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1

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TITLE TOWARDS AN ANALYTIC SOLUTION OF QCD; THE GLUEBALL MASS GAP

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TOWARDS AN ANALYTIC SOLUTION OF QCD; THE GLUEBALL MASS GAP*

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In spite of the apparent simplicity of the QCD Lagrangian only meagre progress has thus far been made towards an analytic understanding of its low energy behavior. Indeed, almost all efforts these days have shifted to the area of numerical simulation. In this talk I would like to review certain general features and beliefs concerning QCD with the view to seeing (a) whether (following 't Hooft)[1] the theory makes sense and (b) whether we can determine its physical spectrum. The sorts of ideas that we have in mind include the following:**

i) A typical Green's function can be represented, as a function of the coupling constant (g), as an expansion around minima of the action;

ii) Each term in such an expansion is divergent and therefore requires renormalization;

iii) Even after renormalization, each of the series generated by an expansion around a minimum is divergent and therefore requires a summability procedure to make sense;

iv) The resulting physical Green's functions must remain causal.

Notice that statements (i) and (iii) imply certain analytic properties in g whereas (iv) implies certain analytic properties in momentum (q). On the other hand, renormalisability (ii) tells us that the behavior in q^2 and g^2 are not, in fact, independent of each other. This rather curious state of affairs will be of crucial importance.

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*Talk given at the Irvine Conference on Non-perturbative Methods in Field Theory, Laguna Beach, CA. January 7, 1987.

**For a general review of some of these ideas as well as the standard notation see, for example, ref. 2.

Typically, no matter how the theory is initially defined, the momenta and couplings are treated, to start with, as completely independent parameters. Yet, after introducing an arbitrary, though consistent, procedure for sweeping inevitable infinites under the rug, they no longer remain so. Concurrently the resulting finite Green's functions must still respect general principles such as gauge invariance and causality, for example. Indeed, since the renormalization group (RG) requires that g and q generically come in the combination $q^2e^{1/g2}$ it is clear that analyticity in one cannot easily coexist with analyticity in the other. The tension between the behavior in these two variables is highlighted by the presumed existence of a perturbative regime (i.e. of the Feynman series) which naively would suggest a finite region of analyticity around $g^2 = 0$. This is clearly in conflict with a finite region of analyticity in q^2 around $q^2 = 0$. A major theme of this talk will be to examine in what way this makes sense and how the theory deals with the apparent conflict with analyticity in q^2. We shall try to show that the resolution of this apparent dilemma can lead to a calculation of the mass gap in QCD and, more generally, to the spectra of Green's functions.

To be specific and to keep things as simple as possible we shall consider the case of the scalar glueball in pure QCD defined in the usual way, by the Lagrangian $\frac{1}{4}(F^a_{\mu\nu})^2$. The interpolating field for the glueball will be taken to be

$$\Theta(x) = \frac{\beta(g)}{g} (F^a_{\mu\nu})^2 \tag{1}$$

where $\beta(g)$ is the standard β-function. Notice that $\theta(x)$ is gauge invariant and, because of the trace anomaly,[2] has no anomolous dimension. After a scaling of the gauge-fields $A^a_\mu \to A^a_\mu/g$, its propagator is defined by

$$G(q^2) = \int d^4x e^{iq.x} \frac{\int DA^a_\mu \, e^{iS/g^2} \theta(x)\, \theta(0)}{\int DA^a_\mu \, e^{iS/g^2}} \tag{2}$$

where complications due to gauge fixing, ghosts, etc. have bren suppressed. Let us now examine points (i) - (iv) in more detail.

i) Expansion around Minima of S

The conventional systematic way of dealing with (2) is to expand it around local minima of S (S_m, say) using a stationary phase or, in Euclidean space, a saddle point technique. The generic form for the expansion is

$$G(q^2) \simeq g^4 \sum_{m,n=0}^{\infty} A_{mn}(q^2)\, e^{-S_m/g^2}\, (g^2)^{n-\nu_m} \tag{3}$$

The only known minima of S, apart from the trivial one $S = 0$, occur in the Euclidean region. In Minkowski space, these are identified as tunneling events between different vacua.[2] Thus, each m can be thought of as characterising a different topological sector. The standard Feynman series ($m = 0$) corresponds to $S_0 = 0$ and, for the glueball, $\nu_0 = 0$. The "bare" expansion represented by (3) is pathological in two separate ways: first, each coefficient $A_{mn}(q^2)$ is divergent and second, each series, summed over n, is divergent (i.e. has zero radius of convergence). The first of these is fixed up via renormalization whereas as the second by a summability technique.

