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Integral Approximants for Functions of Higher Monodromic Dimension*

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In addition to the description of multiform, locally analytic functions as covering a many sheeted version of the complex plane, Riemann also introduced the notion of considering them as describing a space whose "monodromic" dimension is the number of linearly independent coverings by the monogenic analytic function at each point of the complex plane. I suggest that this latter concept is natural for integral approximants (sub-class of Hermite-Padé approximants) and discuss results for both "horizontal" and "diagonal" sequences of approximants. Some theorems are now available in both cases and make clear the natural domain of convergence of the horizontal sequences is a disk centered on the origin and that of the diagonal sequences is a suitably cut complex-plane together with its identically cut pendant Riemann sheets. Some numerical examples have also been computed.

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INTEGRAL APPROXIMANTS FOR FUNCTIONS OF HIGHER MONODROMIC DIMENSION

Integral approximants[1] are a special case of Hermite-Padé approximants[2] of the Latin type. The integral approximants to a formal power series $f(z) = \Sigma_{j=0}^{\infty} f_j z^j$. One defines an integral approximant as follows: First let

$$\sum_{j=0}^{m} Q_j^{(\vec{q})}(z)\, f^{(j)}(z) + p^{(\vec{q})}(z) = O(z^{s+1}) \tag{1.1}$$

Define by the accuracy through order of principal the polynomials $Q_j^{(\vec{q})}$, $p^{(\vec{q})}$ of degrees less than or equal to q_j, p where

$$\vec{q} = (q_0, q_1, \ldots, q_m, p),\ s = m + p - 1 + \sum_{i=0}^{m} q_i \quad . \tag{1.2}$$

with the convention $q_i = -1$ implies $Q_i \equiv 0$ These $(\vec{Q}, P)$ always exist and $\vec{Q} \neq 0$, by a lemma of Baker and Lubinsky.[3] If $Q_m^{(\vec{q})}(0) \neq 0$, then they are unique. In contrast to the Padé case[4], we cannot be sure that we can usefully find an infinite subsequence where $Q_m^{(\vec{q})}(0) \neq 0$ as 0 might possibly be a singular point on a different Riemann sheet from the first one where we have the expansion given for f(z) as a power series of z. From the polynomials thus defined, we may next compute y(z) from

$$\sum_{j=0}^{m} Q_j^{(\vec{a})}(z)\, y^{(j)}(z) - p^{(\vec{a})}(z) = 0 \quad . \tag{1.3}$$

$$y(0) = f(0),\ y'(0) = f'(0),\ \ldots,\ y^{(m-1)}(0) = f^{(m-1)}(0) \tag{1.4}$$

This solution is the integral approximant to $f(z)$ and is denoted by

$$[p; a_0, a_1, \ldots, a_m]_f(z) \equiv y(z) \tag{1.5}$$

If $Q_m^{(\vec{a})}(0) \neq 0$ then this solution will exist. If however $Q_m^{(\vec{a})}(0) = 0$, then there may be a restriction on the initial conditions to achieve a regular solution. In the work which I will report here this problem is overcome by the proofs of convergence in cases where the limiting value of $Q_m(0) \neq 0$.

Next let us define the concept of monodromic dimension. We will begin with a functional element,

$$f(z) = \sum_{j=0}^{\infty} f_j z^j \quad . \tag{1.6}$$

which converges in some neighborhood of $z = 0$. It defines the complete monogenic analytic function. This function consists of a finite or at most denumerably infinite number of coverings of the complex plane. At this point, for reference, I remind you of the

<u>Monodromy theorem</u>.[5] If $f(z)$ is regular in a simply connected region G, then $f(z)$ is uniform (single valued) there.

The simplest extension of the situation described in the monodromy theorem is to a multiply connected region. Here the situation is quite different. Let me give a couple of examples.

$$f(z) = z^{1/m} \quad , \qquad f_2(z) = e^{2\pi i/m} f_1(z) \quad , \tag{1.7}$$

$$f(z) = \ln z \quad , \qquad f_n(z) = f_1(z) + 2\pi i(n-1) \tag{1.8}$$

In the example of Eq. (1.7) the function has exactly m sheets. In the example of Eq. (1.8) the function has an infinite number of sheets. Riemann[6] had the idea of classifying these functions according to the number of linearly independent coverings generated by the initial functional element. The example of Eq. (1.7) has just one such linearly independent covering. The example of Eq. (1.8) has exactly two linearly independent coverings even though it has an infinite number of Riemann sheets. Obviously the number of linearly independent coverings is less than or equal to the number of Riemann sheets so it can be a more efficient description or function classification.

<u>Definition</u>.[7] The monodromic dimension of a functional element is the number of linearly independent coverings of the complex plane generated by the associated monogenic analytic function.

