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TITLE: SOME PROGRESS IN STATISTICAL TURBULENCE THEORY

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SOME PROGRESS IN STATISTICAL TURBULENCE THEORY

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An examination is made of relationships among several approaches to analytical turbulence theory: approximation by finite sets of moments, renormalized perturbation theory, decimation under symmetry constraints, renormalization-group methods, and the upper-bounding of transport under integral constraints. Most of the discussion assumes isotropic turbulence. Decimation under symmetry constraints plays a unifying role. It promises a rational basis for renormalized perturbation theory and provides links to the other named approaches

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1. INTRODUCTION

The basic hope of statistical mechanics is that the overall behavior of large and complicated systems can be disentangled from the details of their dynamics to some adequate approximation. Navier-Stokes (N-S) turbulence is a particularly challenging candidate system. The dynamics are highly nonlinear and unstable and the statistical states of physical relevance differ strongly from the absolute equilibrium for which classical methods of statistical mechanics are most powerful.[1] A combined consequence of nonlinearity and dissipative nonequilibrium is the appearance of plastically ordered structures in the midst of the randomness traditionally assigned to turbulence.

This combination of order and randomness is perhaps the most perplexing feature of turbulence for theorists. There is a fundamental dilemma concerning honesty of representation. If structures are guessed at the start (whether the guess is shear layers, horseshoe vortices, smoke rings, or whatever), there is danger that the subsequent theoretical construction may have its most important predictions built in. If a neutral underlying representation is adopted, like uncorrelated Fourier modes, the danger is that the effects of characteristic structures of the real flow may be totally lost.

The present assessment is mostly in the context of an idealized part of statistical turbulence theory: attempts to treat isotropic turbulence by analytical procedures applied to the N-S equation. Here the representation is all too honest: typically a neutral Gaussian statistical state is assumed at an initial time and then the N-S equation is switched on. The analytical approaches are mostly based on truncation of one or another series or sequence. Thereby they face the embarrassment that turbulence at high Reynolds numbers offers no obvious small expansion parameter by which to justify the truncation.

The approaches discussed in this paper are direct moment approximations (truncation of a moment representation of the joint probability distribution),[2-10] renormalized perturbation theory,[11-17] decimation under symmetry constraints,[7] renormalization-group methods,[18-32] and the upper-bounding of transport under integral constraints.[33-36] Decimation under symmetry constraints plays a central role in the discussion. In this approach some subset of the totality of degrees of freedom is followed by explicit equations of motion, while the effects of all other modes (the implicit modes) are expressed by a constrained stochastic forcing in the equations for the explicit modes. The constraints express the statistical symmetries among or within classes of modes.

The explicit set of modes may be formed by choosing a few sample modes in each dense neighborhood of wavevector space. A small parameter thereby arises, namely the ratio of sample size to total neighborhood population. In the limit where this ratio goes to zero (strong decimation), the imposition of a basic symmetry constraint yields the direct-interaction approximation. Higher approximations arise if successively more symmetry constraints are imposed. They are related to renormalized perturbation theory approximations but, in contrast to the latter, are expected to form a convergent sequence.

Instead, the explicit set of modes may be all the modes below a cutoff wavenumber and the implicit set all modes above the cutoff. In this case the symmetry constraints express an extrapolation which relates moments of the implicit modes to those of the explicit modes. The result is a sequence of subgrid-scale representations. If this is done for an infinite Kolmogorov inertial range, the extrapolation simply expresses the Kolmogorov scaling of moments. The resulting nonperturbative analytical framework is related to renormalization-group approaches.

Finally, if the imposed symmetry constraints are limited to overall integral properties, the resulting approximations are similar in spirit to the upper-bounding analysis for turbulent transport developed by Busse, Howard, Malkus, and others.[33-36]

It is unclear to what extent any of the approximations here examined can succeed in representing the dynamical effects of organized flow structures. Certainly it is not expected that the explicit geometry of such structures in physical space can be inferred from any description limited to low order moments or other low-order statistical descriptors.[37] But it may be hoped that the essential effects of structures on energy and momentum transport and other averaged properties of the flow can be captured to fair approximation.

It is of particular interest to assess what order of improvement can be expected from analytical approximations above the presently studied level of 2nd-order renormalized perturbation theory. Certain features of turbulence are first captured at the level of 4th-order perturbation theory, or at the level of constraints involving 4th-order moments in the decimation method of Sec. 4. These include intermittency effects, possible force-free ordering,[38] certain effects of helicity on turbulent diffusion[39,40] and, in magneto-hydrodynamics, negative diffusivity effects.[39]

The most straight-forward approximations examined in this paper are the moment-related ones described at the end of Sec. 2. They are guaranteed to converge. What then is the motivation for examining more complex and uncertain approximation methods like renormalized perturbation theory, decimation, and the renormalization group? First it should be said that the simple moment-related approximations may deserve deeper exploration than they have had. But the complex approaches offer the possibility of more faithful representation of the physics at a given level of approximation. They all involve probing of the dynamics by examination of the effects of perturbations. This is a powerful tool.

2. MOMENT APPROXIMATIONS AND REALIZABILITY

Consider the evolution in time of an incompressible velocity field which obeys cyclic boundary conditions and has an isotropic, spatially homogeneous Gaussian statistical distribution at time $t=0$. Problems of existence of solutions of the N-S equation can be sidestepped, and the number of degrees of freedom made finite, if the field and the N-S equation are expressed in terms of wavevector amplitudes and truncated at some cutoff wavenumber k_{max}. If this cannot be done without destroying the physics, then the validity of the N-S equation itself is questionable. The evolution problem can be made wholly finite by replacing the N-S equation with a finite-difference form and evaluating the amplitudes only at the discrete times t_s.

