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TITLE BARYON AND LEPTON VIOLATION IN THE WEINBERG-SALAM THEORY

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BARYON AND LEPTON NUMBER VIOLATION IN THE WEINBERG-SALAM THEORY

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1. Introduction

Baryon and lepton number (B and L) conservation is a striking general feature of nature. Whatever the complications of the interaction, whether its rate is characteristic of a strong, weak, or electromagnetic process, and whatever other quantities are not conserved (isospin, parity, CP) all experiments to date are consistent with conservation of baryon and lepton number.

Notwithstanding this complete lack of experimental evidence for nature violating B and L, their absolute conservation has always been viewed suspiciously by theorists. Before the advent of the Weinberg-Salam theory unifying the weak and electromagnetic interactions, there were two basic reasons for this suspicion. First of all, unlike electric charge which we believe is absolutely conserved, baryon and lepton number are not coupled to a local gauge field. It is this gauge invariant coupling which guarantees electic charge conservation, and gives rise to a long range interaction (Coulomb's law), corresponding to a strictly massless photon in electromagnetism. There is no evidence for such a long range interaction between (electrically neutral) baryons or leptons. Hence, B and L conservation seem rather "accidental" and not protected by a deeper gauge principle, as electric charge conservation is.

The second reason for scepticism about exact conservation is an obvious observational fact about the universe. Everywhere you look there are baryons and leptons, but scarcely any antibaryons or antileptons: the universe is completely asymmetric in preferring baryons over antibaryons and leptons over antileptons. If, as quantum theory tells us, there is complete symmetry between particles and antiparticles in their fundamental interactions, and B and L are exactly conserved, we have no way whatsoever of understanding this extraordinary asymmetry of the universe in the large. It would simply have to be postulated as an initial condition of the big bang, precluding any dynamical explanation.

In the Weinberg-Salam electroweak theory these misgivings are subtantiated in that baryon and lepton number are NOT exactly conserved. The nonconservation of B and L can be traced to the existence of parity violation in the electroweak theory, together with the chiral current anomaly, which is where we should begin.

For simplicity let us first consider a familiar Abelian gauge field theory, spinor electrodynamics. In addition to the ordinary electromagnetic current j^μ one may also consider the chiral current,

$$j_5^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi,$$

where $\gamma_5 = \pm 1$, depending on whether the fermion spin is aligned parallel or antiparallel to its momentum. Taking the divergence of j_5^μ, and using the Dirac equation, one finds that

$$\partial_\mu j_5^\mu = 2m\overline{\psi}\gamma_5\psi, \qquad (classically). \tag{1.1}$$

The fact that the divergence is proportional to the mass of the fermion suggests that in the limit of zero mass the chiral current would be conserved, and this is certainly the case classically. However, in the full quantum theory the Dirac field becomes an operator, and the product of two such operators at the same spacetime point is generally singular. Hence, the right hand side of (1.1) must be examined carefully in the limit $m \to 0$ to be certain that there are no singular contributions in this limit. In fact there is a singular contribution to the operator product $\overline{\psi}\gamma_5\psi$ coming precisely from one graph in one loop perturbation theory. This is the famous triangle graph of Fig. 1, and the contribution to the right side of (1.1) is such that even in the limit $m \to 0$, a well defined finite term remains. This is the ABJ triangle, or chiral current anomaly[1]:

$$\partial_\mu j_5^\mu = 2m\overline{\psi}\gamma_5\psi + \frac{g^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}, \tag{1.2}$$

where the singular terms have now been removed from the operator product $\overline{\psi}\gamma_5\psi$.

By now this "anomaly" is well understood theoretically. It has even been verified experimentally in the decay of the π^0 meson into two photons, in the sense that the large decay rate observed experimentally requires the finite contribution of the triangle diagram of Fig. 1 (where the fermion lines represent the quarks in the pion), in the chiral limit, $m \to 0$. In the non-Abelian case there is also a chiral anomaly so that the chiral currents of the Weinberg-Salam theory have gauge invariant anomalous divergences analogous to (1.2). The reason that this has anything at all to do with B and L nonconservation is that the gauge fields of the electroweak theory couple asymmetrically to the fermions of the theory. In particular, the $SU(2)$ gauge field couples only to left-handed quark and lepton doublets. Writing the B and L currents as sums of left and right handed components, and taking account of the anomalies in the chiral currents leads to the following result for their divergence:

$$\partial_\mu b^\mu = \partial_\mu \ell^\mu = \frac{N}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\left[-g_2^2 G^a_{\mu\nu}G^a_{\alpha\beta} + g_1^2 F_{\mu\nu}F_{\alpha\beta}\right] \tag{1.3}$$

where $G^a_{\mu\nu}$ and $F_{\mu\nu}$ are the field strength tensors for the $SU(2)$ and $U(1)$ hypercharge gauge fields of the Weinberg-Salam theory, g_2 and g_1 are the corresponding gauge coupling constants, and N is the number of sequential generations of quarks and leptons.

Since the total baryon or lepton number is the three-space integral of the fourth component of the respective local current, b^μ or ℓ^μ, we immediately conclude that B and L are not conserved in the standard electroweak theory, although the quantity B - L is conserved. Since we started with the observation that there is no evidence for any B or L violation in nature, we must ask how big the violation predicted by eq. (1.3) is, and is it in contradiction to experiment? This requires a detailed understanding of the anomalous terms appearing on the right side of eq. (1.3), particularly their relationship to the nontrivial topological vacuum structure of non-Abelian gauge theories[2]. These lectures are intended to be pedagogical, so we will consider the anologs of nontrivial vacuum structure in some simpler models, and return to the actual Weinberg-Salam theory and the anomaly (1.3) only after understanding these simpler models in some detail.

2. Periodic Vacua in Quantum Mechanics: The Simple Pendulum Model

By far the simplest model of a quantum mechanical system which exhibits a periodic ground state ("vacuum") is the simple pendulum. The Lagrangian is:

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta). \tag{2.1}$$

Defining

$$\begin{aligned} \eta &\equiv \frac{\theta}{2}; \\ \alpha &\equiv \frac{\frac{1}{2}\hbar\omega}{2mg\ell} << 1; \\ \omega^2 &\equiv \frac{g}{\ell} \end{aligned} \tag{2.2}$$

and

$$p_\eta \equiv \frac{\partial L}{\partial \eta} = \frac{\hbar\dot{\eta}}{\alpha\omega} \tag{2.3}$$

leads to the Hamiltonian,

$$H = \frac{\alpha\omega}{2\hbar}p_\eta^2 + \frac{\hbar\omega}{2\alpha}\sin^2\eta. \tag{2.4}$$

Upon making the canonical replacement,

$$p_\eta \to -i\hbar\frac{d}{d\eta}$$

we obtain the Schrödinger equation for the simple pendulum:

$$\frac{\hbar\omega}{2}\left(-\alpha\frac{d^2}{d\eta^2} + \frac{1}{\alpha}\sin^2\eta\right)\psi_n(\eta) = E_n\psi_n(\eta). \tag{2.5}$$

Since the potential is periodic in η the exact wave function solutions to (2.5) will be as well. However, perturbation theory corresponds to expanding about a *single minimum* of the potential:

$$\sin^2\eta = \eta^2 - \frac{\eta^4}{3} + \dots \tag{2.6}$$

Keeping only the first term in this expansion yields an harmonic oscillator potential with zeroth order Gaussian wavefunctions,

$$\psi_n^{(0)} \sim H_n(\eta) e^{-\eta^2/2\alpha} \tag{2.7}$$

peaked around one minimum of the potential (at $\eta = 0$) and containing no information about the periodic structure of the potential. This is contained in the higher orders of (2.6) which have been neglected in this lowest order result. The Gaussian approximation to the energy eigenvalues,

$$E_n^{(0)} = \hbar\omega(n + \frac{1}{2}) \tag{2.7}$$

is valid only in the limit that this energy is small compared to the height of the potential barrier between neighboring minima, *i. e.* if and only if

$$\alpha n << 1. \tag{2.8}$$

The quantity α has the dual role of controlling the validity of the perturbation expansion of the potential (2.6) about the Gaussian wavefunctions (2.7), and of giving the ratio between the quantum zero point energy to the height of the classical barrier between neighboring minima of the potential. Even for very small (but finite) α the perturbative expansion eventually breaks down for harmonic oscillator occupation numbers $n \sim \frac{1}{\alpha}$. The analog of this statement will be very important in the non-Abelian gauge theory.