Suppose we consider a function with monodromic dimension m and exactly n singular points in the whole complex plane. At some regular point z_0 from each of the m linearly independent coverings we can define m functional elements, and therefore m monogenic analytic functions, $y_1, \ldots, y_m$. If we encircle the $i^{\underline{th}}$ one of the n branch points then we get

$$y_j \to \sum_{k=1}^{m} M_{jk}^{(i)} y_k \tag{1.9}$$

This equation defines n mxm matrices M_{jk}. There is one for each of the n singularities. Form Eq. (1.9) must hold as a consequence of the supposition that there are at most m linearly independent coverings. In addition one can prove using Cauchy's theorem that for an appropriate order

$$M^{(1)} \cdots M^{(n)} = I \tag{1.10}$$

Thus the monodromy matrices, $M^{(i)}$ generate the Monodromy Group M of the $\vec{y}$ system. Now define the class

$$Q\left(\begin{matrix} a_1 & & a_n & \\ M^{(1)} & & M^{(n)} & z \end{matrix}\right) \tag{1.11}$$

of all $\vec{y}$-systems with these monodromy properties, plus the added property that there are no singularities of infinite order, that is to say, there exist A and r such that as z tends to a_i

$$\left|\frac{y_k}{y_k}\right| \leq \frac{A}{|z-a_i|^r} \tag{1.12}$$

holds for all i, k.

<u>Theorem (Riemann[6])</u>. For any m+1 systems $\vec{y}_j$ j = 1, . . m+1 belonging to the same class Q there exists a linear homogeneous relation with polynomial coefficients in z, the independent variable, such that

$$\sum_{j=1}^{m+1} A_j(z)\vec{y}_j(z) = 0 \tag{1.13}$$

<u>Corollary</u>. If $\vec{y}$ is an element of class Q then

$$\sum_{j=0}^{n} A_j(z)\, \vec{y}^{(j)}(z) = 0 \qquad (1.14)$$

Proof: If y is an element of class Q then so also are y', y'', as can be seen by differentiating the monodromy group equations. Thus the corollary follows directly from the theorem.

We can conclude therefore that for functions of class Q the integral approximants for large enough q_i, (p=0) are exact! This approximation procedure will, if carried to adequate order, yield the exact answer (higher orders are degenerate and essentially are equal to the exact answer). Thus it seems reasonable to study the theory of integral approximants in the context of the idea of monodromic dimension. Later on we will prove a theorem which shows that integral approximants cannot converge outside this class, in the sense that they cannot converge on more Riemann sheets simultaneously than can be accomodated by the monodromic dimension of the solution of the differential equation defining integral approximant as understood by the standard theory of differential equations.

For integral approximants there is a great variety of possible sequences of $(\vec{q}) \to \infty$ to consider in studying the convergence behavior of approximants defined by Eq. (1.1). I will first discuss "horizontal" sequences where the q_i $i=0, \ldots, m$ are all fixed and finite and p tends to ∞. Later I will discuss the diagonal sequences where all the q_i and p tend to ∞ together.

Section II The Horizontal Sequence Convergence Problem

Ordinary Padé approximants[4] are defined by

$$Q(z)f(z) - P(z) = O(z^{q-1}) \quad , \quad Q(0) = 1.00 \tag{2.1}$$

They have been proven to converge pointwise, except at poles of $f(z)$, for an appropriately selected set of degrees q of $Q(z)$ when $q \to \infty$ if $f(z)$ is meromorphic (theorem by de Montessus de Ballore)[8] in a set of nested disks about the origin. Plainly[3] this conclusion implies that sequences can be found which converge in the whole complex plane (i.e., on any compact subset) except for poles of $f(z)$ and ∞.

Are the same results also true for "horizontal" sequences of integral approximants? The answer, as we shall see, is yes, if care is used. To investigate this question we need first to define a differential multiplier.

$$\vec{Q} = (Q_0, Q_1, \ldots, Q_m) \quad . \tag{2.2}$$

is called the differential multiplier of type $\vec{m} = (q_0, \ldots, q_m)$ for $f(z)$ in $|z| < R$, if $\vec{Q}$ has degree at most $\vec{m}$ and $\vec{Q} \not\equiv 0$ is analytic in $|z| < R$. We say that $\vec{Q}$ is a unique differential multiplier type $\vec{m}$ if it is essentially unique, i.e. if any $\vec{Q}$ with these properties can be related to any other $\vec{Q}'$ by $\vec{Q} = c\vec{Q}'$ where $c \neq 0$ is a constant. $\vec{Q}$ is called pole matching if $Q_m(z) = 0$ only at poles of $f(z)$.

I now give what I call Baker-Lubinsky conditions.[3] Let f be analytic at 0 and meromorphic in $|z| < R$ $(0 < R \leq \infty)$ with ℓ distinct poles $z_1, \ldots, z_\ell$ of multiplicities $p_1, \ldots, p_\ell$ respectively. Next define

$$D = \sum_{j=1}^{\ell} D_j \quad , \quad S_1(z) = \prod_{j=1}^{\ell} (z-z_j) , \quad S(z) = \prod_{j=1}^{\ell} (z-z_j)^{D_j} \tag{2.4}$$

Let $m \geq 0$, be an integer and $a_1, \ldots, a_m \geq -1$, also be integers. Define

$$\vec{m} = (a_0, \ldots, a_m) \quad , \quad M = \sum_{j=0}^{m} (a_j+1) -1 = D + m\ell \tag{2.5}$$

By use of these conditions Baker and Lubinsky have been able to prove a number of theorems. I review here some of them.

