partial S = T/C_{\eta}$. Finally, the strain derivatives of the temperature are

$$\partial T/\partial \epsilon_{ij}^E = - T \gamma_{ij} \quad , \quad (2.15)$$

$$\partial T/\partial \omega_{ij} = 0 \quad . \quad (2.16)$$

One can understand why γ_{ij} appears in both the equation for ∂T and the equation for $\partial\tau_{ij}$ when these equations are expressed in terms of partials of U . Interchanging the order of differentiation yields the same derivative of U in both cases.

Now we can write down the final equations:

$$d\tau_{ij} = B_{ijkl} d\epsilon_{kl}^e + d\tau_{ij}^\omega - \rho\gamma_{ij} TdS \quad , \quad (2.17)$$

$$dT = - T\gamma_{ij} d\epsilon_{ij}^e + TdS/C_\eta \quad . \quad (2.18)$$

As an interesting exercise, one might think of how, for a single crystal of tetragonal or lower symmetry, one would measure C , and also how one would measure C_η . Show that a measurement of γ is contained in the first experiment, and a measurement of γ_{ij} is contained in the second experiment.

A common circumstance in treatises on elastic-plastic strain is that only Hooke's law is considered. In the present work, the first term on the right of equation (2.17) is Hooke's law, in differential form. Omitting consideration of the rotation term, there is still another contribution to equation (2.17), and that is the term in TdS . That term is necessary for the specification of $d\tau_{ij}$. Even if one changes variables, the equivalent of the TdS term will always be present. In other words, stress can never be expressed in terms of strain alone.

3. Approximation of Small Anisotropy

Thermoelasticity was developed in the first place to describe experiments, especially stress-strain experiments and wave-propagation experiments, on single crystals in the presence of applied stresses. Applications of thermoelasticity can become quite complicated, especially for crystals of low symmetry. On the other hand, in plastic flow experiments, it is often possible to consider the solid under study as nearly isotropic. A polycrystalline aggregate, for example, is approximately isotropic for many

purposes. It is therefore useful to simplify the thermoelastic equations for the case of an isotropic solid. Of course, as soon as an anisotropic stress is applied, the solid undergoes anisotropic elastic strain, and in this state the solid is not isotropic. But if the anisotropy of the elastic strain is small, then the material anisotropy will be small, and this will be the basis of our approximation. Ultimately, we rely on plastic flow to keep the anisotropy of elastic strains small, or equivalently, to keep the stress close to a pure pressure. There will always be some solids for which the approximation is not acceptable.

The small-anisotropy approximation consists in replacing the second-order thermoelastic coefficients C_η , γ_{ij} , and B_{ijkl} by what they would become if the anisotropic strains were changed to have the same volume dilation with no other strains. The approximation is easily visualized in terms of an equilibrium thermodynamic process. While the process goes along a line in anisotropic-state space, having volume V and entropy S at any point, a unique image point moves along a line in isotropic-state space, always having the same V and S . The second-order coefficients are evaluated at the image point.

The approximation leads to the following replacements: $C_\eta \rightarrow C$, $\gamma_{ij} \rightarrow \gamma \delta_{ij}$, and the B_{ijkl} are replaced by the isotropic $B_{\alpha\beta}$, which have the following symmetry:

$$\begin{array}{ccccccc}
 B_{11} & B_{12} & B_{12} & & & & \\
 B_{12} & B_{11} & B_{12} & & & & 0 \\
 B_{12} & B_{12} & B_{11} & & & & \\
 & & & B_{44} & 0 & 0 & \\
 & & & 0 & 0 & B_{44} & 0 \\
 & & & 0 & 0 & B_{44} & ,
 \end{array}$$

where we have written B_{ijkl} in the Voigt notation form $B_{\alpha\beta}$ where the Greek indices take values 1-6 which correspond to the index pairs xx, yy, zz, yz or zy, xz or zx, and xy or yx, respectively. This replacement is allowed because B_{ijkl} is symmetric in ij and symmetric in kl. There are only two independent $B_{\alpha\beta}$, because of the additional relation $B_{44} = 1/2 (B_{11} - B_{12})$. One can express B_{11} as $\lambda + 2\mu - P$, where P is the pressure of the isotropic state and λ and μ are the Wallace Lamé elastic coefficients. $B_{4\mu}$ is equal to $\mu - P$.

We will write the equations for dU, $d\tau_{ij}$, and dT in the small anisotropy limit. It will be convenient to change the stress variables from τ_{ij} to the set \bar{P} , s_{ij} , where the s_{ij} are stress deviators defined by

$$s_{ij} = \tau_{ij} + \bar{P}\delta_{ij} \quad . \quad (2.19)$$

We will also use conservation of mass in the form $d \ln V = d\epsilon_{11}^e + d\epsilon_{22}^e + d\epsilon_{33}^e$. This equation introduces a new notation, in which Cartesian indices are written as numbers 1,2,3, and repeated numbers are not summed. There are times

when the algebra is simplified by enumerating the Cartesian indices, and explicitly writing out all the terms in a calculation. The small-anisotropy equations are the following:

$$dU = -\bar{P}dV + TdS + Vs_{1j}d\epsilon_{1j}^e, \quad (2.20)$$

$$d\bar{P} = -B d \ln V + \rho\gamma TdS, \quad (2.21)$$

$$dT = -\gamma T d \ln V + TdS/C, \quad (2.22)$$

$$ds_{11} = 2G[de_{11}^e - (1/3)d \ln V] + ds_{11}^\omega, \quad (2.23)$$

$$ds_{12} = 2G de_{12}^e + ds_{12}^\omega, \quad (2.24)$$

where the remaining diagonal and off-diagonal stress deviators are given by obvious relabeling of (2.23) and (2.24), respectively. The rotational terms are

$$ds_{11}^\omega = 2(s_{12}d\omega_{12} + s_{13}d\omega_{13}), \quad (2.25)$$

$$ds_{12}^\omega = (s_{22} - s_{11})d\omega_{12} + s_{23}d\omega_{13} + s_{13}d\omega_{23}, \quad (2.26)$$

and the remaining rotation terms are given by cyclic permutation of indices in (2.25) and (2.26) above.