In order to recover information about the periodic structure of the pendulum potential, we need either to consider very high orders of perturation theory (again of order $\frac{1}{\alpha}$), or treat the system in a different approximation scheme: the semiclassical limit. This corresponds to the WKB approximation to the Schrödinger equation (2.5), so that tunnelling between the minima of the potential can occur, and the periodic nature of the potential may be taken into account. The first step in this approach is to look for nontrivial solutions of the classical equations of motion in *imaginary* time,

$$t = -i\tau. \tag{2.9}$$

The constant of the motion corresponding to the energy in imaginary time is:

$$\epsilon = \frac{\hbar}{2\alpha\omega}\left(\frac{d\eta}{d\tau}\right)^2 - \frac{\hbar\omega}{2\alpha}\sin^2\eta. \tag{2.10}$$

This is equivalent to changing the sign of the potential term in the equations of motion.

At zero temperature the pendulum is in its ground state, so we look for a solution with $\epsilon = 0$ that interpolates between neighboring minima of the periodic potential (maxima of the inverted potential):

$$\begin{aligned} \eta &\rightarrow 0 \quad as \quad \tau \rightarrow -\infty; \\ \eta &\rightarrow \pi \quad as \quad \tau \rightarrow +\infty. \end{aligned} \tag{2.11}$$

Such a solution (the "instanton") is easily found by integrating (2.10) with $\epsilon = 0$:

$$\sin \overline{\eta}(\tau) = \mathrm{sech}[\omega(\tau - \tau_0)]. \tag{2.12}$$

The classical Euclidean action for this solution S is given by:

$$\frac{S}{\hbar} = \frac{1}{2\alpha} \int_{-\infty}^{+\infty} \left[\frac{1}{\omega} \left(\frac{d\overline{\eta}}{d\tau} \right)^2 + \omega \sin^2 \overline{\eta} \right] d\tau = \frac{2}{\alpha}. \tag{2.13}$$

The action for the excursion from $\eta = 0$ to $\eta = \pi$ plus that of the return trip (the "bounce") is twice this or $\frac{4}{\alpha}$.

According to the usual semiclassical analysis of the Feynman path integral, the tunnelling rate from one minimum to the neighboring one is proprortional to the fundamental frequency of oscillation, ω and the exponential of the negative of this bounce action,

$$\Gamma \sim \omega e^{-\frac{2S}{\hbar}} = \omega e^{-\frac{4}{\alpha}} <<< \omega. \tag{2.14}$$

For a macroscopic pendulum, $\alpha \sim 10^{-34}$, this is an incredibly small tunnelling rate. Even when α becomes of order 10^{-2}, corresponding to the fine structure constant which enters the Weinberg-Salam model, the rate is still negligibly small. We conclude that perturbation theory, which neglects the periodic features of the potential is an excellent approximation for a weakly coupled theory at zero temperature, when the system is in its ground state. Furthermore as long as the thermal energy, kT is small compared to the excitation energy of the quanta of the system, $\hbar\omega$, so that the higher excited states of the oscillator are hardly excited, it is clear that the above picture remains valid. However, it is equally clear that when the temperature becomes large enough so that the pendulum has enough thermal energy to completely surmount the potential barrier at $\eta = \frac{\pi}{2}$, it need not wait for the very rare quantum tunnelling event with the rate we have just estimated. Instead, the purely *classically allowed* process of jumping over the barrier will have a vastly larger, unsuppressed rate. This is what we wish to estimate next.

3. Tunnelling at Finite Temperature and Classical Thermal Activation

The finite temperature transition rate may be calculated by an extension of the semi-classical method used to arrive at (2.14). We simply look for Euclidean solutions not with $\epsilon = 0$ or the boundary conditions (2.11), but with finite Euclidean periodicity, $\beta \equiv \frac{\hbar}{kT}$,

$$\eta(\tau) = \eta(\tau + \beta). \tag{3.1}$$

For the simple pendulum problem, such solutions ("finite temperature instantons," or "calorons") may be found explicitly in terms of elliptic functions. However, all the information we shall need is contained in the Euclidean action, given by the same integrand as (2.13) above, but taken over the fundamental period of the solution, $\tau = 0$ to $\tau = \beta$. Substituting the definition of ϵ, eq. (2.10), and rewriting the integral over τ as an integral over η gives:

$$S = -\frac{\epsilon\beta}{2} + \frac{\hbar}{\alpha\omega}\int_{\eta_{min}}^{\eta_{max}} d\eta \left(\frac{2\alpha\epsilon\omega}{\hbar} + \omega^2 \sin^2\eta\right)^{\frac{1}{2}} \tag{3.2}$$

Differentiating this expression with respect to β yields:

$$\frac{\partial S}{\partial \beta} = -\frac{\epsilon(\beta)}{2} > 0, \tag{3.3}$$

since $\epsilon < 0$. Thus, viewed as a function of temperature, S is a *decreasing* function as the temperature is raised. This means that the exponential, $e^{-\frac{2S}{\hbar}}$ and the rate Γ are an increasing functions of temperature: the thermal population of the higher excited modes of the oscillator at $\eta = 0$ makes it easier and easier to get over the barrier to the vicinity of the neighboring ground state ("vacuum").

As the temperature is increased, (β decreased) the turning points of the classical motion in (3.2), η_{min} and η_{max} approach each other at the midpoint $\eta = \frac{\pi}{2}$. The turning points coincide when $\beta = \frac{2\pi}{\omega}$, the period of the harmonic oscillator motion in the inverted potential at $\eta = \frac{\pi}{2}$. At this value of β, the second term in (3.2) vanishes and $-\frac{2S}{\hbar}$ becomes equal to

$$\frac{\epsilon\beta}{\hbar} = -\frac{\omega}{2\alpha}\beta = -\frac{\pi}{\alpha}. \tag{3.4}$$

Thus, the transition rate estimate of (2.14) becomes

$$\Gamma \sim \omega e^{-\frac{2S}{\hbar}} = \frac{kT}{2\pi\hbar}e^{-\frac{\pi}{\alpha}} = \frac{kT}{2\pi\hbar}e^{-V_{max}/kT} \tag{3.5}$$

at this temperature.