<u>Theorem (Existence of the approximants, Baker-Lubinsky[3])</u> Assume the Baker-Lubinsky conditions hold, then there exists a differential multiplier $\vec{Q}$ of type $\vec{m}$. If in addition this multiplier is unique, then for L large enough there exists essentially unique integral approximant polynomials $P_L, \vec{Q}^{(\vec{q})}$ of type $\vec{q}$, with suitable normalization,

$$\lim_{L\to\infty} Q_j^{(\vec{q})}(z) = Q_j(z), \; j = 0,1, \ldots, m \quad , \tag{2.6}$$

$$\lim_{L\to\infty} P_L^{(\vec{q})}(z) = P(z) \quad , \tag{2.7}$$

uniformly on compact sets of $|z| < R$. P is analytic in $|z| < R$ and satisfies

$$\sum_{j=0}^{m} f^{(j)}(z)\, Q_j(z) - P(z) = 0 \quad , \; |z| < R \tag{2.8}$$

<u>Theorem (Convergence, Baker-Lubinsky[3])</u>. Assume the Baker-Lubinsky conditions hold and that the differential multiplier is unique and pole matching for f in $|z| < R$, then for L large enough $[L-a_0, \ldots, a_m]$

exists and is uniquely defined in a neighborhood of $z = 0$. It may be analytically continued to a single valued analytic function in any open simply connected set in $|z| < R$ whose closure does not contain any of the set of point $z_1, \ldots, z_\ell$. Also, uniformly on compact subsets of $|z| < R$ $\{z_i\}$,

$$\lim_{L\to\infty} [L\, c_0, \ldots, c_m] = f(z) \quad . \tag{2.9}$$

<u>Theorem (Existence of a unique, Differential Multiplier, Baker-Lubinsky[3])</u>. Assume the Baker-Lubinsky conditions hold. Then there exists a unique differential multiplier of type $\vec{m}$ for f in $|z| < R$, if

$$\vec{m} = (p, \ell-1, \ldots, \ell-1) \quad , \tag{2.10}$$

or if for some $1 \le t \le n \le m$,

$$\vec{m} = (p-1, \ell-1, \ldots, \ell-1, -1, \ldots, -1, s\ell, \ell-1, \ldots, \ell-1) \tag{2.11}$$

In Eq. (2.11) the terms $\ell-1$ are repeated $n-t$ times, the terms -1 are repeated $t-1$ times and the terms $\ell-1$ are repeated $m-n$ times.

<u>Theorem (Existence of a unique, Pole Matching, Differential Multiplier and the Rate of Convergence, Baker-Lubinsky[3])</u>. Assume the Baker-Lubinsky conditions hold, then there is a unique differential multiplier $\vec{Q}$ of type $\vec{m}$ for f in $|z| < R$ that is pole matching if for some $1 \le t \le m$

$$\vec{m} = (p-1, \ell-1, \ldots, \ell-1, -1, \ldots, -1, t\ell) \tag{2.12}$$

In Eq. (2.12) the term $\ell-1$ is repeated $m-t$ times and the term -1 is repeated $t-1$ times [Eq. (2.12) is a special case of the previous theorem where $n=m$] The differential multiplier has the form

$$\vec{q} = (Q_0, Q_1, \ldots, Q_{m-t}, 0, \ldots, 0, S_1^t) \qquad (2.13)$$

If we normalize so $Q_m^{(\vec{q})}$ is a monic polynomial, then in any compact subset K of $\mathbb{C}$

$$\limsup_{L\to\infty} \| Q_j^{(\vec{q})} - Q_j \|_K^{1/L} \le (\max_K |z_K|)\, R, \quad j = 0, \ldots, m \qquad (2.14)$$

while for the case $K \subset \{|z| < R\}$,

$$\limsup_{L\to\infty} \| P_L^{(\vec{q})}(z) - P(z) \|_K^{1/L} < \|z\|_K\, R \qquad (2.15)$$

and if K contains no poles of $f(z)$,

$$\limsup_{L\to\infty} \| [L/q_0, \ldots, q_n] - f(z) \|_K^{1/L} < \|z\|_K\, R \qquad (2.16)$$

The results of these theorems give us the same type of pointwise convergence for horizontal sequences of integral approximants (properly selected) as the de Montessus theorem gave for Padé approximants to functions meromorphic in a disk.

It is an easy corollary[3] to construct, just as it was from the horizontal convergent sequences of Padé approximants, an appropriate convergent sequence which is made up of integral approximants of $m^{\underline{th}}$ order which converges uniformly on any given compact set in the complex plane not containing a pole of $f(z)$ when $f(z)$ is a meromorphic function with no limit point of poles for any finite point. In this respect, we have not lost any ground going from Padé approximants to integral appoximants. Though, of course, integral approximants are more general than Pade approximants, it is not a foregone conclusion without proof

that the integral approximants would also work in a case where the Padé approximants are known by previous results to work.