Once again, consider an equilibrium thermodynamic process, in which the solid moves along a line whose points are specified by (\vec{x}, S) , while its image moves along a corresponding line specified by (V, S) . The image is undergoing a process in isotropic-state space, and hence is governed by the equations of

pressure-volume thermodynamics. Let us use the subscript I, for isotropic, to denote variables belonging to the image. Then comparison of the small-anisotropy equations with their isotropic limits, and remembering that $V(\vec{x}) = V$, gives the results

$$\bar{P}(\vec{x}, S) = P_I(V, S) \quad , \quad (2.27)$$

$$T(\vec{x}, S) = T_I(V, S) \quad , \quad (2.28)$$

$$U(\vec{x}, S) = U_I(V, S) + U_A(\vec{x}, S) \quad , \quad (2.29)$$

where U_A is the anisotropic elastic energy of the solid, and is given by

$$U_A(\vec{x}, S) = \int V s_{1j} d\epsilon_{1j}^e \quad . \quad (2.30)$$

The integral is along the path of the process, from the initial state to the current state. The above equations show that the approximation of small anisotropy reduces the thermoelastic process to a "minimally anisotropic" one: the mean compressive stress and the temperature can be obtained from the isotropic equation-of-state state, and the presence of anisotropy is contained only in the existence of nonzero stress deviators, and of the anisotropic elastic energy. Of course, when thermoelasticity is coupled to continuum mechanics, the presence of these anisotropic terms will affect the course of the process, at each timestep.

As an exercise, the student can imagine how, for a given polycrystalline solid, he would make the best currently-possible estimates of the quantities

B, C, γ , and C, for pressures up to 100 kbar (10^5 atmospheres) and for temperatures from room temperature to melting.

LECTURE III - PLASTIC FLOW AND IRREVERSIBLE THERMODYNAMICS

In this lecture we emphasize the irreversible thermodynamical aspects of plastic flow. We assume general knowledge of the yield and flow conditions, but give a summary of the Prandl-Reuss-von Mises theory for completeness and to establish the notation. Some standard textbook references for this theory, which is the simplest for isotropic solids, are the works by Hill⁵, Mendelson⁶, and Kachanov.⁷ Techniques for computer calculation of elastic-plastic flow problems in one- and two-dimensions are discussed by Wilkins.⁸

1. Plastic Yield

Before describing the plastic yield condition of von Mises, we discuss a few useful aspects of stress in a solid. Since the stress tensor τ_{ij} is a real symmetric tensor, a coordinate system exists, called the principle axis system, in which the stress is diagonal. In this axis system the three diagonal elements of the stress tensor can be taken as the coordinates of a vector; i.e. the stress can be represented as a vector. Given a plane, the stress acting on it can be resolved into a normal component and a tangential component. In principal axes, the spherical stress line (also called the hydrostatic line) is the octahedral line $\tau_{11} = \tau_{22} = \tau_{33}$, where these diagonal components are to be plotted as a vector as described above. Planes perpendicular to this line (octahedral planes) are planes of constant spherical stress, which is constant mean compressive stress in our notation. An arbitrary stress is decomposable into a projection onto the spherical stress line, and a projection onto the octahedral plane, in this principle axis

space. In our notation, these projections are proportional to the mean compressive stress, and the shear stress. Specifically, the stress vector in principal axes has the length $\sqrt{(\tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2)}$, has a projection of length $-\sqrt{3} \bar{P}$ along the spherical stress line, and has the the projection of length $\sqrt{(2S_2)}$ in the octahedral plane, where S_2 is the second invariant of the stress deviators, $(1/2) s_{ij}s_{ij}$.

The concept of yield is as follows. If the stresses applied to an elastic-plastic solid are increased from zero, the response of the solid is initially elastic, and when the anisotropic stresses reach a certain "point," plastic flow begins. The point reached is a point on the "yield surface." The yield surface depends on the state of the solid when the experiment is performed; it depends, for example, on the temperature, the pressure, and the dislocation density. For the moment, let us consider merely the stress dependence: the yield surface is given by an expression of the form $f(\tau_{ij}) = 0$. For an isotropic solid, the yield surface should depend on the stress only through the rotational invariants. From the decomposition of the stress in principal axes, and arguing that the yield surface should not depend on \bar{P} , because it is the anisotropic part of the stress that drives plastic flow, one expects the yield surface to depend only on S_2 . This is the approximation of von Mises: yield occurs when the component of tangential stress on the octahedral plane reaches a fixed value.

In the present work we define a rotationally invariant effective shear stress τ , according to

$$\tau^2 = (3/4) S_2 = (3/8) s_{ij}s_{ij} \quad , \quad (3.1)$$

where τ is equal or greater than 0. The von Mises yield condition is then $\tau = K$, where K is a material property which determines the yield strength of the material.

2. Plastic Flow

The plastic strain (or flow) will be denoted by a superscript p , as $d\epsilon_{ij}^p$. For a process in which elastic and plastic strains are going on simultaneously, it is possible to add the two kinds of strain in infinitesimal form. The total strain $d\epsilon_{ij}$ is then $d\epsilon_{ij}^e + d\epsilon_{ij}^p$. Since the rotation is rigid, involving neither elastic nor plastic strain, increments of the displacement gradients are

$$du_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p + d\omega_{ij} . \quad (3.2)$$

Now a problem arises. Consider the transformation α , which transforms the initial configuration \vec{X} to the current configuration \vec{x} , such that the function $\alpha_1(\vec{X})$ gives x_1 . The functions $\alpha_{ij}(\vec{X})$, defined by $\partial\alpha_1(\vec{X})/\partial X_j$, give the gradients of the current configuration \vec{x} with respect to the original configuration \vec{X} , and show how the current configuration has evolved from the initial configuration. The α_{ij} relate the incremental vector $d\vec{X}$ at the initial configuration with the incremental vector $d\vec{x}$ of the current configuration as follows:

$$dx_i = \alpha_{ij} dX_j . \quad (3.3)$$

Increments $d\alpha_{ij}$ can be calculated from du_{ij} from the equation $d\alpha_{ij} = du_{ik} \alpha_{kj}$,

which can be proved from the definitions above. Hence, these increments can be integrated along the material process to give $\alpha(\vec{x} \rightarrow \vec{x})$ for any configuration \vec{x} . Thus the total configuration is known, for every mass element and for every time. Furthermore, it is possible to write equations for $d\alpha_{ij}^e$ and $d\alpha_{ij}^p$ separately, under the condition that $\alpha_{ij} = \alpha_{ij}^e \alpha_{ij}^p$. The total α_{ij} have the same meaning as before, but the component matrices α_{ij}^e and α_{ij}^p have no independent physical meaning, because they do not commute. In other words, when elastic and plastic strains are going on simultaneously, it is not possible to integrate the strains along the process to find separately the current elastic and plastic configurations in any objective sense. Some workers⁹ have used the α_{ij}^e and α_{ij}^p and given them the following interpretation. α_{ij}^p takes the initial configuration and applies a pure plastic strain which rearranges the atoms but does not generate any stresses. The resulting condition is thought of as an intermediate unloaded state. Then, the α_{ij}^e strains are applied which load the material elastically to the current stresses. This scheme is workable, but the intermediate state has no objective physical meaning. The material as a whole cannot exist in this state without breaking apart. To make the scheme work, one must think of the intermediate state as existing for each piece of material individually, not in connection with those of the neighboring pieces. Other, different schemes for decomposing the total strain into elastic and plastic part are possible¹⁰ and all suffer from similar problems.