Although the exponent in Γ has only changed from $-\frac{4}{\alpha}$ at zero temperature to $-\frac{\pi}{\alpha}$ at $T = \frac{\hbar\omega}{2\pi k}$, the significance of the finite temperature instanton having collapsed to a single point, $\overline{\eta} = \frac{\pi}{2}$ is that at this temperature (still much lower than the barrier height) the pendulum does not need to quantum mechanically tunnel to the neighboring minimum of the potential. Rather it may *jump over the barrier* by (classically) receiving a large enough kick from the bath of thermal fluctuations into which it is immersed. At all higher temperatures this classical activation transition dominates over quantum tunnelling. In the last form of (3.5) it is easy to see what happens as the temperature is raised still further: the Boltzmann suppression factor, $e^{-V_{max}/kT}$ simply grows larger and larger until it eventually becomes of order unity, at $kT \sim V_{max}$. Then there is no suppression whatsoever, as the pendulum now has so much thermal energy on average that it swings freely over the potential barrier. Thus V_{max} is a critical energy scale which may be identified with the energy of the degenerate *static* bounce solution of the classical equations, namely the trivial solution $\bar{\eta} = \frac{\pi}{2}$. This unstable, finite energy static solution is called a "sphaleron" for the simple pendulum model.

Before leaving this instructional example of the simple pendulum for more realistic field theories, consider the quantity

$$Q = \frac{1}{2}\int d\tau \left(\frac{d\eta}{d\tau}\right) \sin\eta = \frac{1}{2}\int d\eta \sin\eta, \tag{3.6}$$

called the winding number. If the limits on the latter integral are 0 to π, $Q = 1$, corresponding to the pendulum winding once about its pivot and returning to its ground state configuration. By considering the manifestly non-negative Euclidean integrals,

$$\int_0^\beta d\tau [\frac{d\eta}{d\tau} \pm \omega \sin\eta]^2 \geq 0$$

it easy to prove that

$$\frac{2S}{\hbar} \geq \frac{4|Q|}{\alpha}, \tag{3.7}$$

where the limits on the integral for Q are η_{min} and η_{max}. If we somehow blundered into forgetting these limits, and continued to regard Q as an integer winding number, we would quickly conclude that the Euclidean action is bounded from below by the zero temperature instanton action, and that the transition rate is always bounded from above by (2.14), even at finite temperature. Of course, this conclusion would be completely incorrect, as physical intuition and the finite temperature semiclassical methods sketched above make quite clear.

The point is that even though $Q = 1$ for a classical transition over the barrier to the neighboring minimum, most or all of this transition may be accomplished in *real* time without any quantum merchanical tunnelling. Thus, the bound on the *imaginary* time Euclidean action, (3.7) plays no role whatsoever in the correct estimate of the classical transition rate, or equivalently, the Euclidean Q appearing in (3.7) may be arbitrarly small or zero, for a finite temperature transition, and the bound loses its force.

4. Calculation of the Rate: General Theory

Having reviewed tunnelling at zero and finite temperature in a simple quantum mechanical model with only one degree of freedom, we turn now to the general path integral method of analyzing the decay rate of an unstable phase at finite temperature in field theory, once the static sphaleron solution has been found[3]. The power of this method is that it does not depend on the details of the potential, as we shall see shortly.

To illustrate the general method consider a single scalar field in $d+1$ dimensions with action

$$S[\Phi] = \int_0^\beta d\tau \int d^d x \Big[\frac{1}{2}\big(\frac{\partial \Phi}{\partial \tau}\big)^2 + \frac{1}{2}(\nabla\Phi)^2 + \mathcal{U}(\Phi)\Big]. \tag{4.1}$$

We set $\hbar = 1$ for notational simplicity where it causes no confusion to do so. Let $\Phi = \phi(x)$ be a static (sphaleron) solution of the equation,

$$-\nabla^2\phi + \frac{\partial \mathcal{U}}{\partial \phi} = 0 \tag{4.2}$$

and expand S to second order in $\Phi - \phi$. The Gaussian fluctuation operator is

$$\mathcal{G} = -\frac{\partial^2}{\partial \tau^2} - \nabla^2 + V(x), \quad V(x) = \frac{\partial^2 \mathcal{U}}{\partial \Phi^2}\Big|_{\Phi=\phi(x)}. \tag{4.3}$$

The eigenfunctions of this operator have the general form, $e^{2\pi i n\tau/\beta}\psi_p(x)$ and

$$H\psi_p = \big[-\nabla^2 + V(x)\big]\psi_p(x) = \epsilon_p^2\psi_p(x). \tag{4.4}$$

The corresponding eigenvalues are:

$$\big(\frac{2\pi n}{\beta}\big)^2 + \epsilon_p^2. \tag{4.5}$$

Now the path integral expression for the partition function is:

$$Z = e^{-\mathcal{F}/kT} = \int [\mathcal{D}\Phi] e^{-S[\Phi]} \tag{4.6}$$

If $\phi(x)$ is an isolated stationary point (except for zero modes which we discuss below) then we may approximate Z by

$$Z \simeq Z_0 + Z_1 \tag{4.7}$$

where Z_0 is the contribution to Z from the (perturbative) vacuum solution $\Phi = \phi_0$. In the Gaussian (semiclassical) limit

$$\begin{aligned} Z_1 &= e^{-\beta E[\phi]} det^{-\frac{1}{2}}\mathcal{G} \\ &= e^{-\beta F_1} \end{aligned} \tag{4.8}$$

where formally

$$F_1 = E[\phi] + \frac{1}{2\beta} Tr(\log \mathcal{G}). \tag{4.9}$$

The trace in (4.9) is over all eigenvalues labelled in (4.5) by n and p. For fixed ϵ_p the contribution of the mode ψ_p to (4.9) is just that of a simple harmonic oscillator with frequency ϵ_p (provided $\epsilon_p^2 > 0$). This contribution to the second half of eq.(4.9) is:

$$-\frac{1}{\beta}\log\big[\sum_{l=0}^{\infty} e^{-(l+1/2)\beta\epsilon_p}\big] = \frac{1}{\beta}\log\big[2\sinh\big(\frac{\beta\epsilon_p}{2}\big)\big] = \frac{\epsilon_p}{2} + \frac{1}{\beta}\log\big(1 - e^{-\beta\epsilon_p}\big) \tag{4.10}$$

The first term is the zero point energy of the oscillator while the second is the finite temperature contribution to the free energy coming from the mode p.

If $d = 0$, everything reduces to quantum mechanics and may be applied directly to the pendulum example of the preceeding two sections. The static "sphaleron" is simply the classical solution, $\eta = \frac{\pi}{2}$. The operator in (4.4) collapses to simple multiplication by the second derivative of the pendulum potential at this value of η, so that there is only one value of the index p. The corresponding value of ϵ^2 is negative, corresponding to the instability of the sphaleron solution atop the potential barrier. Eq. (4.10) continues to hold via an analytic continuation to imaginary ϵ, as described below, and gives rise to an imaginary part to the free energy which may be interpreted as the rate of classical activation over the potential barrier.

Consider now the case $d = 1$. If the classical solution is well localized and approaches the vacuum solution, ϕ_0 fast enough as $|x| \to \infty$, H will have a continuous spectrum and the corresponding scattering solutions ψ will have the asymptotic forms (for $d = 1$)

$$\psi_{\epsilon(p)}(x) \to A^{(\pm)}(p)e^{ipx} + B^{(\pm)}(p)e^{-ipx} \tag{4.11}$$

as $x \to \pm\infty$. In fact we shall be interested specifically in the case that the coefficients $B^{(\pm)}$ vanish identically. In one dimension this occurs only for some very special potentials. If $d > 1$ and $\phi(x)$ is spherically symmetric, we need to reformulate the scattering problem in terms of radially outgoing partial waves for unit flux incident from the left. For such scattering wave solutions the coefficients analogous to $B^{(\pm)}$ always vanish. For simplicity we concentrate on the $d = 1$ case for details of the computation, though with the above remarks, everything in the remainder of this section may be extended to any d.