Of course, it is not just to approximate meromorphic functions that one is interested in integral approximants, but for the approximation of functions which have branch points. We turn now to more general function classes. I next give a theorem, whose proof is only a minor variant of Riemann's original proof of his monodromy theorem.

<u>Definition</u>[9]. Given a convergent Taylor series $f(z)$ about $z = 0$ and a disk $D = \{z \mid |z| \leq R\}$ we say $f(z)$ has local monodromic dimension m if analytic continuation along all paths in D generates exactly m linearly independent coverings of D.

<u>Disk Monodromy Theorem (Baker, Oitmaa and Velgakis[9])</u>. Let $f(z)$ be a convergent Taylor series about $z = 0$ and of local monodromic dimension m in a disk $D = \{z \mid |z| \leq R\}$. Further let there be exactly $n < \infty$ singular points a_k of finite order in D, and $|a_k| < R$, $k=1, \ldots, n$. Then

$$\sum_{j=0}^{m} p_j(z)\, f^{(j)}(z) = 0 \quad . \tag{2.17}$$

where $p_m(z)$ is a polynomial of finite degree and $p_j(z)$, $j=0, \ldots, m-1$ are analytic in D.

I now sketch the proof because the ideas, although old, appear recurrently in this work.

<u>Proof (Sketch)</u>. Suppose $y_j(z)$ are the m linearly independent coverings of D generated by the functional element $f(z)$ in the neighborhood of $z = 0$.

At the singular point a_k we can introduce a change of basis

$$u_\ell(z) = \sum_{j=1}^{m} U_{k,\ell j}\, y_j(z) \quad . \tag{2.18}$$

where U_k is a constant matrix. This matrix is chosen such that

$$\Lambda_k = U_k\, M^{(k)}\, U_k^{-1} \quad . \tag{2.19}$$

where the matrix Λ_k is a diagonal matrix and the matrix $M^{(k)}$ is the monodromy matrix at the point $z = a_k$. With this change of basis we have,

$$\Lambda_k\, u_\ell = \lambda_{k;\ell}\, u_\ell \quad . \tag{2.20}$$

where the $\lambda_{k;\ell}$ are the eigenvalues of the matrix $M^{(k)}$ If the eigenvalues $\lambda_{k,\ell}$ are not degenerate, then

$$u_\ell = (z-a_k)^{\nu_{k;\ell}}\, h_{k;\ell}(z) \quad , \tag{2.21}$$

where the $h_{k;\ell}(z)$ is uniform (single valued) and

$$\nu_{k;\ell} - \frac{1}{2\pi i} \log \lambda_{k;\ell} = \text{whole number} \tag{2.22}$$

If the singularity is of the first order [r=1 in Eq. (1.12)], then $h_{k;\ell}(z) \neq 0$ or ∞ and is analytic at $z = a_k$. If $r > 1$ or the eigenvalues are degenerate the proof is more complex but follows classical lines. [It is easy to get an idea of the results which are to be expected here by thinking of a nearby case where the λ s are not degenerate and $(z-a_k)^{-r} \cong \Sigma_{j=1}^{r} \beta_j/(z-b_j)$, where b_j and β_j are chosen so that the $\Sigma \to (z-a_k)^{-r}$ as the b_j tend to a_k. The continuity of the

solution of a differential equation as a function of its coefficients away from any singularity gives then the idea of the results Of course the proof runs in the opposite direction]

Now since $\vec{u}$ and its derivatives all belong to the same monodromy group it is elementary to show that

$$u_\ell^{(j)}(z) = (z-a_k)^{\nu_{k,\ell}-j}\, n_{j,k,\ell}(z) \quad . \tag{2.23}$$

where $n_{j,k;\ell}(z)$ is uniform in the neighborhood of a_k.

Next let us consider,

$$\sum_{j=0}^{m} c_j\, y_i^{(j)}(z) = 0 \ , \ i = 1, \ldots, m \quad . \tag{2.24}$$

Cramer's rule gives the result,

$$c_{j_0} = \Delta_{j_0}(z) = \det\left|y_i^{(j)}(z)\right| \quad . \quad \begin{matrix} i = 1, \ldots, m \ ; \\ j = 0, \ldots, m, \ j \neq j_0 \end{matrix} \quad . \tag{2.25}$$

and that $\Delta_{j_0}(z) \not\equiv 0$ because, by hypothesis, the y_i are linearly independent.