We prefer to deal with the situation as follows. First, the incremental strains $d\epsilon_{ij}^e$ and $d\epsilon_{ij}^p$ are perfectly good differential variables for the coupled differential equations of a flow process; these equations can still be

integrated step by step. However, we do not have the elastic configuration available as an integrated state variable. But the integrated stresses τ_{ij} can serve as state variables for thermoelasticity. These and one other variable, say S , completely specify a thermoelastic state, including the elastic configuration. For the plastic strain, we will introduce a non-negative scalar measure of the total plastic strain increment, and the integral of this will provide a measure of the total strain which has occurred in the material.

The simplification of plastic strain is accomplished by the Prandtl-Reuss approximation: it is assumed that plastic strain increments are proportional to the corresponding stress deviators.^{11,12} That is, $d\epsilon_{ij}^P = s_{ij} d\lambda$, where $d\lambda$ is some scalar infinitesimal. To identify an effective plastic strain in $d\lambda$, we divide by the effective shear stress τ , and write the Prandtl-Reuss approximation as

$$d\epsilon_{ij}^P = 3s_{ij} d\psi / (4\tau) \quad , \quad (3.4)$$

where $d\psi$ is the effective plastic strain increment. With the Prandtl-Reuss approximation, there is no volume change due to plastic strain, i.e. $d \ln V^P = d\epsilon_{11}^P = 0$, because $s_{11} = 0$. This property is in accord with the experimental observation that plastic strain is volume conserving.

Another property of the effective plastic strain results from the physical nature of the process, namely that plastic strain is a stress-relaxing process. For any incremental process which occurs at constant total configuration, i.e. $d\epsilon_{ij} = 0$, the set of plastic increments $d\epsilon_{ij}^P$ must be such as

to reduce the magnitude of each and every s_{ij} . With the Prandtl-Reuss approximation, this will be the case if and only if $d\psi > 0$. The simplest way to show that $d\psi > 0$ is to calculate the incremental plastic work done on a mass element. This is given by $\rho dW^P = \tau_{ij} d\epsilon_{ij}^P$. Thus, the plastic work is work done by the stresses against the plastic strains (there is no restriction on the total $d\epsilon_{ij}$ here). With τ_{ij} replaced by $s_{ij} - \bar{P}\delta_{ij}$, and with the Prandtl-Reuss approximation, the plastic work is

$$\rho dW^P = (3/4\tau) (s_{ij} - \bar{P}\delta_{ij})s_{ij}d\psi \quad (3.5)$$

From this it is seen that the mean compressive stress does no work in plastic strain, because $s_{ii} = 0$. With the definition of τ , equation (3.1), the plastic work becomes $\rho dW^P = 2\tau d\psi$. Now in any mass element, plastic strain cannot proceed in such a way as to do work on the surroundings. Therefore, we must have $dW^P > 0$, and since $\tau > 0$, it follows that $d\psi$ must be greater or equal to zero. This condition is not to be regarded as a constraint, because it should be automatically satisfied in any correct calculation. As a result of $d\psi > 0$ for any mass element, the integrated plastic strain ψ is a nondecreasing function of time. As an informative exercise, one can prove, for a process at constant total configuration, with the Prandtl-Reuss approximation and with the approximation of small anisotropy, that the magnitude of every s_{ij} decreases if and only if $d\psi > 0$.

With the Prandtl-Reuss approximation, the constitutive behavior of a solid is almost completely specified. It remains to specify the behavior of

the effective shear stress τ . This alone is an extremely difficult undertaking, because of the complications of plastic flow behavior in real solids. We will merely try to list the variables which control the behavior of τ , and that for an isotropic solid only. The von Mises constant K is generalized to a flow surface, i.e. τ still equals K , but K is no longer constant. K should depend on the thermoelastic state, which for an isotropic solid is specified by V and S . A dependence on pressure is implied here. In contrast to von Mises' original point of view, which neglects the dependence of K of \bar{P} , because \bar{P} does not drive plastic flow, we may want to apply the theory at pressures high enough to alter the material properties. Hence the dependence on pressure is kept. The effect of work hardening is represented by a dependence of K on the plastic strain ψ , and also K should depend on the plastic strainrate $\dot{\psi}$, which is the Lagrangian time derivative of ψ , $\dot{\psi} = \partial\psi/\partial t]_{\chi}$. Hence in the plastic flow regime, i.e. when $\dot{\psi} > 0$: $\tau = K(\psi, \dot{\psi}, V, S)$. When the effective shear stress is inside the flow surface, the plastic flow ceases and the solid is in the elastic regime, with $\dot{\psi} = 0$: $\tau < K(\psi, \dot{\psi}, V, S)$. These two equations represent a common way of expressing the plastic constitutive behavior of an isotropic solid. However, the relations are presumably unique, which means invertible, and it is logically simpler to think of the plastic strainrate as the dependent variable. This approach also has better stability properties for numerical calculations. Hence one can write the above two equations together as the plastic constitutive relation: $\dot{\psi} = \dot{\psi}(\tau, \psi, V, S)$.

3. Constitutive Relation for Heat Transport

For most problems of thermoelastic-plastic flow in solids, the propagation of thermal energy is not important. Heat is generated locally, by plastic dissipation, and the important process of heat transport is heat conduction. With this approximation, the constitutive equation for heat transport reduces to Fourier's law of heat conduction:

$$\vec{j} = -k\vec{\nabla}T \quad . \quad (3.6)$$

This is consistent with an infinite velocity of propagation of heat which then occurs by diffusion.