The initial and evolved joint probability density (JPD) for the surviving and discretized Fourier amplitudes can be represented by the set of all moments. Basic questions are what comprises sufficient conditions for completeness of this representation and how to construct convergent approximation sequences which involve only finite sets of moments. Consider the space in which the independent real and imaginary parts of the Fourier amplitudes of all the surviving modes are Cartesian coordinates. Let these amplitudes be represented by the vector $\mathbf{y}$ with components $y_i(t_s)$, where i is associated with some 1-dimensional ordering of the Fourier amplitudes. The sufficient condition for completeness of moment representation is that the JPD fall off exponentially or faster at infinity along any direction in this space.[7]

If this condition is violated, as it may be for a physically interesting JPD that is sufficiently intermittent, a complete representation can still be built from weighted moments, defined as follows. If $\rho(\mathbf{y})$ is the normalized JPD, a general moment then has the form

$$\left\langle y_\alpha y_\beta \cdots \right\rangle \equiv \int (y_\alpha y_\beta \cdots)\rho(\mathbf{y})d\mathbf{y}, \tag{2.1}$$

where α, β,... represent particular values of the index pair (i,s). A general weighted moment can be defined by

$$\left\langle y_\alpha y_\beta \cdots \right\rangle_\omega \equiv \int (y_\alpha y_\beta \cdots)\omega(\mathbf{y})\rho(\mathbf{y})d\mathbf{y}, \tag{2.2}$$

where $\omega(\mathbf{y})$ is positive weight function. If $\rho(\mathbf{y})$ vanishes at infinity along any ray, then exponential falloff of $\omega(\mathbf{y})$ in all directions is sufficient to insure that the weighted moments are a complete representation.[7]

The existence of a complete moment representation of the JPD implies that it is possible to form converging sequences of approximants constructed from finite sets of moments. There are pitfalls in doing this. Thus the expansion of the JPD into cumulants is nonconvergent in general. An example is the well-behaved 1-dimensional density $\rho(x) = (2\pi)^{-1/2}x^2\exp(-x^2/2)$, whose cumulants grow like $(2n)!$. [The unphysical results in isotropic turbulence theory obtained by the approximation of setting 4th-order cumulants to zero[4,5] are apart from the convergence difficulty at high orders.]

One way of constructing convergent approximants to $\rho(\mathbf{y})$ from finite sets of its moments is by the orthogonal expansion

$$\rho_M(\mathbf{y}) = w(\mathbf{y}) \sum_{j=0}^{M} b_n p_n(\mathbf{y}), \tag{2.3}$$

where the $p_n(\mathbf{y})$ are a complete set of polynomials in the $y_i(t_s)$, orthonormal with respect to the positive weight $w(\mathbf{y})$ and placed in some 1-dimensional order. The b_n are fixed by

$$b_n = \left\langle p_n(\mathbf{y}) \right\rangle \tag{2.4}$$

and involve the values of moments only up to the order of p_n. The approximants converge in mean square as $M \to \infty$ if $\int [\rho(\mathbf{y})]^2 [w(\mathbf{y})]^{-1} d\mathbf{y}$ exists. If the moments are complete, this condition can be satisfied with a $\delta(\mathbf{y})$ which falls off exponentially, and thereby has a set of p_n which are complete. In this case the mean square convergence of the approximants is to $\rho(\mathbf{y})$. If $\rho(\mathbf{y})$ falls off so slowly that its moments are incomplete, then the expansion (2.3) can be replaced by a similar expansion for $[w(\mathbf{y})]^{1/2}\rho(\mathbf{y})$ and convergents to $\rho(\mathbf{y})$ in mean square are constructed from finite sets of weighted moments.[7]

When $\rho(\mathbf{y})$ is represented by its moments, it is important to express the condition that $\rho(\mathbf{y})$ be positive everywhere. Obvious neccssary conditions are the infinite set of moment inequalities

$$\left\langle [P_n(\mathbf{y})]^2 \right\rangle \geq 0, \tag{2.5}$$

where P_n is any real polynomial. If the moments are a complete representation, (2.5) is also sufficient. The finite approximants $\rho_M(\mathbf{y})$ satisfy (2.5) up to a finite degree of polynomial, but in general are not positive everywhere. Also, it should be noted that satisfaction of (2.5) for all $p_n(\mathbf{y})$ whose degree is $\leq J$, does not assure that every positive real polynomial of degree $2J$ has a positive average over ρ_M; some positive polynomials are not expressible as sums of squares of lower-degree polynomials. If the ordinary moments are incomplete, (2.5) is replaced by corresponding inequalities for weighted moments.[7]

Consider now the construction and moment representation of a JPD which approximately satisfies the N-S equation. Let the discretized form of the latter be represented by

$$L_{is}(\mathbf{y}) = 0. \tag{2.6}$$

$L_{is}(\mathbf{y})$ is a 2nd-degree polynomial. Thus the mean-square of the N-S equation

$$\left\langle [L_{is}(\mathbf{y})]^2 \right\rangle = 0 \tag{2.7}$$

may be considered a limiting case (equality) of one of the relations (2.5). If all the relations (2.5) are satisfied, and (2.5) is sufficient for realizability, then the entire moment-equation hierarchy

$$\left\langle P_n(\mathbf{y}) L_{is}(\mathbf{y}) \right\rangle = 0 \tag{2.8}$$

follows from (2.7) by virtue of Schwarz' inequality

$$\left\langle P_n(\mathbf{y}) L_{is}(\mathbf{y}) \right\rangle^2 \leq \left\langle [P_n(\mathbf{y})]^2 \right\rangle \left\langle [L_{is}(\mathbf{y})]^2 \right\rangle.$$

$P_n(\mathbf{y})$ here is any polynomial.

If (2.7) is satisfied and any equation in the hierarchy (2.8) is not, then the moments represent a JPD which is not positive-definite. It can be shown that the entire hierarchy (2.8) in fact represents a subset of (2.5), specialized to polynomials $L_{is}(\mathbf{y}) + \delta P_n(\mathbf{y})$, with δ infinitesimal.[7] Any approximate set of moments which satisfies (2.8) for all first and second degree polynomials P_n satisfies (2.5), since L_{is} is of 2nd degree. Therefore such an approximation either is an exact solution of the entire hierarchy or cannot be realized.