All important information about the scattering solutions resides in the phase shift $\delta(p)$, defined by the transmission coefficient corresponding to unit flux incident from the left in the Schrödinger eq. (4.4)

$$\delta(p) = \arg\big[\frac{A^{(+)}(p)}{A^{(-)}(p)}\big] \tag{4.12}$$

By differentiating (4.4) with respect to ϵ it is not difficult to show that

$$\begin{aligned} Trf(H) &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dp}{2\pi} f(\epsilon^2(p))\psi_{\epsilon(p)}(x)\psi^*_{\epsilon(p)}(x) \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{1}{2\epsilon(p)} f(\epsilon^2(p)) W\big[\frac{d\psi_{\epsilon(p)}}{d\epsilon}, \psi^*_{\epsilon(p)}\big]_{-\infty}^{\infty} \end{aligned} \tag{4.13}$$

where $W[u,v]_a^b$ is the Wronskian of the functions u, v evaluated between a and b. As $a \to -\infty, b \to +\infty$, (4.4) and (4.11) - (4.13) with $B^{(\pm)} = 0$ give:

$$Trf(H) = (b-a)\int_{-\infty}^{\infty} \frac{dp}{2\pi} f(\epsilon^2(p)) + \int_{-\infty}^{\infty} \frac{dp}{2\pi} f(\epsilon^2(p)) \frac{d\delta(p)}{dp} \tag{4.14}$$

for any function $f(\epsilon^2)$. In deriving eq. (4.14) we have used the fact that $d\epsilon^2 = dp^2$, which holds provided $\phi(x) \to \phi_0$ fast enough as $|x| \to \infty$.

We now apply (4.14) to (4.10), i.e. we take

$$f(\epsilon^2) = \frac{\epsilon}{2} + \frac{1}{\beta}\log(1 - e^{-\beta\epsilon}) \tag{4.15}$$

and subtract the same quantity for the vacuum case, $\Phi = \phi_0$. Then the linear volume divergence in (4.14) cancels and we obtain

$$Tr[f(H) - f(H_0)] = \frac{1}{\pi}\int_0^{\infty} dp f(\epsilon^2(p)) \frac{d\delta(p)}{dp} \tag{4.16}$$

This will give the contribution to (4.8) of the positive eigenvalues of H. It is clear that formula (4.8) breaks down if $\mathcal{G}$ has zero or negative eigenvalues. This can only be the case if H does. The zero eigenmodes of H are easy to treat, since they are just harmonic oscillator modes with zero oscillator frequency, i.e. free modes. The contribution to Z_1, of each free mode is thus just the factor

$$\int \frac{dqdp}{2\pi\hbar} e^{-p^2/2kT} = \frac{1}{\hbar}\sqrt{\frac{kT}{2\pi}} \int dq \tag{4.17}$$

where q is the coordinate in this direction and p the corresponding canonical momentum (we take the mass to be unity). That is, the projection of the general linear fluctuation $\delta\Phi$ onto its zero mode subspace is given by:

$$(\delta\Phi)_0 = (\Phi - \phi)_0 = g\psi_0(x)q(t) \tag{4.18}$$

and a factor of the coupling constant $g \sim \hbar$ has been exhibited explicitly. If $(\delta\Phi)_0$ can be related to some symmetry of the action S, not shared by the solution $\phi(x)$, then

$$(\delta\Phi)_0 = \frac{\partial\phi}{\partial a}\delta a \tag{4.19}$$

where a is the parameter that breaks the symmetry (such as translational invariance). If $\psi_0(x)$ is normalized by

$$\int d^dx|\psi_0(x)|^2 = 1. \tag{4.20}$$

we make use of eqs. (4.18) - (4.20) to secure:

$$\int dq = \Delta q = \frac{\Delta a}{g}\Big[\int d^dx\Big|\frac{\partial\phi}{\partial a}\Big|^2\Big]^{1/2} \tag{4.21}$$

Hence, the zero mode factor (4.17) is

$$\frac{\Delta a}{g\hbar}\left[\frac{kT}{2\pi}\int d^d x\left|\frac{\partial\phi}{\partial a}\right|^2\right]^{1/2} \tag{4.22}$$

If in addition H has a negative eigenvalue $\epsilon_-{}^2 = -|\epsilon_-{}^2| < 0$ then $\mathcal{G}$ does as well. If

$$kT > \frac{|\epsilon_-|}{2\pi} \tag{4.23}$$

$\mathcal{G}$ has only one negative eigenvalue. Then we may interpret the negative mode as giving rise to an imaginary part in Z, according to the prescription,

$$\frac{1}{2\sinh\left(\frac{\beta\epsilon_-}{2}\right)} \to \frac{1}{2}\frac{i}{2i\sin\left(\frac{\beta|\epsilon_-|}{2}\right)} \tag{4.24}$$

The additional factor of 1/2 in (4.24) arises from the distortion of the non-Gaussian contour over half of its range[3]. This means the free energy $\mathcal{F}$ defined by (4.6) picks up an imaginary part from the unstable stationary point $\Phi = \phi(x)$:

$$\mathrm{Im}\mathcal{F} = -\frac{1}{\beta}\mathrm{Im}\log Z \simeq -\frac{1}{\beta}\frac{1}{Z_0}\mathrm{Im}Z_1 \tag{4.25}$$

Reassembling the various contributions (4.16), (4.22) and (4.24), we find

$$\mathrm{Im}\mathcal{F} = +\frac{1}{4\beta}\frac{1}{\sin\left(\frac{\beta|\epsilon_-|}{2}\right)}\mathcal{N}\mathcal{V}\, e^{-\beta(F_1-F_0)} \tag{4.26}$$

where

$$F_1 - F_0 = E[\phi] - E[\phi_0] + Tr[f(H) - f(H_0)] \tag{4.27}$$

and $\mathcal{N}\mathcal{V}$ is the product of the normalized volume factors (4.22) for each zero mode.

According to Langer[3], the imaginary part of the free energy function $\mathcal{F}$ is to be interpreted as giving rise to a decay rate of the perturbative vacuum ϕ_0 according to

$$\Gamma_0 = \frac{|\kappa|}{\pi kT}\mathrm{Im}\mathcal{F} \tag{4.28}$$

where κ is a damping constant, namely, the real time rate of decay of the configuration $\Phi = \phi(x)$ in the heat bath. All the dynamics of the heat bath are buried in this one quantity. For a weakly coupled theory the interaction with the heat bath does not affect the decay of the configuration $\phi(x)$,which is determined purely by its negative eigenvalue, ϵ_-^2. That is, if $g^2 \ll 1$ we are always in the underdamped limit and

$$|\kappa| = |\epsilon_-|. \tag{4.29}$$

Hence,

$$\Gamma_0 = \frac{1}{4\pi} \frac{|\epsilon_-|}{\sin\left(\frac{\beta|\epsilon_-|}{2}\right)} \mathcal{N}\mathcal{V} \; e^{-\beta(F_1 - F_0)} \tag{4.30}$$

In the high temperature limit, $\beta \to 0$, this becomes

$$\frac{kT}{2\pi\hbar} \mathcal{N}\mathcal{V} \; e^{-\beta(F_1 - F_0)} \tag{4.31}$$

For the simple pendulum there are no factors $\mathcal{N}\mathcal{V}$ since there are no zero modes as $\beta \to 0$, and we find that the previous estimate for the rate, (3.5) is in fact exact.