4. Irreversible Thermodynamics

Consider processes which are sufficiently close to equilibrium so that the total center-of-mass energy U can be identified as the thermodynamical internal energy E . Then dE which equals $dW + dQ$ is equal to dU . Writing dW as the sum of elastic and plastic contributions then puts the energy equation in the form

$$dU = dW^e + dW^p + dQ \quad . \quad (3.7)$$

The key step in completing the irreversible thermodynamic description is the identification of the entropy production. Heat transport produces entropy dS^K , according to the usual relation $TdS^K = dQ$. Also, the plastic work is assumed to be totally dissipated, so $TdS^K = dW^p$. The total is $dS^K + dS^\psi$, and the energy equation reads

$$\rho dU = \tau_{ij} d\epsilon_{ij}^e + \rho T dS \quad . \quad (3.8)$$

With $2\tau d\psi$ for the plastic work dW^P , the total entropy increment is given by

$$\rho T dS = \rho dQ + 2\tau d\psi \quad . \quad (3.9)$$

At this point, it is useful to make a list of the complete set of equations which describe the irreversible-thermodynamic elastic-plastic flow process in a solid.

Conservation of mass (2.1)

Conservation of linear momentum (2.2)

Conservation of energy (3.8)

Entropy production (3.9)

Stress tensor increments (2.17)

Temperature increment (2.18)

Plastic constitutive

Heat-transport constitutive (3.6)

For most problems, this system can be simplified considerably. Further, it is always possible to eliminate the elastic strains $d\epsilon_{ij}^e$, in favor of the total strains $d\epsilon_{ij}$ and the plastic strain $d\psi$. This is done merely by writing $d\epsilon_{ij}^e = d\epsilon_{ij} - d\epsilon_{ij}^P$, and then using the Prandtl-Reuss approximation to replace $d\epsilon_{ij}^P$ with $d\psi$. This is a useful step, because the equations of motion determine the total $d\epsilon_{ij}$. Let us make a sketch of how the system of equations

works in numerical calculation. Suppose that all the field functions are known in the current state $\{\vec{x}, S\}$. Given a time increment dt , we want to evaluate the functions at the next state $\{\vec{x} + d\vec{x}, S + dS\}$. The equations of motion (conservation of mass and momentum) give the increments $d\epsilon_{ij}$ and $d\tau_{ij}^{\omega}$. The plastic constitutive equation gives $d\psi = \dot{\psi}dt$. Then the heat transport equations and the entropy production equation give the total dS . With these increments, the changes in the internal energy, the stresses, and the temperature can be calculated, and all the functions are then determined in the next state.

It should be emphasized that the present theory can also be used to advantage in analyzing experiments. For most experimental conditions, all the material properties in the theory are reasonably well known, except for the plastic constitutive behavior. Hence the complete set of equations, together with a well-designed set of experimental measurements, will determine the plastic constitutive behavior of a given solid. This procedure was used in analyzing nonsteady shock profiles, to determine the plastic behavior of an aluminum alloy, for plastic strains up to 5%, and for plastic strainrates up to 10^7 s^{-1} .

In assigning the entropy production, it was assumed that the plastic work is totally dissipated. Experimental support for this in metals goes back to the work of Farren and Taylor,¹³ and of Taylor and Quinney,¹⁴ who found that approximately 90% of the plastic work is dissipated, for strains greater than a few percent. Subsequent research on the energy stored in cold working was reviewed by Tichener and Bever.¹⁵ This stored energy goes into the defect structure of the solid, most notably the dislocation structure, and the energy

is in fact recoverable. However, this energy is not included in ordinary thermoelastic theory, and an explicit accounting of it would require a redefinition of the thermoelastic quantities. For example, the presence of stored energy will give rise to a negative contribution to the heat capacity, which is strongly dependent on the heating rate, because of annealing. Hence the assumption of total plastic dissipation, besides being a good approximation as far as the total plastic work is concerned, also simplifies the complete theory. In the same vein, the remaining thermoelastic coefficients, namely the anisotropic Gruneisen parameters and the stress-strain coefficients, are presumed to be independent of the defect structure introduced by plastic flow. To the extent of the author's knowledge, this is in accord with experimental observation. On the other hand, the significant effect of the stored part of the plastic work, namely work hardening, is contained in the theory, through the presence of the plastic strain ψ in the plastic constitutive equation. Finally, an observation can be made regarding experiments designed to increase our understanding of high-strainrate processes in solids. If one is studying a process in which the temperature rise due to dissipation is important, then a measurement of the temperature can be helpful in extracting constitutive behavior from the experiment.

5. Application to Uniaxial Compression in a Uniaxial Solid

The purpose of this section is to further illustrate the general theory. The example of uniaxial flow is chosen because of its geometrical simplicity, and also because it includes planar wave processes. The restriction to compression is done merely to avoid worrying about the sign of the shear stress:

it is always positive in compression. The case of rarefaction, or tension, requires changing a few signs in the equations of the first two sections. While the general theory of the shock process is beyond the scope of this monograph, the basic equations underlying recent progress in shock theory are constructed.

In a Cartesian coordinate system, material motion is in the x-direction only. For a given mass element, the transverse surfaces do not move. A subtle point arises immediately. For uniaxial compression, when plastic flow occurs, atoms actually move in transverse directions. But this transverse motion of atoms does not give rise to a net transport of mass, or a net momentum, so the transverse motion is not seen in the continuum-mechanic equations for conservation of mass and momentum. The occurrence of plastic flow is accounted for by the continuum variable ψ ; the boundary conditions on the material motion, in the present case that transverse surfaces do not move, are instrumental in controlling the amount of plastic flow which takes place. The same observations apply to continuum elastic-plastic flow in other geometries as well.

For uniaxial flow, the first step is to write down the simplified list of continuum-mechanic variables. We let x be the laboratory coordinate, $X = x_a$ be the Lagrangian coordinate, and $v = \partial x / \partial t$ at constant X be the material or particle velocity. The transformation matrix is $\alpha_{xx} = \partial x / \partial X \Big|_t$ and equals ρ_a / ρ . α_{yy} and α_{zz} equal one and $\alpha_{ij} = 0$ for i not equal to j . No summation is implied by repeated indices x, y , or z in the above. The symmetric infinitesimal strains are $d\epsilon_{xx} = d \ln V$, with all other $d\epsilon_{ij} = 0$. The stress tensor

has only two independent components, the normal compressive stress σ , and the shear stress τ , where

$$\begin{aligned}\sigma &= -\tau_{xx} \\ (\sigma - 2\tau) &= -\tau_{yy} = -\tau_{zz}\end{aligned}\quad (3.10)$$

and $\tau_{ij} = 0$ for i not equal to j . The components σ and τ are both positive in compression. The τ here is the effective shear stress defined earlier for plastic flow. Also note there is no rotation in the case of uniaxial flow. As a consequence, the elastic and plastic decomposition of α_{ij} mentioned earlier can be done unambiguously in this case, i.e. the α_{ij}^e and α_{ij}^p commute.