ii) Renormalization Group Constraint

The finite renormalized coupling $g(\mu)$ must be defined at some arbitrary mass scale μ thereby introducing a mass into the massless theory. The resulting renormalized Green's function, (2), should therefore be considered as a function $G(q^2,\mu^2,g^2(\mu))$. As such, it still has an expansion of the form (3) but with g replaced by $g(\mu)$. The renormalization group (RG) expresses the invariance of G to the choice of μ; explicitly[3] (recall that Θ has no anomolous dimension)

$$\left[\mu \frac{\partial}{\partial\mu} + \beta(g) \frac{\partial}{\partial g}\right] G(q^2,\mu^2,g^2) = \sum_{m=0}^{2} \mu^{2(2-m)}\, q^{2m}\, I_m(g) \tag{4}$$

The inhomogeneous terms on the right come from extra subtractions needed to remove divergences arising from the composite nature of $\theta(x)$. The general solution to (4) is given by

$$G(q^2,\mu^2,g^2) = \mu^4 e^{-4K(g)}\, F\left[\frac{q^2}{\mu^2}\, e^{2K(g)}\right] + \sum_{m=1}^{2} \mu^{2(2-m)}\, q^{2m}\, \phi_m(g) \tag{5}$$

where $K(g) = \int^g \frac{dg}{\beta(g)}$ and $F(z)$ is an "arbitrary" function of the single variable $z = (q^2/\mu^2)e^{2K(g)}$. The functions $\phi_m(g)$ are related to $I_m(g)$; in perturbation theory, it turns out that $\phi_1(g) \simeq O(g^4)$ whereas $\phi_2(g) \sim \int I_2(g)/\beta(g) \simeq O(g^2)$. At first sight this is somewhat surprising since in perturbation theory $G \simeq O(g^4)$. However, it is easy to see that the cancellation of the $O(g^2)$ term in $\phi_2(g)$ requires both

$$F(z) \overset{z\to\infty}{\sim} z^2(\ln z)^{-1} \tag{5a}$$

<u>and</u>

$$G(q^2,\mu^2,0) \equiv A_0(q^2) = q^4 \ln q^2/\mu^2 \text{ for } \underline{\text{all}}\ q^2. \tag{5b}$$

The first of these (5a) is none other than the asymptotic freedom result for G and implies that $G(q^2,\mu^2,g^2) \overset{q^2\to\infty}{\simeq} q^4 [\ln q/\mu^2]^{-1}$. The second (6b) is simply its normalization.

Notice, incidentally, that $\rho(q^2,\mu^2,g^2) \equiv \mathrm{Im}\, G(q^2,\mu^2,g^2)$ satisfies the conventional homogeneous renormalization group equation and so must have the structure

$$\rho(q^2,\mu^2,g^2) = \mu^4 e^{-4K(g)}\ f\left[\frac{q^2}{\mu^2}\, e^{2K(g)}\right] \tag{7}$$

where $f(z) \equiv \mathrm{Im}F(z)$. It is not difficult to show that the perturbative requirement that $\rho \sim O(g^4)$ for small g^2 leads to $\rho(q^2,\mu^2,0) \equiv a_0(q^2) = q^4$ and $\rho(q^2,\mu^2,g^2) \overset{q^2\to\infty}{\simeq} q^4[\ln q^2/\mu^2]^{-2}$ in agreement with the results for G, Eq. (6).