The existence in general of converging moment-based approximants to JPD's suggests that it should be possible to construct converging moment-based approximants to ensembles of solutions of (2.6). One way to seek such approximants is by Galerkin methods: use the representation (2.3) in a subset of the hierarchy (2.8) large enough to fix the b_n and assert a subset of the realizability conditions (2.5) which grows with M. Convergence is not totally assured by the general convergence of the $\rho_M(\mathbf{y})$. It is also required that the exact $\rho(\mathbf{y})$ to which convergence is sought have certain stability properties under small perturbations of (2.6).[7]

An alternative way of constructing approximants which have assured convergence, and automatically satisfy (2.5), is the following. Write

$$\rho_M(\mathbf{y}) = [\psi_M(\mathbf{y})]^2$$

and expand $\psi_M(\mathbf{y})$ in the form

$$\psi_M(\mathbf{y}) = \sum_{n=0}^{M} [w(\mathbf{y})]^{1/2} c_n p_n(\mathbf{y}). \tag{2.9}$$

Then $\rho_M(\mathbf{y})$ is positive-definite and (2.7) is sufficient to ensure satisfaction of (2.6) in all but zero-measure set of the realizations that comprise the ensemble. The left side of (2.7) is a quadratic form in the c_n. The latter may then be determined variationally in order to minimize some positive linear sum taken over (2.7) for all i and s, subject to whatever other constraints (for example, initial conditions) may be appropriate.[10]

The c_n in (2.9) do not have a relation to moment values as simple as that for the b_n in (2.3). Another scheme which produces converging approximants to exact solution JPD's, and automatically satisfies realizability inequalities, is a variational procedure based on the truncated expansion of the $y_i(t_n)$ in powers of Gaussian random processes (many-time Wiener-Hermite expansion). The coefficients of this expansion again determine the moments, but according to a still more complicated structure.[8-10]

Approximations based on (2.3) or (2.9) appear to have been little explored and may deserve further study. The rates of convergence are not known, but it is plausible that the order of approximation required for given accuracy stays finite as Reynolds number becomes infinite.

3. RENORMALIZED PERTURBATION THEORY

A formal solution for the time evolution of the wavevector amplitudes $u_i(\mathbf{k},t)$ of the velocity field may be developed by straightforward perturbation-iteration treatment of the Navier-Stokes equation. Let $u_i^0(\mathbf{k},t)$ be the solution of the linear problem posed by striking out the nonlinear terms of the N-S equation, and let $G_{ij}^0(\mathbf{k};t,t')$ be the Green's or response function of the linearized equation. Then

$$u_i^0(\mathbf{k},t) = u_j^0(\mathbf{k},t')G_{ij}(\mathbf{k};t,t'), \qquad G_{ij}^0(\mathbf{k};t,t') = \delta_{ij}G^0(k;t,t'), \tag{3.1}$$

where

$$G^0(k;t,t') = \exp[-\nu k^2(t-t')] \tag{3.2}$$

and ν is kinematic viscosity. Then the reintroduction and iteration of the nonlinear terms yields $u_i(\mathbf{k},t)$ as a functional power series in all the initial values $u_i(\mathbf{p},0) = u_i^0(\mathbf{p},0)$ and zeroth-order response scalars $G^0(p;s,s')$.

Assume that the initial state is homogeneous, isotropic, and multivariate Gaussian. The power-series expansion together with well-known reduction rules for Gaussian moments yields a formal expression for any moment of the evolved wavevector amplitudes as a functional power series in the G^0 and the defining scalars $U^0(p;t,t')$ of the zeroth-order 2-time covariances. Also, similar expressions may be obtained for the Green's functions that measure the ensemble-averaged response of the full N-S equation to infinitesimal perturbations. In particular, this may be done for the defining scalar $U(k;t,t')$ of the covariance of the exact amplitude $u_i(\mathbf{k},t)$ and for the defining scalar $G(k;t,t')$ of the average response tensor for a statistically sharp perturbation of a single Fourier mode.

Renormalization of these primitive perturbation series is motivated by the plausible argument that the actual covariance scalar $U(k;t,t')$ and actual response scalar $G(k;t,t')$ are more physically relevant than the zeroth-order quantities. The primitive expansion is essentially an expansion in powers of Reynolds number, which is not a small number in most cases of interest. However, straightforward truncations of the primitive expansion can give surprisingly good results for the initial-period evolution.[41,42] Renormalization can be carried out by a variety of methods.[11-17] The best known is summation of diagrams (classes of terms in the primitive expansion).[11-13] Perhaps more flexible is term-by-term reversion of the expressions for $U(k;t,t')$ and $G(k;t,t')$ as functional power series in the U^0 and G^0.[17,43] This yields $U^0(k;t,t)$ and $G^0(k;t,t')$ as functional power series in the exact U and G. In turn these last series can be substituted into the primitive expansion for any moment of the exact Fourier amplitudes to yield a reworked expansion in which appear only the exact U and G. These reworked expansions for all moments constitute the complete line-renormalized perturbation apparatus.

Approximants to U and G may be constructed by truncating the renormalized expansions for those moments which, according to the moment hierarchy equations, express the time derivatives of $U(k;t,t')$ and $G(k;t,t')$ The hierarchy yields

$$(\partial/\partial t + \nu k^2)U(k;t,t') = S(k;t,t'),$$

$$(\partial/\partial t + \nu k^2)G(k;t,t') = H(k;t,t'), \tag{3.3}$$

where S is a triple moment expression and H involves the covariance of a mode amplitude with an unaveraged response tensor. The leading term in the renormalized expansion for S has the structure GUU. That is, it is a time and wavevector integral over an integrand containing geometric factors, one

G function factor and two U function factors. The higher terms have the structures GGGUUU, GGGGGUUUU, etc. The terms in the expansion for H have the structures GGU, GGGGUU, GGGGGGUUU, etc.[11]

If the renormalized expansions for H and S are truncated at some order, the result is closed integrodifferential equations for G and U. There is no reason to believe that successive approximants constructed in this manner converge. In fact, there is evidence to the contrary from model problems. If extrapolation from few-mode bilinear systems is valid, both the primitive and the renormalized perturbation expansions have zero radius of convergence either in t or in the strength of the nonlinearity (that is, in Reynolds number). In a 3-mode bilinear system, the radius of convergence in a typical realization is finite, and the averaging over the Gaussian initial conditions yields zero radius of convergence for the moment expansions.[44] The divergence of the renormalized expansion appears not less severe at high orders than that of the primitive expansion. In fact it is more dangerous, because the renormalized expansions are used via (3.3) to form a closed system. Padé methods and other acceleration techniques may help with the covergence problems for both primitive and renormalized expansions, but the results are uncertain.[43,45]

If H and S are consistently truncated at any order (the same for both), the resulting approximant formally conserves energy transfer by nonlinearity, in accord with the exact dynamics, and if $\nu = 0$ it gives formal absolute equilibrium distributions which obey detailed balance and yield the fluctuation-dissipation relations of the exact dynamics.[11]

The lowest truncation retains only the GGU terms in H and the GUU terms in T. Substitution of the result into (3.3) yields the so-called direct-interaction approximation (DIA), which has special properties. The DIA can be obtained, in two separate ways, as an exact consequence of certain model amplitude equations, independently of the perturbation and renormalization analysis. These model representations show that the approximation is self-consistent in the sense that $U(k;t,t')$ obeys the 2nd-order realizability inequalities. In particular, $U(k;t,t)$ is non-negative. This property, together with energy conservation, is sufficient to assure that there exist healthy solutions to the DIA equations. In contrast, it is known from examples that solutions for higher truncations of the expansions of H and S can blow up catastrophically.