The Lagrangian equation (2.1) for conservation of mass reduces to

$$(\rho_a / \rho^2) \partial \rho / \partial t]_X + \partial v / \partial x]_t = 0 \quad (3.11)$$

Conservation of linear momentum in Lagrangian form becomes

$$\rho_a \partial v / \partial t]_X + \partial \sigma / \partial x]_t = 0 \quad (3.12)$$

To proceed to the thermoelastic equations, we need to find expressions for the elastic and plastic components of strain. We will make repeated use of the symmetry in y and z . The stress deviators follow from (3.10). s_{xx} is $-4/3 \tau$ and $s_{yy} = 2/3 \tau$. The Prandtl-Reuss approximation gives $d\epsilon_{xx}^p = -d\psi$ and $d\epsilon_{yy}^p = 1/2 d\psi$. Then solving the total strain increments for the elastic

parts gives $d\epsilon_{xx}^e = d \ln V + d\psi$ and $d\epsilon_{yy}^e = -1/2 d\psi$. Hence in uniaxial compression with plastic flow, there are only two independent strain measures, which we will take to be $d \ln V$ and $d\psi$. The energy equation now reduces to

$$\rho dU = -\sigma d \ln V - 2\tau d\psi + \rho T dS \quad , \quad (3.13)$$

and the entropy production is

$$\rho T dS = -\partial J / \partial x \Big|_t dt + 2\tau d\psi \quad . \quad (3.14)$$

The heat current J is in the x -direction. The quantity of heat Q has been eliminated in favor of J , since this generally simplifies computations. A tricky point should be noted and remembered: elastic strain and plastic strain are not separately apparent in our equations, because they have been coupled through the boundary condition. In (3.13), for example, the first two terms on the right, involving both $d \ln V$ and $d\psi$, are just the elastic work $\tau_{ij} d\epsilon_{ij}^e$.

An isotropic solid under biaxial elastic strain has tetragonal symmetry. The adiabatic stress-strain coefficients $D_{\alpha\beta}$ have the following symmetry:

$$\begin{array}{ccccccc}
 B_{11} & B_{12} & B_{12} & & & & \\
 & B_{21} & B_{22} & B_{23} & & & 0 \\
 & & B_{21} & B_{23} & B_{22} & & \\
 & & & & & B_{44} & 0 & 0 \\
 & & & & & & & 0 & B_{66} & 0 \\
 & & & & & & & & 0 & 0 & B_{66}
 \end{array}$$

The coefficients B_{44} and B_{66} do not enter the equations for the stresses and temperature, because $d\epsilon_{ij}^e = 0$ for $i \neq j$. Voigt notation will also be used for the anisotropic Gruneisen parameters: $\gamma_{ij} = \gamma_\beta$. The tetragonal symmetry is accounted for by writing $\gamma_{xx} = \gamma_1$ and γ_{yy} and γ_{zz} equal γ_2 . With these symmetries, the earlier equations for the stresses and the temperature are evaluated to give

$$d\sigma = \rho\gamma_1 T dS - B_{11} d \ln V - (B_{11} - B_{12}) d\psi, \quad (3.15)$$

$$\begin{aligned}
 d\tau = & 1/2 \rho(\gamma_1 - \gamma_2) T dS - 1/2 (B_{11} - B_{21}) d \ln V \\
 & - 1/2 (B_{11} + 1/2 B_{22} + 1/2 B_{23} - B_{12} - B_{21}) d\psi. \quad (3.16)
 \end{aligned}$$

$$dT = -\gamma_1 T d \ln V - (\gamma_1 - \gamma_2) T d\psi + T dS/C_n \quad . \quad (3.17)$$

We have now written the set of equations for uniaxial flow. The energy equation is uncoupled from the rest, and can be discarded. Further, whenever the transport of heat can be neglected, then $\rho T dS = 2\tau d\psi$, and dS can be eliminated from the set of equations. It was this latter set of equations, with $J = 0$, which was used to extract plastic constitutive data from weak-shock profiles for an aluminum alloy.¹⁶ Following that, it was possible to calculate equation-of-state data from the shock measurements, explicitly accounting for the nonsteady nature of the shock profiles.¹⁷ This procedure gives more accurate equation-of-state information than does the customary method of using Hugoniot jump conditions for the shock analysis, since the jump conditions hold only for steady waves.

We mention an interesting aspect of the weak-shock work. In such shocks the strains, elastic or plastic or total, are small. They were less than 5% in the work cited. This allowed the equations (3.15), (3.16), and (3.17) for $d\sigma$, $d\tau$, and dT to be expanded in the strain. Both elastic and plastic strains are of roughly the same magnitude in weak shocks and were considered to be joint expansion parameters in the expansion. Terms up to the second order were kept. This proved to be enough terms for the weak shocks of interest. The B_{ijkl} were thus expressed in terms of second and third order elastic constants, both of which had been experimentally measured for the 6061T6 Aluminum material investigated. Heat conduction was neglected which is a good approximation for weak shocks. Hence the entropy production was due entirely to the plastic work: $\rho T dS = 2\tau d\psi$. Since both τ and $d\psi$ are of first order (both

depend in a "linear" fashion on the expansion parameters), the entropy is of second order. Hence in equation (3.15) for $d\sigma$, the term $\rho\gamma_1 TdS$ can be replaced by $\rho\gamma_a TdS$, where γ_a is the zeroth order initial value, since TdS is already of second order. Also, in the equation (3.16) for $d\tau$, the TdS term disappears when terms of second order are kept, since the factor $\gamma_1 - \gamma_2$ is of second order. Thus the shear stress has only elastic contributions to this order of approximation.

6. Steady Shock in an Isotropic Solid

The risetime of weak shocks in metals is long, and the nonsteady wave profile can be observed in detail with VISAR optical interferometric techniques.¹⁸ As the shock strength increases, the risetime decreases, until at a hundred kbar or so, the risetime can no longer be resolved. However, as far as it is known experimentally, a planar shock always travels at a constant velocity. Hence for moderately strong shocks, specifically for overdriven shocks, one expects a shock to propagate as a steady wave. The steady-wave condition allows the equations of motion to be integrated, which profoundly simplifies the complete set of equations for the flow process.