The expansion of the β-function $\beta(g) = -b_1g^3 + b_2g^5 + \ldots$ leads to $K(g) = 1/b_1g^2 - b'\ln g^2 + \ldots\ldots$ where $b' = b_2/b_1^2$; thus $K(g) \to \infty$ when $g^2 \to 0$. From Eq. (5) this is equivalent to $q^2 \to \infty$ and therefore to asymptotic freedom. On the other hand, if $g^2 \to 0^-$ (i.e. from below) then $K(g) \to -\infty$ which is equivalent to $q^2 \to 0$. Thus <u>if</u> G were analytic in g^2 at $g^2 = 0$ (so that it doesn't matter how the free field limit is approached) then its infrared behavior ($q^2 \to 0$) would be identical to its ultraviolet ($q^2 \to \infty$). Of course, this is not so, precisely because, as we shall demonstrate below, even the perturbation series is not analytic at $g^2 = 0$. Nevertheless this observation illustrates the potential power of the use of arguments based upon analyticity. The apparently innocuous (though erroneous) assumption that perturbation theory be analytic at $g^2 = 0$ would lead to a determination of the infrared as well as ultraviolet behavior of the theory! A valid statement, however, is that the difference between the infrared and ultraviolet is a measure of the non-analytic behavior at $g^2 = 0$.

We should also remark at this point that, according to 't Hooft[1], there exists a renormalization scheme where $K(g)$ is completely determined by its first two perturbative terms: $K(g) = 1/b_1 g^2 - b' \ln g^2$. The claim is that in a given scheme there always exists a transformation of g which reduces $K(g)$ to this form. This is motivated by the observation that b_1 and b_2 are the only terms in $\beta(g)$ which are both scheme and gauge invariant. Furthermore, they are the only terms in $K(g)$ that are singular when $g^2 \to 0$.[4]

iv) Causality and Analyticity

The fact that the implicit commutator in (1) is causal and therefore vanishes outside of the forward light cone is the physical reason that G is analytic everywhere in the complex q^2-plane except possibly along the real axis as shown in Fig. 1. In a physical channel the first singularity must

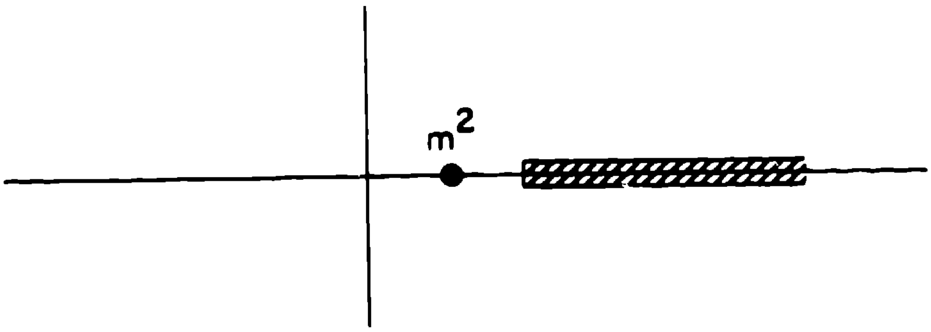

Fig. 1 Expected singularity structure in the q^2-plane for a physical channel.

occur at $q^2 = m^2$ where $m^2 > 0$. It is usually assumed, or implied, that in order for this to be interpreted as representing a free particle or asymptotic state it must be an isolated pole $\sim(q^2 - m^2)^{-1}$. Recall, however, that for the electron in QED (which is certainly a physical particle) this is not the case; its singularity is a cut whose strength is in fact, gauge dependent.[5] This, of course, is a reflection of the fact that an electron, being charged, is always attended by a cloud of zero mass photons. Naively one would not expect this to be the case for the glueball since it is colorless and, furthermore, gluons unlike photons are supposedly unobservable. Nevertheless it is presumably possible that physical colored singlets are always accompanied by a sea of gluons which are also in a singlet state. Notice, incidentally, that regardless of the nature of the singularity the existence of a mass gap means that G must be analytic in q^2 at $q^2 = 0$. The RG, Eq. (5), therefore, forbids G from being analytic in g^2 at $g^2 = 0$.

Analyticity can be expressed and exploited in many different ways, the most primitive of which is a Taylor series expansion in q^2 for $q^2 < m^2$;

$$G(q^2,\mu^2,g^2) = \mu^4 \sum_{n=0}^{\infty} C_n(g^2)\left(\frac{q^2}{\mu^2}\right)^n \tag{8}$$

Unlike the expansion in g^2, Eq. (3), this series has a non-zero radius of convergence, namely m^2. Furthermore, this series must be expressible in RG invariant form as in Eq. (5). It is easy to see that this determines the full g^2 dependence of the coefficients $C_n(g^2)$ viz:

$$C_n(g^2) = \bar{C}_n\, e^{(n-2)2K(g)} + \delta_{n1}\,\phi_1(g) + \delta_{n2}\,\phi_2(g) \tag{9}$$

where the $\bar{C}_n$ are numbers independent of g^2.