To recapitulate, the weak coupling limit ensures the validity of the Gaussian approximation used in deriving this formula and also leads to the weak damping limit (4.29). Other than $g^2 \ll 1$ the only additional assumption made in deriving (4.30) is that the stationary point $\Phi = \phi(x)$ is isolated, except for zero modes related to symmetries in the theory. If the solution is not isolated in this sense, there will be additional "accidental" zero modes of $\mathcal{G}$ which will cause (4.30) to break down. This is just what happens as $kT \to \frac{|\epsilon_-|}{2\pi}$, for instance. For temperatures not satisfying (4.23) the static solution $\Phi = \phi(x)$ does not contribute to $\mathrm{Im}\mathcal{F}$ or the decay rate Γ_0, which are dominated by *non*-static, instanton-like configurations. It is in this way that the high temperature analysis matches onto the low temperature instanton analysis, just as in the simple pendulum model considered previously. A quantitative method of implementing this matching has been desribed by Affleck[4].

I elected to present this path integral derivation of the rate because of its compactness. However, there is no need to resort to path integrals or the analytic continuation in the negative mode direction implied in (4.24). Eq. (4.30) could have been derived, as in Langer's original paper, by consideration of the probability flow in one direction over the saddle point $\Phi = \phi(x)$. The main point is that (4.30) is a formula based solely on *classical statistical mechanics* and correctly accounts for entropy effects through the free energy function $F_1 - F_0$. If there were something pathological about the sphaleron, such as a large entropy suppression it would have to show up in the expression (4.27). I turn now to an explicit evaluation of (4.26) and (4.27) for the sphaleron solution of an instructive field theoretic model in 1 + 1 dimensions.

5. An O(3) Non-Linear Sigma Model

In the pendulum example it is very clear that the instanton suppression does not persist at sufficiently high temperatures, because thermal activation comes to dominate over quantum tunnelling. However, the pendulum differs from the Weinberg-Salam theory in at least one important respect, namely it is a model with only one degree of freedom. In such a model it is evident that heating the system must imply greater kinetic energy available to leap the potential barrier: there is nowhere else for the energy to go. In a field theory there are infinitely many degrees of freedom and the class of configurations that interpolate between vacuum states of different winding number may be very special and very few. Heating this system also increases the available energy, but it is by no means clear that the incoherent thermal energy can organize itself into the special configuration(s) necessary to leap the barrier. In other words, a significant entropy suppression is possible. We need to consider the free energy, not just the classical energy of the sphaleron. In addition, a one dimensional model like the simple pendulum cannot couple to fermions, so there is no analog of the chiral current anomaly which is central to the issue of B and L violation.

In the following, I shall present a field theoretic model of tunnelling at finite temperature: the O(3) nonlinear sigma model in 1+1 dimensions[5]. The completely symmetric model has been studied before for its remarkable similarities to non-Abelian gauge theories. We shall need to modify the symmetric model by introducing a term in the action that breaks the O(3) symmetry down to O(2), in order that a sphaleron solution exist.

Although this is an explicitly broken global symmetry, unlike the spontaneously broken local symmetry of the Weinberg-Salam theory, it shares many properties with the latter. Its main virtue is the fact that we will be able to attain closed form results for fermion number violating processes, and visualize what is going on in a geometrical fashion that builds upon the intuitions garnered from the pendulum model.

In 1+1 dimensions, the action of the O(3) non-linear sigma model is:

$$S_0 = \frac{1}{2g^2}\int d^2x\,(\partial_\mu\hat{n}\cdot\partial_\mu\hat{n})\,; \qquad \hat{n}^2(x) = 1 \tag{5.1}$$

This model possesses some remarkable similarities with non-Abelian gauge theories in 3+1 dimensions, and for that reason has been much studied[6]. The most important features which concern us here are the following:

(i) Scale Invariance of the Classical Action;
(ii) Renormalizability and Asymptotic Freedom in the coupling constant g;
(iii) Existence of a Toplogical Winding Number, Instantons and a Chiral Anomaly when coupled to Fermions.

The first property is obvious and the second well known [7]. The winding number will be evident if we identify the points at infinity of the Euclidean plane. Then the plane has topology S^2. Since $\hat{n}$ is also constrained to lie on S^2, the $\hat{n}$ field is a map from S^2 to S^2. This mapping can be characterized by an integer winding number, given explicitly by:

$$Q = \frac{1}{8\pi}\int d^2x\,\epsilon_{\mu\nu}\,\hat{n}\cdot(\partial_\mu\hat{n}\times\partial_\nu\hat{n})\,. \tag{5.2}$$

By forming the quantity,

$$\int d^2x \left[\partial_\mu \hat{n} \pm \epsilon_{\mu\nu} (\hat{n} \times \partial_\nu \hat{n})\right]^2 \geq 0 \tag{5.3}$$

it is easy to see that the Euclidean action for any $\hat{n}$ *obeying the boundary condition at infinity* is bounded from below:

$$S_0 \geq \frac{4\pi}{g^2} |Q|. \tag{5.4}$$

The bound is saturated by the instanton solutions which can be given explicitly in terms of the complex function

$$w = \frac{n_1 + in_2}{1 - n_3} \tag{5.5}$$

of the complex variable $z = x_1 + ix_2$. In terms of w, S_0 and Q become proportional to

$$\int d^2x \frac{1}{(1+|w|^2)^2} \left[\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}} \pm \frac{\partial w}{\partial \bar{z}}\frac{\partial \bar{w}}{\partial z}\right] \tag{5.6}$$

respectively. Thus the bound (5.4) is saturated when one of the terms in brackets vanishes and (anti)instanton solutions are simply meromorphic functions of the complex variable $z(\bar{z})$. In particular,

$$w_n = c \prod_{l=1}^{n} \frac{z - a_l}{z - b_l} \tag{5.7}$$

has $Q = n$ and $S_0 = 4\pi n/g^2$.

Yet another formulation of the model is obtained by defining a two component field $\chi = \binom{\chi_1}{\chi_2}$ such that

$$\hat{n} = \chi^\dagger \vec{\sigma} \chi \tag{5.8}$$

where $\vec{\sigma}$ are the Pauli matrices. The condition $\hat{n}^2 = 1$ now becomes

$$\chi^\dagger \chi = 1 \tag{5.9}.$$

In terms of χ,

$$S_0 = \frac{2}{g^2} \int d^2x \left[(\partial_\mu \chi^\dagger)(\partial_\mu \chi) - (\chi^\dagger \partial_\mu \chi)(\partial_\mu \chi^\dagger \chi)\right] \tag{5.10}$$