Using our change of basis, we find,

$$\Delta_{j_0}(z) = \det U_k \det\left|u_\ell^{(j)}(z)\right| \quad \begin{matrix} \ell = 1, \ldots, m \ ; \\ j = 0, \ldots, m, \ j \neq j_0 \end{matrix} \tag{2.26}$$

From our representation of $u_\ell^{(j)}(z)$ we see that

$$\prod_{\ell=1}^{m} (z-a_k)^{-\nu_{k;\ell}+m}\, \Delta_{j_0}(z) \tag{2.27}$$

is analytic in the neighborhood of $z = a_k$ If we repeat the same argument as given above for each a_k then we may conclude

$$P_{j_0}(z) = \left[\prod_{k=1}^{n} \prod_{\ell=1}^{m} (z-a_k)^{-\rho_{k,\ell}-m}\right] \Delta_{j_0}(z) \qquad (2.23)$$

is analytic at $z = a_1, \ldots, a_k$ and hence in D by construction. The P's of Eq. (2.28) exist and are not identically equal to 0 in D. By standard theorems $P_m(z)$ has only a finite number of zeros in D. We may therefore factor it as $\rho_m(z)\,Q(z)$ where $\rho_m(z) \neq 0$ for $|z| > R$ and $Q(z) \neq 0$ for $|z| \leq R$. The division by $Q(z)$ completes the proof.

Remark. Once $\rho_m(z)$ is fixed, the linear independence of the y_i fix the other ρ_j uniquely. If fewer than m y's were independent the same argument as given above would lead to an equation of lower order which could be added to the one we have just arrived and thereby destroy uniqueness.

To make further progress with the theory of horizontal sequences of integral approximants, it is useful to note the following key property.

<u>Separation Property (Baker, Oitmaa, and Velgakis[9])</u>. If a function $f(z)$, possibly multiform, can be written as $f(z) = f_1(z) - f_0(z)$, where for a disk $D = \{z \mid |z| \leq R\}$, $f_0(z)$ is analytic for all $z \in D$ and every analytic continuation of $f_1(z)$ is analytic for all z in the finite complex plane outside D, then $f(z)$ has the separation property with respect to D.

The simplest example would be the class of meromorphic functions, with respect to an appropriate sequence of R's. We have already discussed the theory at length in this case.

<u>Separation Property Theorem (Baker, Oitmaa and Velgakis[9])</u>. If $f(z)$ has the separation property with respect to a disk D, a finite number of

singular points a_i, $|a_i| > 0$ for all i, in the interior of D and none on the boundary of D. Assume further that all these singular points plus the point at ∞ for $f_i(z)$ are of finite order. Let $f_i(z)$ be of exact monodromic dimension m. Then, (i) there exists an essentially unique differential multiplier for disk D whose coefficients are polynomials of degrees $q_0, \ldots, q_m$ (q_m is chosen as the minimum possible) (ii) the integral approximants $[L/q_0, \ldots, q_m]$ converges as $L \to \infty$ on compact subsets of $D - \{a_k\}$ to $f(z)$.

Remarks: By Riemann's monodromy theorem[6] there exists A_j polynomials such that

$$\sum_{j=0}^{m} A_j(z)\, f_i^{(j)}(z) = 0 \tag{2.29}$$

If we add $f_0(z)$ to f_i we get

$$\sum_{j=0}^{m} A_j(z)\, f^{(j)}(z) = \phi(z) \quad . \tag{2.30}$$

where $\phi(z)$ is analytic in D. The existence of an essentially unique $A_m(z)$ of minimum degree follows from a proof by contradiction. The convergence part follows by arguments of standard type.

For that class of functions with the separation property, we see by this theory that the de Montessus[8] type theorem just given assures us that the integral approximants converge in a pointwise manner much as would have been expected from the analogy with corresponding results for the Padé approximants to the meromorphic function class. In fact it is, I think, this class of functions which is the correct analogy to meromorphic functions for the theory of horizontal sequences of integral approximants. This class, as we saw, is equivalent to f such that

$$\sum_{j=0}^{m} p_j(z)\, f^{(j)}(z) = \phi(z) \quad . \tag{2.31}$$

where the p_j are polynomials and $\phi(z)$ is analytic in the disk D. The general class corresponding to f(z) of local monodromic dimension m plus a uniform analytic background would have $p_m(z)$ a polynomial and $p_j(z)$, $j = 0, \ldots, m-1$ and $\phi(z)$ analytic in D.

To illustrate the case when one does not have the separation property, I look at the simplest possible, non-trivial case. Suppose f(z) is regular in a disk $|z| \leq \rho$, $\rho > 1$ except for a regular singular point at $z = 1$ and further that f(z) in $|z| \leq \rho$ consists of a uniform background plus an m = 1 piece. Then, further simplifying to a first order singularity,

$$(1-z)\, f'(z) + G(z)\, f(z) = H(z) \quad . \tag{2.32}$$

where G and H are analytic in $|z| \leq \rho$. The solution for f(z), assuming further that $G(z) = z\, G_1(1-z)^{\gamma}$ and that G_0 is not an integer ($\gamma = -G_0$), is

$$f(z)\, A(z)(1-z)^{-\gamma} + B(z) \quad . \tag{2.33}$$

where A and B are analytic in $|z| \leq \rho$.

Let us use the following notation for the [L,M-1,1] approximant,

$$[\lambda^{(L)}(1-z) + \alpha^{(L)}]\, f'(z) + g^{(L)}(z)\, f(z) = h^{(L)}(z) + O(z^{L+M-1}) \tag{2.34}$$

where $g^{(L)}(z)$ is a polynomial of degree M-1 and $h^{(L)}(z)$ is a polynomial of degree L.