The expansion Eq. (8) with coefficients given by (9) is an exact statement expressing causality and the renormalisability of the theory. It can be thought of as the complement to the original expansion in g^2 given in Eq. (3). This immediately raises the question as to how these apparently different representations (3), (5) and (8) can coexist. In particular, what is the meaning or nature of perturbation theory since the coefficients $C_n(g^2)$ in the "exact" expression (8) are so singular at $g^2 = 0$? To put it explicitly, how, for example, in the 't Hooft scheme can

$$G(q^2,\mu^2,g^2) = \mu^4 \sum_{n=0}^{\infty} \bar{C}_n\, e^{(n-2)/bg^2}(g^2)^{-(n-2)b'}(q^2/\mu^2)^n$$

$$\simeq g^4 \sum_{n=0}^{\infty} A_n(q^2)\, g^{2n} + \ldots\ldots \quad ? \tag{10}$$

To answer this, we need to review the ideas of the summability of asymptotic series. Before doing so, however, let me give the more familiar disperson representation of analyticity since this generalizes the Taylor series expression, Eq. (8).

Assuming that the number of subtractions is determined solely by convergence properties and that these are given by asymptotic freedom, Eq. (6), allows us to write the representation

$$G(q^2,\mu^2,g^2) = G(0,\mu^2,g^2) + q^2\, G'(0,\mu^2,g^2)$$

$$+ \frac{q^4}{2\pi}\int_{m^2}^{\infty} \frac{dq'^2}{q'^4}\quad \frac{\rho(q'^2,\mu^2,g^2)}{(q'^2-q^2)} \tag{11}$$

This can be expressed in RG invariant form, Eq. (5) to give[3]

$$G(q^2,\mu^2,g^2) = \mu^2 q^2 \phi_1(g) + q^4 \phi_2(g) + \mu^4 e^{-4K(g)} F(0) + \mu^2 q^2 e^{-2K(g)} F'(0)$$

$$+ \frac{q^4}{\pi} \int_{z_0}^{\infty} \frac{dz}{z^2} \frac{f(z)}{[z-(q^2/\mu^2)e^{2K(g)}]} \qquad (12)$$

where $m^2 = z_0\, \mu^2 e^{-2K(g)}$; z_0 is a RG invariant number that determines the mass gap. Below, we shall show how it can be calculated. Notice that an expansion of the dispersive part of Eq. (12) in powers of (q^2/μ^2) reproduces the Taylor series, Eq. (8), thereby showing how the $\bar{C}_n$ are related to moments of $f(z)$.

<u>Summability</u>

We have already stressed that a typical series in the expansion (3), and, in particular, the perturbation series, is divergent. At best, we can hope that these series are asymptotic and therefore amenable to some summability technique such as that of Borel[1,2,6]. To review the ideas consider the perturbation series

$$G(q^2,\mu^2 k) \simeq \sum_{n=0}^{\infty} A_n(q^2) k^{-(n+2)} \qquad (13)$$

where, for convenience, we have introduced $k \equiv 1/g^2$. The Borel transform of G is basically just an inverse Laplace transform

$$\mathcal{G}(q^2,\mu^2,\xi) \equiv \int_L \frac{dk}{2\pi i}\, k\, e^{k\xi} G(q^2,\mu^2,k) \qquad (14)$$

where the integral is along a line L running parallel to the imaginary axis in the complex k-plane and standing to the right of all singularities. Substituting the series (13) into (14) gives

$$\mathcal{G}(q^2,\mu^2,\xi) = \sum_{n=0}^{\infty} \frac{A_n(q^2)\xi^n}{n!} \qquad (15)$$

This new series has considerably more convergence than the original one and the idea is that if it has a finite radius of convergence then the resulting sum can be used to reconstruct G via the inverse of (14); namely the Laplace transform

$$G(q^2,\mu^2,k) = k \int_0^\infty d\xi\; e^{-k\xi}\, \mathcal{G}(q^2,\mu^2,\xi) \quad . \tag{16}$$

From a naive point of view, such a scheme may have some chance of success since the series (13) and, more generally (3), have their origins in the asymptotic expansion of the path integral, Eq. (2). In Euclidean space this looks like a functional Laplace transform on e^{-kS}. There are, naturally, many conditions to be satisfied for this procedure to work in a consistent and unique manner. Obviously, we must at least demand that the new series (15) has a finite radius of convergence and that the singularities in ξ for $\xi > 0$ be integrable; furthermore, the integral over ξ must converge and so on.