Evidently there is now a U(1) gauge invariance in this formulation since S_0 is invariant under $\chi \to e^{i\alpha(x)}\chi$. This may be made explicit by introducing a subsidiary gauge field

$$A_\mu(x) = \frac{1}{2i}(\chi^\dagger \partial_\mu \chi - \partial_\mu \chi^\dagger \chi) \tag{5.11}$$

and defining the covariant derivative,

$$D_\mu = \partial_\mu - iA_\mu \tag{5.12}$$

so that

$$S_0 = \frac{2}{g^2}\int d^2x |D_\mu \chi|^2 \tag{5.13}$$

In this language,

$$Q = \frac{1}{4\pi}\int d^2x \epsilon^{\mu\nu} F_{\mu\nu} = \frac{1}{2\pi}\int d^2x \partial_\mu (\epsilon^{\mu\nu} A_\nu) \tag{5.14}$$

Massless fermions may now be added to the system and coupled in the usual way to the $U(1)$ gauge field:

$$S_{fermion} = i\int d^2x \bar{\psi}\gamma^\mu D_\mu \psi \tag{5.15}$$

Such fermions are well known [8] to possess an anomaly in the chiral current

$$j^{\mu 5} = \bar{\psi}\gamma^\mu \gamma^5 \psi; \tag{5.16}$$

namely

$$\partial_\mu j^{\mu 5} = \frac{1}{2\pi}\epsilon^{\mu\nu} F_{\mu\nu} \tag{5.17}$$

The Feynman graph contributing to the anomaly is illustrated in Fig. 2. By integrating this equation over two dimensional space, we obtain the index theorem,

$$\Delta \int dx j^{05} \equiv \Delta N_5 = 2Q = 2\Delta N_{CS} \tag{5.18}$$

where

$$N_{CS} = \frac{1}{2\pi}\int dx A_1 \tag{5.19}$$

is the Chern-Simons number corresponding to the Q of eq. (5.14). Thus, (5.18) relates the change of chiral fermion number to the topological charge, or change of winding number, N_{CS} in going from one vacuum configuration to another. Because of the bound (5.4) on the classical action, and the interpretation of the Euclidean instanton as a tunnelling event (by continuation to imaginary ime), such topology changing events and concommitant fermion number violation are strongly suppressed at zero temperature: the rate is proportional to $\exp(-\frac{2S_0}{\hbar}) \ll 1$. In order to understand what happens in the model at high temperature, we should look for the anolog of the static solution with finite energy analogous to $\eta = \frac{\pi}{2}$ of the pendulum model. However, one can easily see that a scale invariant action such as (5.1) cannot have finite energy solutions. Hence we shall consider a modified action by adding to (5.1) the simple symmetry breaking term, inspired by the pendulum model:

$$S_1 = \frac{\omega^2}{g^2}\int d^2x\,(1+\hat{n}_3) \tag{5.20}$$

The classical energy functional of the model now reads:

$$E = \frac{1}{g^2}\int dx\left[\frac{1}{2}\left(\frac{d\hat{n}}{dx}\right)^2 + \omega^2(1+\hat{n}_3)\right]. \tag{5.21}$$

Now, we shall argue that an unstable static solution to the equations of motion must exist with finite energy (5.21). First let us parameterize the sphere in the following way:

$$\hat{n} = \left(\sin\eta\sin\xi,\ \sin\eta\cos\eta(\cos\xi-1),\ -\sin^2\eta\cos\xi-\cos^2\eta\right). \tag{5.22}$$

This parametrization has the following properties:

(i) it satisfies the constraint $\hat{n}^2 = 1$ and is continuous in its arguments;
(ii) for fixed η , ξ is the azimuthal angle of a circle, S^1;
(iii) for all η, $\hat{n}(\xi=0) = \hat{n}(\xi=2\pi) = (0,0,-1)$;
(iv) for all ξ, $\hat{n}(\eta=0) = \hat{n}(\eta=\pi) = (0,0,-1)$;
(v) each point on S^2 occurs for at least one (η,ξ) and if $\hat{n}$ is not the point $(0,0,-1)$ then $\eta(\hat{n})$ is unique;
(vi) as η ranges from 0 to π and ξ from 0 to 2π the map (5.3) has $Q = 1$.

The angles η and ξ are easily visualized geometrically by the diagram in Figure 3: for given η between 0 and π, $\hat{n}$ lies on the circle S^1 which is the intersection of the unit two sphere with the plane,

$$x_2\sin\eta + x_3\cos\eta = -\cos\eta$$

We are interested in noncontractible loops in configuration space which begin and end at the vacuum. Because of the symmetry breaking term S_1, this is the point $\hat{n}_V = (0,0,-1)$. We may now consider static configurations, $\hat{n}(x)$ at fixed η, with $\xi(x)$ ranging from 0 to 2π as x ranges from $-\infty$ to $+\infty$. Because of (iii) this satisfies the boundary condition for finite energy. Because of (iv) this set of configurations reduces identically to the vacuum at $\eta = 0$ and $\eta = \pi$. Because of (vi) this one parameter (i.e. η) family of loops which begins and ends at the vacuum is noncontractible: that is, the whole sequence cannot be simultaneously continuously deformed to the vacuum. The energy functional (5.21) for fixed η and $\xi = \xi(x)$ is:

$$E = \frac{\sin^2\eta}{g^2}\int dx\left\{\frac{1}{2}\left(\frac{d\xi}{dx}\right)^2 + \omega^2(1-\cos\xi)\right\}. \tag{5.23}$$

Consider now the extremizing of this functional. As a function of the parameter η, E clearly attains its maximum at $\eta = \pi/2$. This is physically obvious from the fact that the energy may be viewed as that of a physical pendulum in a uniform gravitational field: for given $\xi(x)$ the maximal energy is achieved by the furthest excursion from the pendulum's point of rest at $\hat{n}_V = (0,0,-1)$. With η fixed at this maximal value of $\pi/2$, now consider minimizing the positive definite energy functional with respect to $\xi(x)$. The resulting Euler-Lagrange equation for ξ is precisely that of a simple pendulum in Euclidean "time" x. Since ξ varies from 0 to 2π as x varies from $-\infty$ to ∞, the solution of this equation is none other than the instanton solution of the pendulum problem:

$$\sin\left(\frac{\xi_{sph}(x)}{2}\right) = \text{sech}[\omega(x - x_0)]. \tag{5.24}$$

The energy of the sphaleron solution for the sigma model is

$$E_{sph} = 8\omega/g^2. \tag{5.25}$$

6. The Transition Rate in the O(3) Model

Having found the sphaleron solution of the model lets proceed now with the calculation of the one loop corrections to it by an analysis of the small fluctuations about the classical solution. If there is a suppression due to phase space or entropy effects, it should show up in the free energy function given by the finite temperature loop expansion. We begin by parameterizing the fluctuations in a convenient way. Let

$$\hat{n} = \frac{1}{\sqrt{1+u^2}}\left(\sin(\xi_{sph}+v), u, -\cos(\xi_{sph}+v)\right). \tag{6.1}$$

Substituting this form for $\hat{n}$ into the action functional and expanding to quadratic order in (u,v) gives the desired small fluctuation operators. The eigenvalues are determined by solving:

$$\begin{aligned} H_1 u &\equiv \left[-\frac{d^2}{dx^2} + \omega^2(1 - 6\,\text{sech}^2\omega x)\right] u = \epsilon^2 u \\ H_2 v &\equiv \left[-\frac{d^2}{dx^2} + \omega^2(1 - 2\,\text{sech}^2\omega x)\right] v = \epsilon^2 v. \end{aligned} \tag{6.2}$$

It is a special feature of the present model that these equations are just Schrödinger's equations in the Rosen-Morse potentials, $U_0\text{sech}^2\omega x$, whose eigenfunctions are known explicitly. Each of the two scattering potentials in (6.2) satisfies all the conditions postulated in the general discussion of section 4. It is amusing to note as well that the two potentials are supersymmetric partners so that their spectra are closely related.