The symmetry is that of uniaxial compression, as described in the last section. For any function of $g(x,t)$ or $g(X,t)$, the relations between Lagrangian and Eulerian derivatives reduce to

$$\left. \frac{\partial g}{\partial X} \right]_t = \left(\frac{\rho_a}{\rho} \right) \left. \frac{\partial g}{\partial x} \right]_t \quad , \quad (3.18)$$

$$\left. \frac{\partial g}{\partial t} \right]_X = \left. \frac{\partial g}{\partial t} \right]_x + v \left. \frac{\partial g}{\partial x} \right]_t \quad . \quad (3.19)$$

A steady wave is a wave which travels at constant velocity without changing its shape. By this we mean that for any material property $g(x,t)$, the graph of g vs t through the wave profile is the same for all x . Hence $g(x,t)$ depends on only a single variable z , the "laboratory steady-wave variable," $z = x - Dt$, where D is the wave velocity. The steady wave condition is $g(x,t) = g(z)$. Partial differentiation yields $\partial g/\partial x]_t = dg/dz$ and $\partial g/\partial t]_x = -D dg/dz$. With (3.18) and (3.19) above, the Lagrangian derivatives are found to be

$$\partial g/\partial X]_t = (\rho_a/\rho) dg/dz \quad , \quad (3.20)$$

$$\dot{g} = \partial g/\partial t]_X = (v - D) dg/dz \quad . \quad (3.21)$$

As an exercise, one can prove, using conservation of mass, that the steady-wave condition $g(x,t) = g(z)$ implies, and is implied by, the condition $g(X,t) = g(Z)$, where $Z = X - Dt$ is the Lagrangian steady-wave variable.

The state ahead of the shock is assumed to be a thermodynamic equilibrium state, characterized by zero particle velocity v , zero normal stress σ , and zero heat current J . The state behind the shock is the Hugoniot state, denoted by subscript H , and it is also assumed to be a thermodynamic equilibrium state so that $J_H = 0$. In the steady wave analysis, it is convenient to use the compression variable ϵ defined by $\epsilon = 1 - V/V_a$. In terms of ϵ , the equation for conservation of mass is

$$\partial \epsilon/\partial t]_X + \partial v/\partial X]_t = 0 \quad . \quad (3.22)$$

With (3.20) and (3.21) above, and noting that $\epsilon_a = 0$, this integrates to $\epsilon = v/D$. In the same way, the equation (3.11) for conservation of linear momentum integrates to $\sigma = \rho_a Dv$. The curve of normal stress vs compression through the shock process is called the Rayleigh line, and from the last two equations this curve is a straight line:

$$\sigma = \rho_a D^2 \epsilon \quad . \quad (3.23)$$

This equation holds for intermediate points in the wave as well as for the final condition of the wave. It is important to recognize that the Rayleigh line is a straight line as a result of conservation of mass, conservation of momentum, and the steady-wave condition; no more and no less.

The energy and entropy equations, (3.13) and (3.14), for uniaxial flow are already in total differential form, except for the heat current term. With the relations $z = x - Dt$, $\partial g / \partial x \Big|_t = dg/dz$, and $\epsilon = v/D$, this term can be written, for a steady wave,

$$-\partial J / \partial x \Big|_t dt = \rho_a dJ / (\rho_a D) \quad (3.24)$$

Then the energy and entropy equations become

$$dU = \sigma v_a d\epsilon + dJ / (\rho_a D) \quad (3.25)$$

$$TdS = dJ / (\rho_a D) + 2v_a d\psi \quad . \quad (3.26)$$

With σ replaced by $\rho_a D^2 \epsilon$ and with the initial conditions $v_a = \sigma_a = J_a = 0$, the integral of σU is

$$U - U_a = 1/2 D^2 \epsilon^2 + J/(\rho_a D) \quad . \quad (3.27)$$

The Hugoniot is the locus of equilibrium states behind shocks of varying strengths; the Hugoniot exists for steady or nonsteady waves. The shock velocity D serves as a parameter specifying the shock strength. The Hugoniot jump conditions are the statements of conservation of mass, momentum, and energy, across a steady-wave shock. From the preceding integrals of the equations of motion, the Hugoniot jump conditions for a given shock velocity D are

$$v_H = D \epsilon_H \quad , \quad (3.28)$$

$$\sigma_H = \rho_a D v_H \quad , \quad (3.29)$$

$$U_H - U_a = 1/2 D^2 \epsilon_H^2 \quad . \quad (3.30)$$

It is important to remember that these conditions hold only for a steady wave. Since $\psi_H = 0$, the shear stress τ_H is presumably on the (static) yield surface. If one sets $\tau_H = 0$, as an approximation, then σ_H becomes the pressure P_H , and the equations (3.28)-(3.30) become the Hugoniot jump conditions for a fluid. For a fluid, the steady-wave Hugoniot can be constructed from the jump conditions together with the equation of state.

As a result of the steady-wave condition, space- and time-dependence has been eliminated from the equations of motion. This cannot be done for the constitutive equations, in general, but the dependence can be reduced to the single variable z . The heat conduction equation becomes

$$J = -\kappa \, dT/dz \quad , \quad (3.31)$$

and the plastic constitutive equation becomes

$$(v - D) \, d\psi/dz = \psi(\tau, \psi, V, S) \quad . \quad (3.32)$$

We can now make a list of the complete set of equations, called the Rayleigh-line equations, which govern the steady shock process.

Rayleigh line (3.23)

Entropy production (3.26)

Normal stress (3.15)

Shear stress (3.16)

Temperature (3.17)

Heat-transport constitutive (3.31)

Plastic constitutive (3.32)

In these equations, one can think of ϵ as the independent variable, and of course V is equivalent to ϵ . The equations listed are seven equations in the seven dependent variables, $\sigma, \tau, S, T, J, \psi$, and z . Hence for any given D , the steady shock process can be calculated from the seven equations. In fact, the

set of equations can be reduced algebraically. With the equation for the Rayleigh line itself, σ is trivially eliminated from the set. Also, one can use the energy equation to replace S by U , if this change of variables is desired. The real space- and time-dependence of the process can be calculated from $z(\epsilon)$. For example, at a constant x , say $x = 0$, $t(\epsilon) = -z(\epsilon)/D$, and at constant t , say $t = 0$, $x(\epsilon) = z(\epsilon)$.