Some of these questions, and indeed the ones relevant to our calculation of z_0, can be answered from what we have already discussed. The position and nature of the singularities in the Borel plane can, in principle, be determined using the representations (12) or (8). For example, in the 't Hooft scheme

$$\mathcal{G}(q^2,\mu^2,\xi) = \mu^2 q^2 \tilde{\phi}_1(\xi) + q^4\, \tilde{\phi}_2(\xi) + \mu^4 F(0)\; \frac{(\xi-2/b)^{2b'-2}}{\Gamma(2b'-1)}$$
$$+\, q^2\mu^2\, \frac{(\xi-1/b)^{b'-2}}{\Gamma(b'-1)}\, F'(0) + \frac{q^4}{\pi} \int_{z_0}^{\infty} dz\, \frac{f(z)}{z^2} \int_L \frac{dk}{2\pi i}\, \frac{e^{k\xi}\qquad k}{[z-(q^2/\mu^2)e^{k/b}k^{b'}]} \tag{17}$$

where $\tilde{\phi}_m(\xi)$ are the Borel transforms of $\phi_m(g)$. Notice that there are singularities on the positive real axis at $\xi = 1/b$ and $2/b$ but that these are associated only with the real part of G. The singularities in the dispersive part, on the other hand, occur only for $\xi \leq 0$ as can be seen using Eq. (8); this gives a string of singularities

$$\frac{(\xi+n/b_1)^{-nb'-2}}{\Gamma(-nb'-1)} \qquad n \geq 0 \tag{18}$$

Notice that the nature of these singularities is governed by the signs of <u>both</u> b_1 and b', reflecting their role in determining the convergence properties of the integrals in (17). Based upon this we have suggested[6] that if Borel summability be used as a criterion for consistency then only theories with $b_1 > 0$ and $b' < 0$ could qualify. Remarkably, QCD appears unique in satisfying these conditions provided the number of flavors, $n_f \leq 8$. The cognoscenti will recognize the singularities in (18) as those first discovered by Parisi using various approximation schemes such as leading log. or 1/N expansions.[7] These were dubbed renormalons to distinguish them from explicit instanton contributions which would give singularities at $\xi = S_m$.[8] From our "exact" result Eqs. (17) and (18) however, it would appear that adding instantons to renormalons would be double counting, at least in the 't Hooft scheme. This could be an indication that this scheme is inadequate to accommodate explicit non-perturbative effects such as those due to instantons. On the other hand, it is worth pointing out that instantons are a product of Euclidean space whereas our derivation of the Borel singularity structure was strictly valid only in Minkowski space since q^2-analyticity was an essential ingredient. The latter, as already emphasized, follows from causality and, concomitantly, the existence of a light cone both of which require Minkowski space to have meaning. It is therefore conceivable that in Minkowski space with the 't Hooft scheme all non-pertubative effects including those due to instantons are included in the renormalons. This, of course, is implied by equating Eq. (10) with the expansion (3). Some additional credence can be given to this conjecture from the observation that mathematically any instanton-like contribution can be expanded in terms of an infinite series of renormalons (and vice-versa). For example, for a given m

$$e^{-1/b_1 g^2} = -\frac{\nu}{b}\sum_{n=0}^{\infty}\frac{(nS_m - 1/b_1)^{\nu n-1}}{\Gamma(1+\nu n)\sin\nu\pi n}\left[\frac{e^{-S_m/g^2}}{g^{2\nu}}\right]^n \tag{19}$$

This suggests that in an arbitrary scheme there may well be no unique way of dividing a renormalized amplitude into separate renormalon and instanton contributions even though the physical basis for each is quite distinct. From this point of view the 't Hooft scheme represents the extreme case where all instanton-like contributions have been implicitly expanded in terms of renormalons as in Eq. (19). The "conventional" expansion, represented by (3), can be thought of as the exact converse of this in which all renormalons have been expanded in terms of instantons. An alternative possibility to this somewhat radical conjecture, is, as already mentioned, that the 't Hooft scheme is simply incomplete and essentially perturbative in nature.