The first operator, H_1 describes fluctuations in $\hat{n}$, perpendicular to the sphaleron. This operator has exactly one negative eigenvalue, namely $\epsilon_-^2 = -3\omega^2$, associated with the fact that sliding the sphaleron loop on the sphere in the η or u direction must decrease the energy. This we knew already. There is one zero eigenvalue associated with the ability to rotate the sphaleron solution about the $\hat{n}_3$ axis without changing its energy. The angle that $\hat{n}_{sph}$ makes with the x_1 axis is the corresponding parameter a in (4.19) for this zero mode. All the remaining eigenvalues are in the continuum above ω^2.

The second operator, H_2 describes fluctuations in $\hat{n}$ along the direction of the sphaleron (i.e. η remains fixed at $\pi/2$). Its lowest eigenvalue is zero with the corresponding mode associated with translation of the sphaleron position. All other eigenvalues are positive. The one negative mode and two zero mode eigenfunctions are easy to find explicity:

$$\begin{aligned} u_- &= \mathrm{sech}^2(\omega x), \\ u_0 &= \sin \xi_{sph} = 2\mathrm{sech}(\omega x) \tanh(\omega x), \\ v_0 &= \frac{d\xi_{sph}}{dx} = 2\omega \mathrm{sech}(\omega x). \end{aligned} \tag{6.3}$$

For the positive spectral continuum of each operator above ω^2, we evaluate the finite temperature determinants by relations (4.12) and (4.16). For the potentials in H_1 and H_2, the transmission coefficients are known and they lead to the following formulae:

$$\frac{d\delta_1(p)}{dp} = -\frac{2\omega}{p^2+\omega^2}$$

and

$$\frac{d\delta_2(p)}{dp} = -\frac{2\omega}{p^2+\omega^2} - \frac{4\omega}{p^2+4\omega^2} \tag{6.4}$$

where $p^2 + \omega^2 = \epsilon^2(p)$. We may now apply (4.16) and sum over the two orthogonal sets of modes for H_1 and H_2 respectively. The zero point energy contributions from the two operators yield the logarithmically divergent integral,

$$\frac{-2\omega}{\pi} \int_0^\infty dp \sqrt{p^2+\omega^2} \left(\frac{1}{p^2+\omega^2} + \frac{1}{p^2+4\omega^2} \right).$$

Introducing an ultraviolet cut-off, Λ and defining the renormalized coupling constant by

$$\frac{1}{g^2_{ren}(\omega)} = \frac{1}{g_0^2} - \frac{1}{2\pi} \log(\Lambda/\omega), \tag{6.5}$$

we observe that this zero point contribution may be absorbed into the classical sphaleron energy (5.25), provided that we replace the bare $1/g^2$ appearing there by the renormalized running coupling evaluated at ω: $1/g^2(\omega)$. Then we are left with only the second term of (4.15)-(4.16), which gives the finite temperature corrections to the sphaleron's statistical weight. This is summarized succintly by the following function:

$$h(a) \equiv \frac{-4a}{\pi} \int_0^\infty dx \left(\frac{1}{x^2+a^2} + \frac{1}{x^2+4a^2} \right) \log\left(1 - e^{-\sqrt{x^2+a^2}}\right) \geq 0 \tag{6.6}$$

where $a = \hbar\omega/kT$. The limiting forms of this function for $a \to \infty$ and $a \to 0$ are respectively

$$h(a) \to \frac{5}{\sqrt{2\pi a}} e^{-a},$$

and

$$h(a) \to -3\log c + \frac{4}{\pi}(\tan^{-1}a + \frac{1}{2}\tan^{-1}2a - 3a)\log a - C + \mathcal{O}(a),$$

where

$$C = \frac{2}{\pi}\int_0^\infty dx \left(\frac{1}{x^2+1} + \frac{1}{x^2+4}\right) \log(x^2+1) = 6.2515852 \tag{6.7}$$

Turning to the evaluation of the zero mode factors $\mathcal{N}\mathcal{V}$ required, we find that the mode u_0 contributes the factor,

$$\frac{4}{g}\sqrt{\frac{\pi kT}{3\omega}} \tag{6.8}$$

since the range in the parameter corresponding to a in the general formula (6.2?) is 2π for rotations about the $\hat{n}_3$ axis. The translational zero mode contributes the factor,

$$\frac{2L}{g}\sqrt{\frac{\omega kT}{\pi}} \tag{6.9}$$

We are now in a position to give a closed form answer for the rate per unit volume, L of thermal activation over the energy barrier between two topologically distinct vacuum configurations, the height of which is the classical sphaleron energy, $E_{sph} = 8\omega/g^2$. The result of substituting (6.4) through (6.8) into the general formula (4.30), derived previously is:

$$\frac{\Gamma_0}{L} = \frac{2}{\pi g^2}\frac{\omega T}{\sin\left(\frac{\sqrt{3}\omega}{2T}\right)}\exp\left(-\frac{8\omega}{g^2 T} - h\left(\frac{\omega}{T}\right)\right) \tag{6.10}$$

where we set $\hbar = k = 1$.

This transition rate does not lead to any violation of chiral fermion number unless there is an initial asymmetry in fermion number. We may introduce such an asymmetry by adding a chemical potential to the Hamiltonian[9]:

$$H \to H - \mu N_{CS} \tag{6.11}$$

where N_{CS} is the Chern-Simons number introduced in eq. (6.19). The vacuum state which is unique in the gauge invariant description, $\hat{n}_V = (0, 0, -1)$ corresponds to an infinitely degenerate set of states labelled by the topological winding number N_{CS}. This quantity is not gauge invariant but changes in it are. .

We take $\frac{\mu}{T} << 1$ so that we may expand in this small quantity in all that follows. First order perturbation theory then gives $-\frac{\mu}{T}\Gamma_0$ for the transition rate from a state with $N_{CS} = 1$ to one with $N_{CS} = 0$, i.e.

$$\frac{d\langle N_{CS}\rangle}{dt} = -\frac{\mu}{T}\Gamma_0. \tag{6.12}$$

The chemical potential induces the asymmetry in N_5 given by

$$\langle N_5\rangle = \frac{4L}{\pi}\mu \tag{6.13}$$

to first order in μ. Substituting this relation for μ into eq. (6.12) and using (6.10) and (6.18) gives finally

$$\frac{d\langle N_5\rangle}{dt} = -\Gamma_5\langle N_5\rangle \tag{6.14}$$

with

$$\Gamma_5 = \frac{\pi}{2T}\frac{\Gamma_0}{L} = \frac{\omega}{g^2}\frac{1}{\sin\left(\frac{\sqrt{3}\omega}{2T}\right)}\exp\left(-\frac{8\omega}{g^2(\omega)T} - h\left(\frac{\omega}{T}\right)\right) \tag{6.16}$$

In the temperature range where $T >> \omega$ so that the sphaleron induced transitions are dominant compared to those caused by instantons, but $T << \omega/g^2$ so that the semiclassical expansion around a single sphaleron solution is justified, we may employ relations (6.22) and (6.7) to obtain

$$\Gamma_5 = K\frac{\omega}{g^2(T)}\left(\frac{\omega}{T}\right)^2 e^{-\frac{8\omega}{g^2(T)T}} \tag{6.17}$$

with

$$K = \frac{2}{\sqrt{3}}e^C = 10.971766 \tag{6.18}$$

and $g^2(T)$ the temperature dependent running coupling constant evaluated at the temperature T. Thus, the initial asymmetry (6.13) decays exponentially with a rate that is considerably greater than the instanton inferred rate, at temperatures large compared to ω.