As a useful exercise, prove that $dZ = (\rho/\rho_a)dz$. If $z(\epsilon)$ is known, how can you calculate $t(\epsilon)$ at a constant X , and $X(\epsilon)$ at a constant t ?

Note that there are two rate-dependent processes going on simultaneously in the shock, namely transport of heat, and plastic flow. At any point in the shock, the time-dependence of the profile must be simultaneously consistent with both of the dissipative processes. In other words, the shock risetime is consistent with both processes at once.

The above list of Rayleigh-line equations was used for a detailed analysis of the process of overdriven shocks in solids.¹⁹ In doing this, the plastic constitutive equation was purposefully removed from the set, on the grounds that it is a totally unknown quantity under such shock conditions. With one equation removed from the set, it is still possible to learn a great deal about the steady shock process, and to establish several theorems regarding the existence of solutions. For metals, the theory predicts that the shock risetime will decrease to around 10^{-12} s, and will not decrease further, as the shock strength increases.²⁰ Finally note that for shocks stronger than a few Mbar in metals, irreversible thermodynamics breaks down, and a new theory has to be constructed.

LECTURE IV-THE THERMODYNAMICS OF DUCTILE FRACTURE

We discuss a model for ductile fracture in which the material damage proceeds through the creation and growth of small, diffusively distributed voids. This simplified model illustrates many of the general features in a clearer way than a brittle fracture-crack model would. An example of such a model illustrating its complexity was given by Davison and Stevens²¹. We will give the general thermodynamical framework, with a few specific examples of the coefficients involved in the framework. To the author's knowledge, no thorough, detailed treatment of this topic, either for ductile or brittle fracture, is available. We describe the currently popular approximations and relations.

1. The Thermodynamical Framework

Ductile fracture of the kind we have in mind proceeds by the nucleation under tension of small voids in the material which proceed to grow by plastic flow in the material surrounding them. Since plastic flow is involved, we will need the thermodynamical framework for plastic flow already developed and will generalize it to include voids. It is experimentally observed that a tension threshold exists within which no appreciable void growth or nucleation occurs and beyond which these processes proceed rapidly. The situation resembles the yield threshold in plastic flow; which is to be expected since plastic flow is the basic process in the growth of voids. Little is known about the microscopic nucleation process, but it probably has things in common with plastic flow. We note that in shock wave phenomena, the process of rapid

ductile or brittle fracture has the name "spall." This has reference to the thin plates of material that are thrown or "spalled" off from the rear surface of a plate when a shock wave reflects from the rear surface as a tensile wave. The same general phenomena is going on here as in fracture at lower tensions and strain rates; only the magnitudes are different. Our theory will encompass ductile fracture of both kinds.

As before, we consider infinitesimal deformations from an arbitrary, stressed initial condition. For this reason, rotational frame invariance is not always explicit in our equations. One can consult References 22-24 in which non-infinitesimal deformations are used, to see how these appear in the equations.

We will describe the voids by the macroscopic internal variable D , which gives the void volume per unit volume in the unloaded situation. This variable is a perfectly good state variable for describing the thermodynamical equilibrium condition. D is an internal variable because there is no way externally of controlling it and because the processes which change it are much slower than the electron-phonon processes. For this reason, in a given material state, D may well want to change to raise the system's entropy. This change is described by the constitutive relation for D .

The thermodynamic description of ductile fracture will involve adding D to the previous set of equations for plastic flow. The set of independent variables now becomes ϵ_{ij}^e , S , and D . The elastic strain is that which occurs upon loading the initially unstressed, damaged material. It is important to realize that further void expansion occurs upon elastic loading that is not included in D . This expansion should not be confused with D and does not make

the definition of D ambiguous. The thermodynamic equations can be developed in a fashion strictly analogous to that used to develop the plastic flow equations. First, one takes all partials of the specific internal energy U to obtain:

$$\rho dU = \tau_{ij} d\epsilon_{ij}^e + \rho T dS + f dD \quad (4.1)$$

where f is $\rho \partial U / \partial D$ and is analogous to a force. f shows how much reversible energy is stored per unit increment of D . Thus $f dD$ is work associated with dD that is stored and not dissipated. This equation pertains to equilibrium states of the material and does not explicitly include nonequilibrium processes or dissipation. We will discuss these aspects later when dS is discussed.

We proceed to the second hierarchy of the thermodynamical scheme by taking all possible partials of τ_{ij} , T , and f , in turn. The result for $d\tau_{ij}$ is

$$d\tau_{ij} = B_{ijkl} d\epsilon_{kl}^e - \rho T \gamma_{ij} dS + b_{ij} dD + d\tau_{ij}^w \quad (4.2)$$

where b_{ij} is $\partial(\rho \partial U / \partial \epsilon_{ij}^e) / \partial D$ and can be thought of as a "stress-strain" coefficient for $d\tau_{ij}$ and dD . the result for dT is

$$dT = - T \gamma_{ij} d\epsilon_{ij}^e + T dS / C_{\eta} - g dD \quad (4.3)$$

where g is $-\partial^2 U / \partial S \partial D$, and is a "Gruneisen parameter" for S and D . The result for df is

$$df = (b_{ij} - f\delta_{ij} + \tau_{ij}/(1 - D))d\epsilon_{ij}^e - \rho g dS + \beta dD \quad (4.4)$$

where β is a "stress-strain" coefficient for for df and dD . As before, we break the hierarchy here and formally assume that all coefficients in the equations (4.2) to (4.4) above are known, either from experiment or theoretical calculation.

In actual fact, not much at all is known about the coefficients associated with dD , viz. b_{ij} , β , f and g . Here is an area for future research. It may prove that g is small because the thermal effects associated with dD are small. In the spall treatments that the author is aware of,^{25,22} all terms in dD in the equations for dT and $d\tau_{ij}$ are left out and the equation for df is left out entirely.

We now consider nonequilibrium processes in which voids are being grown and created. Until the spall tension threshold is exceeded, D remains fixed and no nonequilibrium processes involving the growth or creation of voids occur. When growth and creation are occurring, neighboring equilibrium states in the process have entropy changes of dissipation associated with the growth and creation, that are in addition to the entropy changes of plastic flow and heat flow. The extra entropy change is due to the dissipated void work $(-\bar{P} - f)dD$. We write

$$\rho dS = dQ/T + \tau_{ij} d\epsilon_{ij}^P + FdD \quad , \quad (4.5)$$

where FdD is the work dissipated by void growth. F must equal $-\bar{P} - f$, which can be seen by considering the total work $(-\bar{P})dD$, done by the void expansion.