There is a final important point that has thus far been ignored and that is the question of an apparent singularity at $\xi = 0$. Going back to Eqs. (12) and (17) notice that since $\phi_2(g) \sim g^2$, $\tilde{\phi}_2(\xi) \sim 1/\xi$. Recall, however, that asymptotic freedom arose from the cancellation of this "non-perturbative" behavior of $\phi_2(g)$ by an exactly similar behavior in F(z) [see Eqs. (5) and (6)]. Obviously, then, the apparent singularity $1/\xi$ in the transformed variable must similarly be eliminated from (17) by the asymptotic behavior of f(z) and, indeed, it is not difficult to verify that this is the case. However, the renormalons singularities given in (18) contain a $1/\xi^2$ term which is in violation of this general result. The problem can be traced to the observation that (18) was derived from the Taylor series representation (8) which can be thought of as an expansion for the dispersive part in (17) and which is valid only when $q^2 < m^2$. It should therefore not be expected to accommodate the $q^2 \to \infty$ behavior needed to cancel the $1/\xi$ term. The uniformity of the Borel singularity structure with respect to the infrared and ultraviolet is a subtle one which we shall address elsewhere. Suffice it to say here that asymptotic freedom ensures the absence of a singularity at $\xi = 0$. In summary, we can say that the series $\mathcal{G}(q^2,\mu^2,\xi)$ can be expected to have a finite radius of convergence ($=1/b_1$ in the 't Hooft scheme but $\min(1/b_1, S_m)$ in any other.)[1] This implies (at least for the perturbative series) that

$$\mathcal{G}(q^2,\mu^2,0) = A_0(q^2) = q^4 \ln q^2/\mu^2 \tag{20a}$$

and

$$\mathrm{Im}\,\mathcal{G}(q^2,\mu^2,0) = a_0\,(q^2) = q^4 \tag{20b}$$

An analogous result is valid for each sector m of Eq. (1). Although $\mathcal{G}$ has singularities on the positive real axis these do not contribute to its absorptive part.

The Glueball Mass

I shall now show how the ideas developed above can be used to determine the position of the first singularity in G. Before doing so however, I want to exploit q^2-analyticity in a somewhat different way than is usually done. Consider the following function

$$\phi(s) \equiv \int_L dz\, z^{s-1}\, F(z) \tag{21}$$

where the line L runs parallel to the imaginary axis in the region of analyticity shown in Fig. 2. The integrand has cuts on the right hand

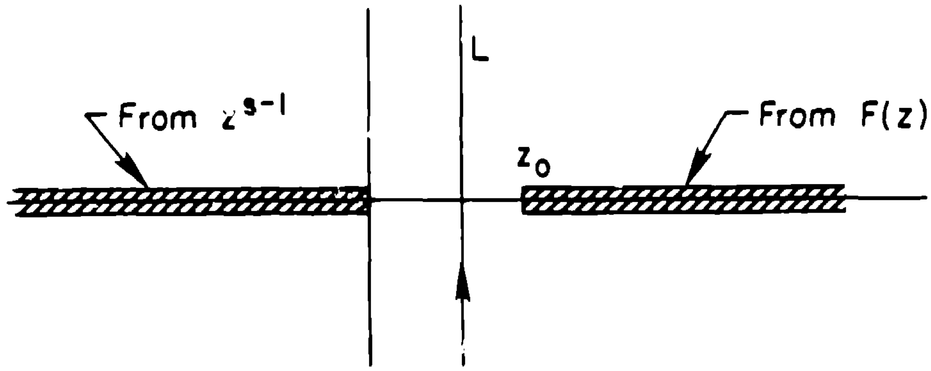

Figure 2. Singlularity structure of the integrand $z^{s-1}F(z)$ of Eq. (21).

positive real axis coming from F(z) -- this is the physics -- and a cut on the left beginning at z = 0 due to z^{s-1}. $\phi(s)$ therefore defines an analytic function of s everywhere the integral converges. This, however, is determined by asymptotic freedom, Eq. (6), so we can conclude that $\phi(s)$ is an analytic function of s provided Re s < -2.