The fundamental dynamical model underlying DIA is obtained by randomizing in a particular way the coefficients in the N-S equations which describe interactions of individual mode triads.[11] The alternative model is a generalized Langevin equation:[45]

$$(\partial/\partial t + \nu k^2)u_i(\mathbf{k},t) + \int_0^t \eta(k;t,s)u_i(\mathbf{k},s) + b_i(\mathbf{k},t) = f_i(\mathbf{k},t). \qquad (3.4)$$

Here the dynamical damping $\eta(k;t,s)$ is statistically sharp and has the structure GU, while $b_i(\mathbf{k},t)$ is a random internal force, with Gaussian statistics, and $f_i(\mathbf{k},t)$ is a possible external force. The model is closed by requiring that the 2-time covariance of $b_i(\mathbf{k},t)$ be identical with that of the total nonlinear term in the N-S equation for $u_i(\mathbf{k},t)$, under the assumption that the Fourier amplitudes $u_j(\mathbf{q},t)$, which appear in the nonlinear term, are exactly statistically independent. Thereby the covariance of b_i has the structure UU. The DIA equations for G and U follow immediately from (3.4) if G is identified with the response scalar of (3.4).

The DIA gives reasonably accurate predictions of the decay of isotropic turbulence at moderate Reynolds numbers, including satisfactory predictions for 2-time quantities.[46,47] It has also had success in some plasma and magneto-hydrodynamics (MHD) applications,[48,49] in turbulent diffusion,[50,36] in anisotropic homogeneous turbulence,[51] and in Boussinesq convection.[52] Applications to shear flows and to convection at lower Prandtl numbers are being developed.[53-55]

If applied to high-Reynolds-number turbulence, the DIA equations exhibit a failing, of broad impact in analytical turbulence theory, associated with the interaction of strongly disparate wavenumbers.[56] This problem arises in a somewhat subtle way from the renormalization; it does not afflict truncations of the primitive perturbation expansion. The origin of the difficulty is the separation between the characteristic time $\tau_d(k)$ for distortion of features with characteristic wavenumber k, and the time $\tau_c(k)$ for decorrelation of the amplitude $u_i(\mathbf{k},t)$ due to convection of the features by the total velocity field. In a Kolmogorov inertial range, $\tau_c \sim 1/(v_0 k)$, where v_0 is the rms value of the total velocity component in any direction, while $\tau_d(k) \sim 1/(\epsilon^{1/3}k^{2/3})$, where ϵ is the energy dissipation rate per unit mass. The ratio is $\tau_c(k)/\tau_d(k) \sim (k_0/k)^{1/3}$, where k_0 is the characteristic wavenumber of the energy-containing range. In the exact dynamics, $\tau_c(k)$ is the decay time of $U(k;t,t')$ as a function of $t - t'$, and this is also true in DIA. However, the DIA energy balance equation, obtained from (3.3) at $t = t'$, effectively substitutes this convective dephasing time for the proper intrinsic distortion time τ_d. The result is a depression of energy transfer and a change of the Kolmogorov spectrum law from -5/3 to -3/2. (In some MHD applications, this trouble does not arise.[57])

Error from the confusion of convection and distortion times persists in every order of the line-renormalized expansions for S. It also persists, but is numerically reduced, in every order of expansions where line renormalization is augmented by vertex renormalization.[56]

The distinction between convection and distortion times can be expressed formally by the property of invariance under random Galilean transformation (RGT).[58] Suppose that the turbulent velocity field is augmented by a spatially uniform velocity which varies randomly over the realizations of the ensemble and has zero mean. The Galilean invariance of the N-S equations insures that stretching and other distortion effects are unaffected by the uniform convection. This is expressed mathematically by the invariance of moments under RGT. In particular the triple moment $\langle u_i(\mathbf{k},t)u_j(\mathbf{p},t')u_m(\mathbf{q},t'')\rangle$, with $\mathbf{k}+\mathbf{p}+\mathbf{q}=0$ is invariant. Any finite truncation of the renormalized expansion for this moment, however, is not invariant, because the correlations among the phase changes induced in the three factors by the RGT are lost.

The underlying physical problem with the renormalized expansion, as it has been formulated, is that the distortion time $\tau_d(k)$ has no simple expression in terms of Eulerian 2-time quantities. But it arises naturally in a Lagrangian representation, where convective effects of large scales are transformed away. The entire line-renormalized apparatus can be recast into a form where Lagrangian as well as Eulerian 2-time correlations enter in a fundamental way.[58,17] The reworked expansion is properly invariant under RGT in every order. The lowest truncation of this expansion yields the so-called Lagrangian-history DIA (LHDIA) in which the convection difficulty disappears, and the -5/3 Kolmogorov spectrum is recovered. An abridged version (ALHDIA) yields excellent absolute agreement with measured inertial-range and dissipation-range spectra, normalized by ϵ and ν.[59] The improvement over DIA is gained at a cost in complication and at the loss of the exact model representations. (But see the work of Kaneda.[60,61])

The LHDI and ALHDI approximations have yielded good qualitative physics in a variety of situations: the $k^{-5/3}$ inertial range in 3D and 2D turbulence, the log-corrected k^{-3} range in 2D turbulence, the $k^{-5/3}$, k^{-1}, and $k^{-17/3}$ ranges in turbulent convection of a passive scalar, the $k^{-3/2}$ MHD inertial range, and the k^{-2} shock-dominated range for Burger's equation.[57,59,62-65] The predicted cascade rate in the $k^{-5/3}$ and k^{-1} scalar ranges and the 2D k^{-3} range is probably too high by a factor of 2 or 3. An alternate version of LHDI has been constructed from Lagrangian reverted series which are based on the strain field rather than the velocity field as the fundamental Lagrangian construct.[66,67] This has yielded numerical constants for the three last-named ranges in fairly good agreement with experiment, without sacrificing agreement in the 3D $k^{-5/3}$ Kolmogorov range.