This work equals that dissipated, FdD , and that which goes reversably into internal energy, fdD . The result for F follows immediately. Our information about f comes from equation (4.4) for df , whose coefficients we optimistically considered as known.

A common practice in actual calculations is to consider all the work of dD to be dissipated, so that F equals $-\bar{P}$ and f is zero. A microscopic void growth model of Johnson²⁵ has this property. The void growth is assumed to involve no volume dilation, and all motion is due to plastic flow. Although the tension increases during void growth, no elastic energy is stored because no elastic strains occur. In this case all the void work is dissipated because the plastic flow of void growth is all dissipated. However, a thermodynamical inconsistency arises for stressed materials from this assumption and the knowledge that void growth changes the elastic moduli and the stress-strain coefficients. (A material full of holes is softer and stretchier, and so the bulk and shear moduli become smaller.) Imagine loading up a sample elastically at a given D . Then load up a sample similar but for a different D . Since the elastic constants are different between the two cases, the resultant total energies must differ. This shows that f must be non-zero for a stressed solid, being more so the higher the stress.

As an aside, we note that this problem does not arise in our model for the plastic flow, since we assume that the thermodynamical state is completely independent of the plastic strain. The plastic strain enters in only as a source of heat. This is not strictly true in practice, as discussed earlier, since a small part of the plastic work is reversibly stored. As a consequence, the stress-strain increment relationship should depend somewhat on plastic

work. It does indeed do so in the plastic flow condition when the work hardening associated with the stored plastic work allows a small increment in both the elastic strain and the stress during plastic flow. With no work hardening, no such increments can occur no matter how large the plastic strain becomes. It is in this sense that the stress-strain increment relationship during plastic flow is independent of the plastic strain.

We can write the total strain equation as

$$d\epsilon_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p + dD \delta_{ij}/3 \quad , \quad (4.6)$$

since the different kinds of strains are infinitesimally additive. We will sketch at this point, how the above equations can be integrated to obtain the complete thermodynamical history of the material during a process. (We assume as known all coefficients in the equations for $d\tau_{ij}$, dT , and df .) At any one point one has knowledge of all quantities and one wants to find them for a small step forward in time. First, the equations of motion, i.e. those for conservation of mass and momentum, will tell one what the total strains and rotational strains are. Then the plastic and void growth/creation constitutive equations will tell one what $d\epsilon_{ij}^p$ and dD are. From these and equation (4.6) for $d\epsilon_{ij}$, one can calculate $d\epsilon_{ij}^e$. From f , $-\bar{P}$, and dD one can calculate the dissipated void work and add it to the dissipated plastic work and the entropy increase due to heat flow and get dS , the total entropy increment. At this point, one knows all the increments of the independent variables, and can

use them in the thermodynamical equations (4.1)-(4.4), to get the increments of all other thermodynamical quantities.

2. Specific Examples

One needs a constitutive relation for the time rate of void growth/creation, or for dD/dt . such a relation serves a role analogous to that for plastic flow. Such constitutive relations are not understood as well as those for plastic flow. The best ones available seem to be the ones derived experimentally by the group at Stanford Research Institute.²⁶ Experiments were done in which materials were loaded to various levels of stress for various times and then unloaded and sectioned to determine the density and distribution of voids. Our theory only recognizes the density and so we will be concerned only with their results expressed in terms of this quantity. They summarized their experiments with the following equations

$$\dot{N} = h(\sigma) = C_N \{ \exp[(\sigma - \sigma_N)/\sigma_1] - 1 \} \quad , \quad (4.7)$$

$$\dot{v}^{(1)} = 3 A(\sigma) v^{(1)} \quad , \quad (4.8)$$

where \dot{N} is the time rate of creation of new voids of volume v_0 per unit volume of solid matter. σ_N is a tension threshold below which no new voids grow, while C_N and σ_1 are material parameters. \dot{N} is to be taken as 0 if the threshold is not exceeded. Equation (4.8) gives the growth rate of volume $v^{(1)}$ of void 1. $A(\sigma)$ is given by $C_G(\sigma - \sigma_G)/\sigma_1$, where C_G is a material parameter and σ_G is a tension threshold for void growth, below which $\dot{v}^{(1)}$ is zero. In

the above σ means $(1/3)\tau_{11}$, which is the "tension". With \bar{v} defined to be the average void size $\sum_i v^{(i)}/N$, where the sum is over all voids surrounded by a unit volume of solid material, we can make the identification $D = N\bar{v}/(1 + N\bar{v})$. Finally, realizing that the total void volume per unit solid material changes in time by both the growth of already present voids, $\sum_i \dot{v}^{(i)}$, and by the creation of new ones, $N\dot{v}_0$, one can with a bit of algebra arrive at a constitutive relation for D :

$$\dot{D} = \{h\dot{v}_0 + 3AD/(1-D)\}(1 - D)^2 . \quad (4.9)$$

Finally we note that for void-damaged material, all the thermodynamic coefficients in the conventional equations for plastic flow seen earlier have a dependence on D . This dependence has not been investigated for the general stressed state. Some approximations exist for the case of an unstressed, isotropic material. We give the following estimates for the bulk modulus B and the shear modulus μ due to Davisson et al²² based on the work of Budiansky²⁷ :

$$B = B_0 \left[1 - (3/2)(1 - \nu_0) D / (1 - 2\nu_0) \right] , \quad (4.10)$$

$$\mu = \mu_0 \left[1 - (15/7)(1 - \nu_0) D / (7 - 5\nu_0) \right] , \quad (4.11)$$

where B_0 , μ_0 , and $\nu_0 = (1/2)[(3B_0 - 2\mu_0)/(3B_0 + \mu_0)]$ are the values of the moduli for undamaged materials. These formulas are approximations for small

damage. Formulas for the thermal conductivity, the thermal expansion coefficient β , the isotropic Gruneisen parameter γ and the heat capacity are also given by Budiansky. The thermal expansion coefficient for the damaged material is the same as for undamaged material. For the damaged material, the heat capacity per unit mass is said to be very close to that of the undamaged material. The Gruneisen parameter for the damaged material can be derived from B , β , and C_v of the damaged material from the standard thermodynamic identity $\gamma = B\beta/C_v$. The formulas are only good for D less than .5 or so. However on page 30, Reference 28 notes that voided material becomes unstable in its motion for D much bigger than this, so perhaps the formulas will serve for most practical cases.

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