Suppose now that the contour is closed in the right hand plane, then $\phi(s)$ reduces to a conventional Mellin transform of f(z) = ImF(z) rather than of F(z) itself:

$$\phi(s) = \int_{z_0}^{\infty} dz\, z^{s-1} f(z) \tag{22}$$

and this must be analytic for Re s < -2. Its inverse is given by[9]

$$f(z) = \int_L \frac{ds}{2\pi i}\, z^{-s}\, \phi(s) \tag{23}$$

where L here is a line standing to the left of Re s = -2. These equations therefore embody the general analytic properties of G as dictated by causality and illustrated in Fig. 1. Notice that they involve f(z) rather than F(z). Because of this it is considerably more convenient to work with $\rho(q^2,\mu^2,g^2)$ than with the full G.

Recall from Eq. (7) that ρ must be of the form

$$\rho(q^2,\mu^2,g^2) = \mu^4\ e^{-4K(k)}\ f\left[\frac{q^2}{\mu^2}\ e^{2K(k)}\right] \tag{7}$$

which, with the Mellin representation (23), can be expressed as*

$$\rho(q^2,\mu^2,g^2) = \mu^4 \int_L \frac{ds}{2\pi i}\ \phi(s)\left(\frac{q^2}{\mu^2}\right)^{-s} e^{-(s+2)2K(k)} \tag{24}$$

Its Borel transform is therefore [see Eq. 16]

$$\mathrm{Im}\ \ (q^2,\mu,{}^2\xi) = \mu^4 \int_L \frac{ds}{2\pi i}\ \phi(s)\left(\frac{q^2}{\mu^2}\right)^{-s} I(s,\xi) \tag{25}$$

where

$$I(s,\xi) = \xi \int_L \frac{dk}{2\pi i}\ e^{k\xi-(s+2)K(k)}\ k \tag{26}$$

Now, for the moment, restrict the discussion to the perturbation series; as already shown, its Borel transform has a finite radius of convergence and is normalized to $\mathrm{Im}\ \mathcal{G}\ (q^2,\mu^2,0) = a_0(q^2) = q^4$. Using this in (25) gives*

$$\phi(s) = \frac{1}{(s+2)I(s,0)} \tag{27}$$

In the 't Hooft scheme $I(s,\xi)$ can be evaluated exactly leading to

$$\phi(s) = \frac{\Gamma[(s+2)b'-1]}{[-(s+2)/b_1]^{(s+2)b'-1}} \tag{28}$$

Notice, that if $b' < 0$, then, as promised, $\phi(s)$ has singularities only in the region Re $s < -2$, (see Fig. 3). This is in agreement with the requirements of q^2-analyticity and the convergence of the Borel series. Given $\phi(s)$ we can use (23) to determine $f(z)$ and thereby, ρ. Although technically it is difficult to evaluate the integrals, it is fairly simple to

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*In order to accommodate the usual "unsummed" perturbation series which has no mass gap, generalized Mellin transforms must be used. These are defined by analogy with generalized Fourier transforms discussed, for example, in Ref. 9.

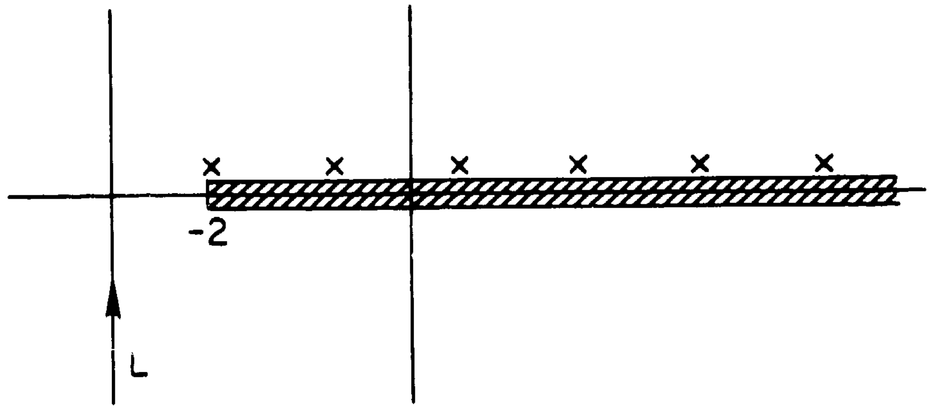

Figure 3. Singularity structure of $\phi(s)$. Eq. (28).