The RGT-invariant Lagrangian form of renormalized perturbation theory is most naturally based on the generalized x-space velocity field $u_i(\mathbf{x},t|s)$, defined as the velocity measured at time s in the fluid element whose trajectory passes through the spacetime point $(\mathbf{x},t)$.[58] The ordinary Eulerian velocity is $u_i(\mathbf{x},t|t)$, while $u_i(\mathbf{x},t_0|t)$ as a function of t is the Lagrangian velocity of a fluid element tagged at time t_0. The full evolution of $u_i(\mathbf{x},t|s)$ is fixed by the N-S equation together with a passive advection equation which determines the t dependence at fixed s. The linearized solutions $u_i^0(\mathbf{x},t|s)$ are independent of s. This makes it possible to revert the series for covariances and Green's functions to express the zeroth-order functions in terms of the exact Lagrangian covariances and Green's functions instead of Eulerian functions. The result is a reworking of the renormalized expansions for moments, in particular triple moments, so that integrations over time histories are back along fluid-element trajectories, instead of at fixed coordinate positions.[17] The tagging time of the trajectories is continually updated.

4. DECIMATION UNDER SYMMETRY CONSTRAINTS

The use of statistical description carries an implicit appeal to redundancy. A system with N degrees of freedom sampled at T time steps in R realizations is described in full detail by NTR numbers. A full statistical description by moments up to order M requires the order of $(NT)^M/M!$ numbers. The statistical description is prohibitively bulky at large NT and moderate M unless the moments change slowly and smoothly with change of mode label and time. If the change is slow, it is sufficient to specify the moments for a relatively small set of strategically chosen modes. This is an exploitation of statistical redundancy (statistical symmetry) within classes of modes. Such symmetry certainly characterizes homogeneous turbulence in a large cyclic box, where the Fourier modes are dense and neighboring modes are statistically similar.

Statistical symmetries can be exploited in a systematic way to yield equations of motion for a reduced set of modes, the explicit set, which are sufficient to characterize the entire system. The rest of the modes (implicit modes) are represented by a constrained stochastic forcing in the equations for the explicit modes. This is a forcing distinct from any external forcing. The constraints are expressions of the underlying statistical symmetries; they relate moments of the implicit modes, and thereby moments of the stochastic forcing, to moments of the explicit modes. This procedure, which will be termed decimation under symmetry constraints (DSC),[7] turns out to have deep connections with renormalized perturbation theory (RPT) and renormalization-group (RNG) approaches, as well as with the moment hierarchy of Sec. 2.

The general structure of the equations of motion for the explicit modes is readily found. In the notation of Sec. 2, the N-S equation has the form

$$(\partial/\partial t + \nu_i)y_i(t) + \sum_{jm} A_{ijm}y_j(t)y_m(t) = f_i(t). \tag{4.1}$$

Here the $y_i(t)$ are independent real and imaginary parts of the Fourier amplitudes (or more generally some modal representation), ν_i represents damping by viscosity, the A_{ijm} are the coefficients of the wavevector triad interactions, and $f_i(t)$ is a possible external force. Let $\mathbf{y}^S(t)$ represent the explicit subset of $\mathbf{y}(t)$ and write the equation of motion for $y_i^S(t)$ as

$$(\partial/\partial t + \nu_i)y_i^S(t) + \sum_{jm}^{S} A_{ijm}y_j^S(t)y_m^S(t) + q_i(t) = f_i(t). \tag{4.2}$$

Here $\sum_{jm}^{S}$ denotes a sum restricted to j and m both in the explicit subset, and the effects of the implicit modes are represented by

$$q_i(t) = q_i^I(t) \equiv \sum_{jm}' A_{ijm}y_j(t)y_m(t), \tag{4.3}$$

where $\sum_{jm}'$ denotes a sum restricted to either j or m or both in the implicit subset.

The function $q_i(t)$ is now to be written as a stochastic forcing term which expresses statistically the effects of the implicit modes. It comprises a primary contribution $b_i(t)$, which is the value of $q_i^I(t)$ with all the explicit amplitudes clamped to zero in the equations of motion for the implicit modes. In addition, there is an infinite series of contributions which express the change in $q_i^I(t)$ induced by the actual nonzero values of the explicit amplitudes. Thus[7]

$$q_i(t) = b_i(t) + \int_0^t ds\,\eta_{ij}(t,s)y_j^S(s) + \int_0^t ds\int_0^t ds'\,\gamma_{ijm}(t,s,s')y_j^S(s)y_m^S(s') + \ldots, \tag{4.4}$$

where

$$b_i(t) = [q_i^I(t)]_0, \qquad \eta_{ij}(t,s) = [\delta q_i^I(t)/\delta y_j^S(s)]_0,$$

$$\gamma_{ijm}(t,s) = [\delta^2 q_i^I(t)/\delta y_j^S(s)\delta y_m^S(s')]_0, \ \ldots\ . \tag{4.5}$$

and $[\ ]_0$ denotes a value taken at $\mathbf{y}^S = 0$. The functions $b_i(t)$, $\eta_{ij}(t,s)$, $\gamma_{ijm}(t,s,s'),\ldots$ are stochastic, with nonzero means and nonGaussian statistics in general. In the present case of isotropic homogeneous turbulence, $b_i(t)$ has zero mean. The $b_i(t)$ for distinct i are not statistically independent in general.

Eqs. (4.2)-(4.5) are a general formalism which can express a variety of physics depending on how the explicit set is chosen. The convergence properties of the formal expansion (4.4), with exact values (4.5) inserted, are not known. A sequence of approximations will now be outlined which plausibly are convergent and are associated with finite truncations of (4.4), but with approximate values of $b_i(t)$, $\eta_{ij}(t,s)$, ...