<u>Theorem (Baker and Graves-Morris[10])</u>. Under the above hypotheses, for L sufficiently large we have, normalizing $\lambda^{(L)} = 1$,

$$\alpha^{(L)} = O(L^{-M-1}) \quad .$$

$$g_i^{(L)} = G_i + O(L^{-M+i}) \quad i = 0,1, \ldots, M-1 \quad . \tag{2.35}$$

therefore,

$$g^{(L)}(z) \underset{L\to\infty}{\to} \sum_{i=0}^{-M+1} G_i(1-z)^i \quad .$$

$$h^{(L)}(z) \underset{L\to\infty}{\to} H(z) - \{G(z) - \sum_{i=0}^{M-1}(1-z)^i G_i\} f(z) \tag{2.36}$$

Remark: The estimates in this theorem show that we get a good representation of $G(z)$ near the singular point $z=1$ but however the polynomial $h^{(L)}(z)$ has a limit function defined by a series which diverges for $|z| > 1$. These results suffice to establish:

<u>Theorem (Baker and Graves-Morris[10])</u>. Under the hypotheses of the previous theorem,

$$\lim_{L\to\infty} [L/M;1] = f(z) \quad , \ |z| < 1 \quad . \tag{2.37}$$

on all Riemann sheets accessible in the disk $|z| \leq \rho$. Further for L large enough

$$\gamma^{(L)} = \gamma + O(L^{-m}) \tag{2.38}$$

Note: The $[L/M;1]$ on the second, Riemann sheets are defined by integrating the approximant around $z = 1$.

Remark: These sequences of approximants are useful in analyzing the closest singularity to the origin, even without the separability condition. For farther singularities from the origin the situation is not proven.

Further results for the case of higher monodromic dimension are known.

Section III DIAGONAL SEQUENCES

For Padé approximants the "diagonal" sequences $[L,M]$, $M \to \infty$, $L/M \to 1$ are much more powerful methods[4] of approximate analytic continuation than "horizontal" type sequences. Unfortunately the convergence theorems are harder to prove and generally for a weaker form of convergence than the pointwise convergence that we have obtained for the "horizontal" sequences. The same situation, so far at least, seems to hold for integral approximants and the problem, as we saw for horizontal sequences, is more complex.

As with the Padé approximants we need global information on the analyticity properties[11] to obtain convergence theorems (so far) The results that follow on diagonal sequences of integral approximants are generalizations of the results of Stahl[12] for Padé approximants

<u>Theorem (limit on the domain of convergence)</u>. A sequence of mth order integral approximants $[p/q_0, \ldots, q_m]$ can not converge simultaneously on $m+2$ coverings of the complex plane to an $f(z)$ which has a structure of a uniform function (may be zero) plus a part with a monodromic dimension greater than m.

<u>Proof</u>: On any Riemann sheet we may write

$$\sum_{j=0}^{m} Q_j^{(\vec{q})}(z)\, f^{(j)}(z^{(i)}) + p^{(\vec{q})}(z) = R^{(\vec{q})}(z^{(i)}) \qquad (3.1)$$

where $\pi(z^{(i)}) = z$ is the projection on the complex plane from the Riemann surface and $R^{(\vec{q})}(z^{(i)})$ is the remainder. If we solve this equation using $m-2$ coverings $f(z^{(i)})$, $i = 1, \ldots, m-2$ for $Q_m^{(\vec{q})}$ in terms of $R^{(\vec{q})}(z^{(i)})$ we get

$$\det \begin{vmatrix} f^{(m)}(z^{(1)}) & f(z^{(1)}) & 1 \\ \vdots & \vdots & \\ f^{(m)}(z^{(m-2)}) & z^{(m-2)}) & 1 \end{vmatrix} = \det \begin{vmatrix} R^{(\vec{q})}(z^{(1)}) & f^{(m-1)} & 1 \\ R^{(\vec{q})}(z^{(m-2)}) & f^{(m-1)} & 1 \end{vmatrix} \Big/ Q_m^{(\vec{q})}(z) \tag{3.2}$$

for example. Solutions for other subsets than m are also possible and are of similar form. If we expand the determinant on the left hand side of Eq. (3.2) along the last row we obtain that it is equal to

$$\det \begin{vmatrix} g^{(m)}(z^{(1)}) & \cdots & g(z^{(1)}) \\ \vdots & & \vdots \\ g^{(m)}(z^{(m-1)}) & \cdots & g(z^{(m-1)}) \end{vmatrix} . \tag{3.3}$$

where we define,

$$g(z^{(i)}) = f(z^{(i+1)}) - f(z^{(i)}) \tag{3.4}$$

Note that the determinant in Eq. (3.3) is independent of $(\vec{q})$ and not identically equal to 0 by our hypothesis. Therefore we can't have $R^{(\vec{q})}(z^{(i)})$, $i = 1, \ldots, m-2$, all vanish simultaneously and hence the conclusion of the theorem follows.