determine the threshold z_0. The important point to note is that the integrand in (23) has the following asymptotic behavior when Re $s \to \infty$:

$$z^{-s}\,\phi(s) \sim \frac{1}{s^{1/2}} \left[\frac{z}{\left(-\frac{e}{b_1 b'}\right)^{-b'}} \right]^{-s} \tag{29}$$

Clearly if $z < (-e/b_1 b')^{-b'}$ this vanishes when Re $s \to -\infty$; in this case we can close the contour in the left hand plane where there are <u>no</u> singularities and thus obtain no contribution. On the other hand if $z > (-e/b_1 b')^{-b'}$ the contour must be closed in the right hand plane where there are singularities thereby giving a non-zero contribution. Thus $f(z)$ is non-vanishing only for $z > z_0 = (-e/b_1 b')^{-b'}$ so that the first singularity in ρ occurs at

$$q^2 = m^2 = z_0\, \mu^2\, e^{-2K(k)} \tag{30}$$

Notice that z_0 can be re-expressed as

$$z_0 = \left(-\frac{eb_2}{b_1}\right)^{-b_2/b_1^2} \tag{31}$$

$$= \left(\frac{88e\pi^2}{51}\right)^{51/121} \quad \text{in the pure gauge theory}$$

$$\simeq 5$$

Thus if $\Lambda^2 \equiv \mu^2\, e^{-2K(k)}$ is the renormalization group invariant mass scale the first singularity in the glueball propagator occurs at $M^2 \simeq 5\,\Lambda^2$.

Remarks and Conclusions

The above calculation can be repeated for each sector m, defined in Eq. (3), if we assume that each such contribution is separately causal and therefore analytic in q^2. In that case one can show that, although the detailed structure of G changes, the position of z_0 remains the same. Thus, the location of the leading singularity of G given by Eqs. (30) and (31) includes the contribution from all possible instanton-like contributions. We caution, however, that these explicit calculations were all carried out in the 't Hooft scheme and, as already discussed, this may be too constraining a scheme to accommodate all non-perturbative phenomena. Nevertheless, it is remarkable, that a unique result for z_0 can actually be derived in this particular scheme. We remind the reader that its major attraction apart from giving an explicit expression for β is that only gauge and scheme invariant contributions are retained. Thus Λ, the only mass scale in the theory, must also have this property. However, it is difficult to compare its value with values in other schemes. The only unambiguous way of comparing calculations is to eliminate Λ by calculating a dimensionless quantity such as the ratio of masses. We have therefore repeated the above calculation with $\theta(x)$ replaced by its axial counterpart $\theta_A(x) \equiv \beta(g)/g\, \tilde{F}_{\mu\nu} F^{\mu\nu}$. Unlike θ, θ_A does have an anomolous dimension, which, of course, changes the details of the calculation. Surprisingly, however, we find that the location of its leading singularity is identical to that found for θ, which suggests that the 0^+ glueball is degenerate with the 0^-. In future investigations we intend to examine other higher spin states as well as the fundamental matrix element $\langle 0|F_{\mu\nu}^2|0\rangle$ which can be related to F(0) via a low energy thereom.[10]

As already remarked we have not yet succeeded in showing that the leading singularity represents an isolated pole. Thus far, we have only been able to show that the singularity must be of the form $(q^2-m^2)^\alpha$ with $\alpha < 0$. It is, of course, conceivable that $\alpha \neq -1$ in which case some thought would have to be given to its interpretation and physical consequences. It is not impossible that pure QCD does not admit of a simple particle interpretation and that it is analogous to a 2-dimensional SU(N) theory in the $N \to \infty$ limit where propagators have cuts but no poles. The physics of such a situation and what it might mean for particle detection have been discussed by McCoy and Wu.[11] Notice, incidentally, that $\alpha \neq -1$ would be very hard to detect numerically. Naturally the inclusion of quarks, especially massive ones, can be expected to change the situation dramatically. The RG equations change and the question of operator mixing becomes important and complex. The variation due to the quark mass parameters means that in the solution to the RG equation the function F now depends on two variables rather than one. A reanalysis in this more general case is therefore considerably more involved and is now being investigated.

In conclusion, we would like to stress that the general constraints of renormalisability, causality and summability are a remarkably tight set which can lead to determination of the glueball mass gap. In principle the analytic structure of all relevant Green's functions in the theory could be unravelled and the physical spectrum determined.

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