The assumption that statistical symmetries relate the implicit set to the explicit set implies that all moments of $\mathbf{q}$ and $\mathbf{y}^S$ can be expressed in terms of moments of $\mathbf{y}^S$ alone. Thus

$$\left\langle q_i(t)\right\rangle = 0, \quad \left\langle q_i(t)y_j^S(t')\right\rangle = M_{ij}^S(t,t'),$$

$$\left\langle q_i(t)y_j^S(t')y_m^S(t'')\right\rangle = M_{ijm}^S(t,t',t''), \quad \ldots , \tag{4.6}$$

where $M_{ij}^S(t,t')$ is a triple moment of the explicit amplitudes alone, $M_{ijm}^S(t,t',t'')$ is a 4th-order moment of the explicit amplitudes alone, ... This follows from the fact that $q_i^I(t)$ is quadratic in the mode amplitudes. Eqs. (4.6) are an infinite set of moment relations. They are not members of the moment hierarchy (2.8) but instead merely express the assumed statistical redundancy between explicit and implicit modes. There are also symmetry constraints for moments of higher order in $\mathbf{q}$. The 2nd-order sequence is

$$\left\langle q_i(t)q_j(t')\right\rangle = N_{ij}^S(t,t'), \quad \ldots , \tag{4.7}$$

where $N_{ij}^S(t,t')$ is a 4th-order moment of the explicit amplitudes alone. The higher members of this sequence involve y factors in the left-side averages.

Now suppose that the sequences (4.6), (4.7), ... are truncated so that only a finite number of the symmetry constraints are imposed. For a concrete example, suppose that the only constraints imposed are the first two of Eqs. (4.6). Then the random process $q_i(t)$ is not uniquely determined by the constraints and initial statistics. Among the solutions is of course the exact solution, since the imposed constraints are a subset of exact constraints. The ambiguity may be resolved by seeking a least-squares solution under the subset of symmetry constraints together with the initial conditions and an appropriate subset of realizability inequalities. The initial conditions are simply

$$q_i(0) = q_i^I(0) \tag{4.8}$$

and the statistics of $q_i^I(0)$ follow trivially from the assumed Gaussian statistics of the $y_i(0)$.

A least-squares solution that satisfies the first two equations of (4.6) must project on $y_j^S(s)$. In order to also satisfy (4.8) it must in addition have a part which initially is statistically independent of $y_j^S(0)$. One is led to infer that the least-squares solution has the form of (4.4) truncated to the first two terms on the right side. Similarly, if more of the symmetry constraints are imposed, one is led to infer that the least-squares form for $q_i(t)$ is a higher truncation of (4.4).

All the symmetry constraints plus the equations of motion for the explicit modes imply the full moment hierarchy (2.8) for the total set of modes, implicit and explicit.[7] This follows from simple substitution. Then solution of the equations of motion for the explicit modes under successively larger sets of the symmetry constraints implies that successively more of the hierarchy equations for the total system are satisfied. As noted in Sec. 2, if an enlarging set of realizability inequalities are also satisfied, and if the exact statistical solutions have certain stability properties, then the resulting approximations converge to an exact solution.

Realizability inequalities that involve the explicit modes alone are automatically satisfied, because actual amplitudes are evolved. But because of weightings built into the symmetry constraints, realizability inequalities that involve the implicit modes are not automatically satisfied in general. This complicates the construction of successive approximations and can complicate the form of the least-squares solutions for the $q_i(t)$.

The approximations outlined above are nonperturbative. But there is a limit in which they can be analyzed accurately by perturbation methods. Suppose that the modes are dense in wavevector space and that the explicit set is formed by choosing a few sample modes from each neighborhood in wavevector space. A small parameter thereby is introduced, the ratio of sample size to total neighborhood population. Consider the limit where this ratio goes to zero (strong decimation). The $\sum^S$ sum in (4.2) then is infinitesimal compared to $q_i(t)$ and may be treated perturbatively. If this is done for the lowest-order least-squares approximation described above, namely with the first two equations of (4.6) taken as the only symmetry constraints, the result is precisely the direct-interaction approximation in the form (3.4).[7] [The 2nd of equations (4.6) taken at $t = t'$ enforces energy conservation in the mean by the nonlinear dynamics.] Thereby DIA is imbedded in a sequence of approximations corresponding to taking successively more symmetry constraints and consequently satisfying successively more of the hierarchy equations. In this way the DSC approach links renormalized perturbation theory to the moment approximations of Sec.

2.

A strong-decimation limit may also be constructed for systems with finite numbers of modes by the device of considering an infinite collection of such systems and performing the manipulations on suitably constructed collective coordinates.[7]

In the strong-decimation limit, the higher DSC approximations again may be treated perturbatively. But in contrast to higher truncations of line or vertex renormalized perturbation expansions, the present approximations are expected to converge. It should be noted that the $\gamma_{ijm}(t,t',t'')$ term in (4.4), the next term beyond the DIA level, is a vertex correction term in the language of renormalized field theory.

A different way to form the explicit set is to put into it all modes with wavenumbers less than some cutoff wavenumber and put all the modes above the cutoff into the implicit set. The symmetry constraints now are extrapolation formulas which express moments of modes above the cutoff in terms of moments of the explicit modes. The truncations of (4.4) associated with least-squares solution under successively more symmetry constraints then form a sequence of subgrid-scale representations. Some techniques for realizing the least-squares solutions in this kind of nonperturbative situation have been described.[7]

The problem of noninvariance under random Galilean transformation which afflicts Eulerian RPT (see Sec. 3) can be handled in the DSC approach without the need for Lagrangian representation. This is because the actual amplitudes of the explicit set of modes are followed. It was noted above that the imposition of just the first two of the symmetry constraints (4.6) led, in the strong decimation limit, to DIA, which is noninvariant under RGT. If the third of the constraints (4.6) is also imposed, the resulting vertex correction terms in the least-squares solution precisely counteract the spurious decay of triple correlations in DIA and restore invariance.[7]

The consistent imposition of all symmetry constraints and realizability inequalities associated with up to 4th-order moments of the explicit amplitudes is of particular interest, because it implies that the fundamental equation (2.7) is satisfied. The relevant constraints are the first three equations of (4.6) and the first equation of (4.7).

5. RENORMALIZATION-GROUP APPROACHES

A variety of analytical approaches have been made to the problem of eliminating the modes above some cutoff wavenumber k_c. Comprehensive citation will not be attempted here. Lindenberg and West have written down an exact formal solution in terms of time-ordered exponentials built from the explicit field ($k < k_c$).[68] They then study a perturbative approximation constructed so as to have desired fluctuation-dissipation properties.