In order to go beyond Riemann's monodromy theorem let us look at a class of functions defined by

$$\sum_{j=0}^{m} E_j(z)\, f^{(j)}(z) + E_{-1}(z) = 0 \quad . \tag{3.5}$$

where the E_j are entire functions. In addition to Eq. (3.5) we need a further assumption in order to ensure that the f(z) so defined are of the full monodromic dimension assumed so that the equation is not reduceable to lower order. In addition to the further assumption that we shall give we also explicitly assume that not all the E_j's are polynomials. We further assume that f(z) is analytic at z = 0 on all Riemann sheets for ease of exposition. In order to ensure irreducibility we assume that starting from the functional element f(z) at z = 0 we may select m + 1 connected sheets on which f(z) is a uniform function plus a function of monodromic degree exactly m. Finally, I assume, that the closure of this domain never produces more than m + 2 coverings of any point of the complex plane.

<u>Theorem (convergence)</u>. Let f(z) belong to the above defined class Then the near-to-diagonal approximants $[p/q_0; \ldots; q_m]$ converge to f(z) in capacity on compact subsets of the complex plane excluding the singular points of f(z) (at least a subsequence).

<u>Proof (sketch)</u>. To make the parallelism to the proof of Stahl easier to follow we will use the expansion about z = ∞ and take the exact diagonal sequences [n/n; ..., n]. This latter is not an essential simplification. The Padé-Hermite equations now are

$$\sum_{j=0}^{m} \hat{Q}_j^{(\vec{q})}(z)\, f^{(j)}(z) + \hat{P}^{(\vec{q})}(z) = \hat{R}^{(\vec{q})}(z) = O(z^{-(m+1)n-1}) \tag{3.6}$$

where

$$\hat{Q}_j^{(\vec{q})}(z) = z^n Q_j^{(\vec{q})}(z^{-1}), \quad \hat{P}^{(\vec{q})}(z) = z^n P^{(\vec{q})}(z^{-1}) \qquad (3.7)$$

The Q and P polynomials are defined in a manner analogous to Eq. (1.1) and we have multiplied them by z^n to give the $\hat{Q}$, $\hat{P}$ as polynomials. Since these polynomials can be multiplied by a nonzero constant we can write

$$\frac{1}{n} \log \max_{j=0,m} \left(|\hat{Q}_j^{(\vec{c})}(z)|, |\hat{P}^{(\vec{q})}(z)| \right) = p(z, \mu_{\vec{q}}), \quad \|\mu_{\vec{q}}\| \leq 1 \quad . \qquad (3.8)$$

where p is the logarithmic potential of a certain measure $\mu_{\vec{q}}$ which is positive because the maximum of subharmonic functions is again subharmonic.[13] By the weak compactness of the unit ball in the space of positive measures, there exists a subsequence $\vec{q}_1 \subseteq \vec{q}$ with the property

$$\lim_{\vec{q}_1} \mu_{\vec{q}_1} = \mu_0 \qquad (3.9)$$

This in turn implies

$$\overline{\lim}_{\vec{q}_1} p(z, \mu_{\vec{q}_1}) = p(z, \mu_0) \qquad (3.10)$$

quasi- everywhere[13] on C

Consider any domain $A \subseteq K$ which has a Green's function $g(z,w,A)$, $z,w \in A$. For a measure ν in A define the Green's potential

$$g(z,\nu) = -\int g(z,w;A)\, d\nu(w) \quad , \; w \in R \quad , \qquad (3.11)$$

on $R \sim A$, $g(z,\nu) \equiv 0$ by the properties of the Green's function. If necessary select another subsequence $\vec{q}_2 \subseteq \vec{q}_1$ we then get.

$$\overline{\lim_{\vec{a}_2}} \frac{1}{n} \log \max_{j=0,m} (|Q_j^{(\vec{a})}|, \ldots, |P^{(\vec{a})}|)$$

$$= g_0(z) = g_0(z, \nu_0) + h_0(z) \tag{3.12}$$

$$\overline{\lim_{\vec{a}_2}} \frac{1}{n} \log (|R^{(\vec{a})}|) = g_1(z) = g_1(z, \nu_0) + h_1(z) \tag{3.13}$$

where h_0 and h_1 are harmonic in $A \subseteq R$. ν_0 is just μ_0 lifted onto A. The g's are independent of A in the sense that they agree for A_1 and A_2 in $A_1 \cap A_2$. The h's depend on A.

<u>Lemma 1</u>. With the sets $I_0 = \{\infty^{(1)}\}$, $I_1 = \pi^{-1}(\infty)$ and $I_2 = I_1 \sim I_0$ we have

(i) $\nu_1(I_0) \geq m + 1$

(ii) $\nu_1(B) \geq 0$ for all Borel set $B \subseteq R \sim I_2$

(iii) If $A \subseteq R \sim I_1$ is a domain (connected!) in which ν is $(m+1)$-valent, then

(a) $0 \leq \nu_0(A) \leq (m+1)|\nu_0(I_0)| \leq m + 1$

(b) $0 \leq \nu_0(\bar{A}) \leq (m+2)\,|\nu_0(I_0)| - \nu_0(A)$

Result (i) follows by definition, the result (ii) follows because R is of the form of a sum of polynomials times derivatives of f and it can only $\to \infty$ outside the set I_2 at those points where f does but this effect is washed out as $n \to \infty$ by the definition of the measure. Result (iii) follows because polynomials have the same number of 0's as poles by Gauss's theorem and there are at most $m + 1$ copies on the surface of A. Finally (b) follows because the boundary under closure produces by hypothesis at most one extra copy and so adds $|\nu_0(I_0)| - \nu_0(A)$ at most

<u>Lemma 3</u>. If the domain A satisfies the assumptions of Lemma 1. (iii) and

$$\nu_0(\vec{a}) = (m-2)\,|\nu_0(i_0)| \quad . \tag{3.14}$$

then $\overline{\pi(a)} = \mathbb{C}$.

<u>Proof</u> Exactly parallel to Stahl's.[12]

Now define

$$d(z) = g_\infty(z) - g_0(z) \quad , \; z \in R \tag{3.15}$$

Note that by definition

$$d(t) \le \lim_{n\to\infty} \frac{1}{n} \log \left\{ \sum_{j=0}^{n} |f^{(j)}| - 1 \right\} = 0 \quad . \tag{3.16}$$

except at singular points of f.

Define the region

$$\tilde{D} = \{z \in R: d(z) < 0\} \tag{3.17}$$

The significance of this definition can be seen by noting the following result

$$\frac{|R|}{\max\left\{\left|Q_j^{(\vec{a})}\right|, P^{(\vec{a})}\right\}} \sim e^{nd(z)} \underset{n\to\infty}{\to} 0 \quad . \tag{3.18}$$

if $d(z) < 0$. As by the Hermite-Padé equation $d(\infty^{(1)}) = -\infty$ we see at once that

$$\infty^{(1)} \in \tilde{D} \tag{3.19}$$

Let us define D to be the connected component of $\tilde{D}$ which contains $\infty^{(1)}$. Since D may have an infinite number of Riemann surfaces we need to prune it before we can complete the proof. First consider $\pi(D) \subseteq D$ we can by hypothesis construct an $m + 1$ sheeted covering of D on which $f(z)$ is represented as a uniform function plus m linearly independent functions. Call this domain $B \subseteq D \subseteq R$. Since $d < 0$ in D and so also in B we use the potential theory notion of flux to get a key inequality

$$\nu_1(\bar{B}) - \nu_0(\bar{B}) \leq 0 \qquad (3.20)$$

As for d, the potential theory sources are the zeros in R that is the positive part of ν_1 and the poles in $\mathrm{Max}(|\hat{Q}_j|, |\hat{P}|)$ the negative parts of ν_0. Likewise the potential theory sinks are the poles in R, i.e., the negative part of ν_1 and the zeros in $\mathrm{Max}(|\hat{Q}_j|, |\hat{P}|)$ the positive part of ν_0. We may re-express Eq. (3.20) as

$$\nu_1(I_0) - |\nu_0(I_0)| - \nu_1(\bar{B} \sim I_0) < \nu_0(\bar{B} \sim I_0)$$
$$\leq (m+2)\,|\nu_0(I_0)| - \nu_0(B \sim I_0) \qquad (3.21)$$

By lemma 1 and our hypothesis on $f(z)$ we get, by a little arithmetic

(a) $\nu_1(I_0) = m + 1$.

(b) $\nu_1(B \sim I_0) = 0$.

(c) $\nu_0(I_0) = -1$.

(d) $\nu_0(\partial B) = (m + 2)|\nu_0(I_0)| = m + 2$.

(e) $\nu_0(B \sim I_0) = 0$

These results show that the zeros cluster on the boundary of the region $\pi(B) = D$ and so by now standard arguments the exceptional set in D is of capacity zero. Property (a) shows that the block size in the Hermite-Padé table tends to ∞ (relative to n).

By a variant of the proof of Riemann or the proof of the disk monodromy theorem, sketched herein we have the result that the $f(z^{(i)})$, $i = 1, \ldots, m-1$ uniquely determine

$$\frac{E_j(z)}{E_m(z)}, \quad j = -1, 0, \ldots, m-1 \quad . \tag{3.22}$$

as uniform functions of z. Hence adapting Stahl's proof of the extremal nature of $\pi(D) = D$ we are able to conclude its uniqueness in terms of the uniquely determined E_j/E_m ratios. This uniqueness of $\pi(D) = D$ allows the conclusions to be extended from a subsequence to any subsequence, again by arguments parallel to Stahl's, except we require $\tilde{Q}_m(z)$ to be of full degree in order that we may solve for the approximant

The knowledge of the global behavior of $f(z)$ has allowed the proof of convergence for diagonal sequences without the separation property, or the limitation to the nearest singularity, required for horizontal sequences. I do not think, using the knowledge we have of Padé approximants, that the assumptions that I have had to make in the results reported here by any means exhaust the range of convergence of the integral approximants.

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