The elimination problem has also been studied by renormalization-group (RNG) methods.[18-20,24-31] The treatments so far worked out explicitly have involved rather drastic approximations. A distant eddy viscosity plays a central role in the lowest approximation to the theory of Yakhot and Orszag.[30,31] It may be defined as the eddy viscosity exerted by modes of wavenumber $k > k_c$ on modes $k << k_c$. An effective extrapolation from the form of this distant eddy viscosity yields in a self-consistent way the total eddy viscosity felt by a mode in the inertial range.[30-32] A description of this approximation and its relation to the full Yakhot-Orszag theory is given by Dr. Yakhot in these Proceedings.[69] Yakhot and Orszag have obtained good predictions of the Kolmogorov constant and other inertial-range parameters. They have successfully applied the inertial-range eddy-viscosity formulas to a calculation of the von Karman constant and to detailed numerical calculations of shear-flow behavior.[30]

The discussion of RNG methods to follow here is not primarily concerned with present applications. Instead, it offers some rather Procrustean speculations on modifications suggested by the nature of the higher RNG approximations. Three principal techniques have been used in the RNG treatment of small scales. The first is elimination of successive thin shells in wavenumber space, starting with the highest wavenumbers, instead of elimination of the modes $k > k_c$ all at once. The second is the use of primitive or renormalized perturbation treatment of the N-S equation, effectively like that of Sec. 3, to accomplish the elimination of each band. The third is the ϵ-expansion: The treatment of the Kolmogorov inertial range with energy spectrum $E(k) \propto k^{-5/3}$ starts with the analysis of a spectrum $E(k) \propto k^{1-2\epsilon/3}$. The Kolmogorov range is then recovered by an power series expansion in ϵ about $\epsilon = 0$; the -5/3 spectrum corresponds to $\epsilon = 4$.[30] Alternatively, this may be regarded as an expansion about dynamics for space dimensionality $D = 17/3$, with the modal intensity $U(k)$ [$U(k) = E(k)/2\pi k^2$ for $D = 3$] held fixed at the Kolmogorov dependence $U(k) \propto k^{-11/3}$. The approach of Yakhot and Orszag envisages approximation sequences based on simultaneous and coordinated inclusion of successively higher perturbation-theory contributions and successively higher terms in the ϵ expansion.[30,31]

Any sequence of approximations which converges to exact elimination of the modes $k > k_c$ must yield, in the limit, renormalized equations of motion for the explicit modes which are at least formally equivalent to (4.2) with (4.4). The infinite series expansion (4.4) may not converge, but basic features expressed by it characterize the exact $q_l(t)$. Thus the exact $q_l(t)$ exhibits contributions from all explicit wavevectors and integrations over past history. It must be assumed that $q_l(t)$ is a complicated nonalgebraic stochastic functional of all the explicit mode amplitudes, most likely not expressible in closed form by any known tools.

It seems likely to the present author that some recasting of RNG approaches is called for if they are to yield physically natural higher approximants that converge to the exact $q_l(t)$. Fundamental questions arise around both the concept of fixed point and the process of successive elimination of thin shells. As RNG methods have been applied to turbulence so far, they assume an underlying power-law spectrum which extends over an infinite ratio in wavenumber. The fixed point which is sought is an invariant behavior, under successive band eliminations, of the suitably rescaled dynamical damping exerted by all the eliminated modes on modes with very small wavenumber. If higher approximations are to be analyzed, this concept of fixed point naturally enlarges to that of invariant behavior of all the (suitably rescaled) stochastic functions $b_i(t)$, $\eta_{ij}(t,s)$, $\gamma_{ijm}(t,s,s')$ which enter (4.4). The dynamic damping exerted on very small wavenumbers is expressed by the $k \ll k_c$ limit of $\eta_{ij}(t,s)$. What scaling is actually correct cannot be assumed in advance. Successive approximations may force a deviation from the -5/3 law.

The fundamental problem with the fixed-point concept is not that it must be enlarged but rather that it may be substantially irrelevant. Actual turbulent flows have finite Reynolds numbers. Even in an infinite inertial range, the really interesting behavior involves semicoherent flow structures, intermittency, and other deviations from the classical Kolmogorov -5/3 scaling. It is likely that these effects are dominated by dynamical interactions among wavevector triads with finite wavenumber ratios, rather than by coherence effects that extend over infinite wavenumber ratios.[26] If this is so, then the interesting dynamics of the infinite inertial range are inessentially different from those of turbulent flows with finite wavenumber ranges. In both cases, it is the structure within a finite range that is important. If that structure were well enough portrayed, then the chaining of finite ranges to yield an infinite inertial range would be a lesser and secondary task.

Suppose that the fixed-point apparatus were eliminated and it were desired to construct a formalism that applies equally to finite as well as infinite Reynolds numbers. The remaining parts of the RNG approach would be the successive band elimination and the ϵ expansion. The relevance of elimination of infinitesimal shells is also questionable in higher approximations. The natural shell thickness is a range in wavenumber over which there extend dynamically significant correlations. At finite Reynolds number the entire spectrum may fall within one such shell. In a infinite inertial range it is unlikely this natural shell thickness is less than a decade in wavenumber.

At the lowest order of RNG treatment, an infinitesimally thick shell on the brink of elimination is assumed to be statistically independent of the currently explicit modes. Then the interaction with a mode of wavenumber $k_1 < k_c$ is introduced and calculated perturbatively. The result is an incremental eddy damping felt at k_1.[30] In this process, interaction of the shell with explicit modes $k_1 < k < k_c$ does not appear. If now higher perturbation orders are included, the dynamical structure within the natural shell thickness begins to show. Already at 4th-order, excitation at k_c is spread over a bandwidth k_1 in the convection time $\tau_c \sim (k_1/k_c)^{1/3}\tau_d$ by interaction with all modes $k_1 < k < k_c$. Here τ_d is the internal distortion (cascade) time at k_c. The implication is that it is more appropriate and less complicated to eliminate the entire natural shell at once instead of artificially breaking it up into infinitesimal shells. This granted, it may be simpler, and as justified, to eliminate <u>all</u> the implicit modes at one blow, and not deal with shells at all.

In any event - thin shells, thick shells, or no shells - the problems of convergence of primitive and renormalized perturbation series arise as they do in Sec. 3. A guessed at or simply estimated simple eddy damping could be introduced at the outset to accelerate convergence in low orders of the approximation sequence. The introduced eddy viscosity could be that determined by the lowest order of the Yakhot-Orszag theory.[30] If this eddy damping were added to the molecular damping and subtracted from the nonlinear term of the N-S equation, it would automatically disappear as the true dynamics were developed by successive approximation. No device of this kind can be expected to help significantly with convergence problems at high orders.[44]

The ε expansion is the most mysterious and intriguing component of the RNG arsenal. The combination of low orders of perturbation expansion and low orders of ε expansion has given good results in the study of critical phenomena. The ε expansion in turbulence theory has a problem of a priori plausibility: elementary arguments give the $E(k) \propto k$ spectrum, about which the expansion is based, a substantially different qualitative physics from that of the Kolmogorov -5/3 spectrum. No clear counterbalancing arguments for the validity of ε expansion have been stated.

It should be noted that the ε expansion logically need not be tied to the band-elimination procedure. For example, the ε expansion could be interlaced with the Eulerian or Lagrangian renormalized perturbation expansions of Sec. 3, since the latter can be applied, at any order, to the $k^{1-2\varepsilon/3}$ spectrum. Or it could be used in conjunction with the DSC approximations of Sec. 4.

If the attempt is made to remodel RNG to handle finite Reynolds numbers, the ε expansion as usually stated becomes less plausible. If $U(k)$ is kept fixed in form and dimensionality changed to 17/3, the initial spectrum $E(k)$ for a decay problem is radically changed, and there is little reason to expect the qualitative physics to survive the change. However there is an alternative procedure which does not have this drawback and which may be interesting to study. Suppose that dimensionality D is increased from $D = 3$ with E(k) rather than U(k) kept fixed. It may be that at a large enough value D_{cr} (especially $D_{cr} = \infty$)[70] the true dynamics and the perturbation-theory dynamics both simplify importantly. An expansion in powers of $D_{cr} - D$ (or 1/D) might be more physically relevant than the original ϵ expansion.

Some remarks should be made about the relation of RNG approximation to the problem of noninvariance under random Galilean transformation, which afflicts Eulerian RPT. In the lowest order of RNG approximation the effective eddy viscosity felt at k is found to have the physically expected relation to the characteristic distortion time $\tau_d(k)$. This is in contrast to the predictions of Eulerian RPT, in particular DIA. However, as noted earlier, the effects of intermediate wavenumbers do not appear in this lowest order. In higher orders of the Eulerian perturbation theory used in the RNG analysis, the effects of convection by intermediate wavenumbers are large. They bring the dominant convection time τ_c into the analysis of distant eddy viscosity. This is so whether the Kolmogorov spectrum is treated directly or the ϵ expansion is used in higher orders. The results will depend very much on how the perturbation treatment is developed. If primitive ordering of perturbation terms is used, problems of random Galilean invariance need not arise.[56] But other orderings of perturbation terms can give noninvariance effects similar to those encountered by Eulerian RPT and described in Sec. 3. In particular, this problem can affect the predicted localness in wavenumber of energy cascade from explicit to implicit modes.

In this connection, a fundamental difference between eddy viscosity and molecular viscosity should be noted. Eddy viscosity can be represented by an absolute equilibrium incompressible motion at thermal velocity on the spatial scale of the molecular mean free path.[71] The Lagrangian and Eulerian correlation times of this motion are of the same order, and this equality is unaffected by convection by the total hydrodynamic motion because the latter has a velocity much smaller than thermal velocity. Thus the effects of convection by intermediate hydrodynamic modes on the molecular viscosity felt at low k are negligible. In contrast, the velocity associated with the inertial-range excitation above k_c is small compared to the velocity in intermediate modes at wavenumbers $\ll k_c$, and the result is Lagrangian and Eulerian correlation times that scale differently with k_c.

In summary, three suggestions have been offered concerning the implementation of higher orders of RNG:

1. Eliminate the implicit modes all at once instead of in successive infinitesimally thick shells.

2. Introduce by hand a simple eddy damping, like that of the lowest order of the Yakhot-Orszag theory, in such a way as to accelerate convergence in low orders of perturbation theory.

3. Modify the ϵ expansion so that the energy spectrum $E(k)$ rather than the mode intensity $U(k)$ is held fixed in form under change of space dimensionality.

With these changes, several methods of successive approximation could be interlocked with the ϵ expansion. They include the primitive and renormalized perturbation expansions of Sec. 3, with or without convergence accelerators, and the decimation approximations of Sec. 4.

6. THE UPPER-BOUNDING APPROACH OF BUSSE, HOWARD, AND MALKUS

The upper-bounding theory for turbulent transport was first formulated by Howard,[34] after a germinal investigation by Malkus.[33] Latter development is due to Busse and others.[35] This approach deserves mention in any survey of analytical methods for turbulence because it uniquely gives rigorous results. The latter are in the form of upper bounds for turbulent transport of momentum and heat, and they are obtained with utmost economy of materials. All that is used of the N-S equations (or the Boussinesq equations for thermal convection) are certain exact integral properties. The bounds are found by extremalizing transport with these properties as constraints. Most of the work has been restricted to single-time constraints, but recently Krommes and Smith have used two-time constraints in a study of heat transport by passive advection.[36] No applications to homogeneous turbulence in an infinite box seem to have been reported, but Sulem and Frisch have developed bounds on energy flux and inertial-range power laws for turbulence of finite energy.[72]

There is a possibility that it may be feasible to combine similar extremalizing techniques with the DSC approximations outlined in Sec. 4. This is because the latter are obtained by imposing a subset of the exact symmetry constraints. In Sec. 4, approximate solutions were sought by constructing least-squares realizations of the stochastic forcing terms $q_i(t)$ under the subset of symmetry constraints. Instead, solutions could be sought that maximize or minimize chosen integral properties, such as the total dissipation by viscosity, or the total transport of energy through a given spherical surface in the wavevector space. The DSC method is not limited to homogeneous turbulence. In inhomogeneous problems, transport could be maximized precisely as in the upper-bounding theory.

The remarkable success obtained with the upper-bounding theory using very few integral constraints suggests that it may be profitable to examine DSC approximations (with or without extremalizing) under broadly similar constraints, rather than the detailed constraints that yield DIA. It may be that integral constraints constructed from several orders of the set of symmetry constraints will yield better results at lower computational cost than detailed constraints confined to the first two orders of (4.6). Such constraints logically would be chosen to express overall conservation and invariance properties.

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