7. The Sphaleron Solution of the Weinberg-Salam Theory

Having discussed a series of simpler pedagogic models of tunnelling, and concomitant fermion number violation, we are ready finally to turn to the actual four dimensional gauge theory. In the case of the group $SU(2)$ the topological charge or winding number is given by

$$Q = \frac{1}{64\pi^2}\int d^4x G^a_{\mu\nu}G^a_{\alpha\beta}\epsilon_{\mu\nu\alpha\beta} \tag{7.1}$$

in Euclidean space. Geometrically, the fact that such a winding number should exist is clear from the following considerations. We are interested in finite action Euclidean configurations. This means that as Euclidean $|x| \to \infty$, the field strength $G_{\mu\nu} \to 0$ and A_μ must approach a pure gauge, $\mathcal{U}\partial_\mu\mathcal{U}^{-1}$. Therefore, the gauge field at Euclidean infinity may be regarded as a mapping from the spatial sphere at infinity, S^3 to the gauge group $SU(2)$, which is also isomorphic to S^3. This is just one dimension higher than the mapping considered in the sigma model case, and falls into topological integer number classes for the same reason as before. Upon writing the integrand of (7.1) as a total divergence, and using Stoke's theorem to convert the volume integral to a surface integral, (7.1) will be recognized as precisely this integer winding number of the map from S^3 to S^3.

By forming the non-negative quantities,

$$\int d^4x(G^a_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^a_{\alpha\beta}) \geq 0, \tag{7.2}$$

we arrive at bound on the Euclidean action analogous to (3.7) and (5.4) for the pure gauge action:

$$\frac{S_0}{\hbar} \geq \frac{8\pi^2}{g^2}. \tag{7.3}$$

The bound is saturated for $Q = 1$ by the finite action instanton solution[10]. Accordingly, one expects that the rate for tunnelling from one vacuum to another topologically inequivalent vacuum is suppressed by a factor of

$$\exp(-\frac{2S_0}{\hbar}) = \exp(-\frac{16\pi^2}{g^2}) = \exp(-\frac{4\pi \sin^2 \Theta_W}{\alpha}) <<< 1. \tag{7.4}$$

Since integration of the anomalous divergence eq. (1.3) tells us that there can be no violation of baryon or lepton number unless the gauge theory winds from one vacuum to the next [recall eq. (5.18) for the $O(3)$ model], we conclude that the rate of B and L violation in the Weinberg-Salam theory is utterly negligible at zero temperature, and this is the conclusion first reached by 't Hooft[10].

Actually this conclusion is not so trivial as there are many complications with the analysis. For one example, strictly speaking, there are no finite action instanton solutions in the electroweak theory because of the existence of the scalar Higgs field, which we have so far ignored. Instanton solutions at zero temperature (infinite periodicity in imaginary time) and finite temperature (periodicity β) do exist in pure non-Abelian gauge theory (without Higgs fields), but there another difficulty arises in that the scale invariant classical theory has instantons of all scale sizes. This instanton scale size must be finally integrated over, but the integration diverges in the infrared. Hence there is no complete, satisfactory instanton analysis in either QCD (where there are additional problems with strong coupling) or Weinberg-Salam theory. This is why simpler field theoretic models are valuable, and why I have concentrated so heavily on the details of the $O(3)$ model to guide our intuition about four dimensional gauge theories such as the Weinberg-Salam model. In the $O(3)$ model also there are no instanton solutions after the symmetry breaking term (5.20) has been added to the action. If one does not add this term, then one also faces an infrared divergence in the integration over the instanton scale size [depending on the parameters a_l and b_l of (5.7)].

Despite the technical difficulties with the instanton analysis, all experience with simpler models leads one to believe that these problems are indeed technical rather than fundamental, and that the estimate (7.4) is basically correct, *at zero temperature*. Since finite temperature instanton solutions to the pure gauge theory with action, $S_0 = 8\pi^2/g^2$ and $Q = 1$ exist also, it was natural to believe that this highly suppressed rate should persist even at finite temperature. However, this depends critically on the reliability of perturbation theory at higher temperatures. We have seen explicitly how perturbation theory breaks down for both the simple pendulum model and the $O(3)$ model at high temperatures, namely when the number of quanta $n \sim \alpha^{-1}$. This is clear because at temperatures of order α^{-1} times the fundamental mass or frequency, (ω in these models), there is enough energy in the system to surmount the classical energy barrier between adjacent minima. What this energy barrier is for the Weinberg-Salam theory was something of a mystery at first, and was answered only later by the work of Manton[11].

Several years after the instanton based estimate of 't Hooft, Manton[11] constructed the parameterization of the noncontractible loop in field configuration space with $Q = 1$, analogous to (5.22) for the broken $O(3)$ sigma model discussed above. Consider the spatial components of the $SU(2)$ gauge potential, $A_i = -i(\frac{\tau^a}{2})A_i^a(\vec{x})$ and the two-component complex Higgs field $\Phi(\vec{x})$ ($\tau^a, a = 1,2,3$ are the Pauli matrices). The first step in Manton's construction is to fix the local gauge freedom. This is done by introducing spherical polar coordinates in the three dimensional space, (r,θ,φ) and demanding that the radial component of A_i vanish:

$$A_r(r,\theta,\varphi) = 0. \tag{7.5}$$

There still remains a global gauge freedom, which we may fix as follows. The Higgs field must approach its vacuum expectation value as $r \to \infty$. Rescaling Φ so that this value is unity, we use the global gauge freedom unfixed by (7.5) to choose:

$$\Phi^{\infty}(\theta = 0) \equiv \Phi(r = \infty, \theta = 0, \varphi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{7.6}$$

Let us also think of the complex two-component Φ as equivalent to a real four-component Φ_{Re}. Since the magnitude of this four-component real column vector must be unity at $r = \infty$, we may regard these components as defining a unit three-sphere. Then, the Higgs field at ∞ may be regarded as a mapping from the spatial two-sphere parameterized by θ and φ to this unit three-sphere.

Since we are interested in noncontractible loops in the gauge-Higgs configuration space, we now introduce the parameter η which varies from 0 to π, just as in the pendulum example or the O(3) nonlinear sigma model. This is the parameter along the loop, at each value of which we have the Higgs field at $r = \infty$ described above, and depending on η as well in such a way so as to satisfy the analogs of properties (i) through (vi) following (5.22). Explicitly this parameterization is:

$$\Phi^{\infty}(\eta,\theta,\varphi) = \begin{pmatrix} \sin\eta \sin\theta e^{i\varphi} \\ e^{-i\eta}(\cos\eta + i\sin\eta\cos\theta) \end{pmatrix} \equiv \mathcal{U}^{\infty} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{7.7}$$

The gauge field at infinity may then be written in the form:

$$A_i^{\infty} = -\partial_i \mathcal{U}^{\infty}(\mathcal{U}^{\infty})^{-1} \tag{7.8}$$

for $i = \theta, \varphi$

As expected, these parameterizations are quite a bit more difficult to visualize than the analogous one for the sigma model, but the basic idea is the same: to construct a noncontractible loop ($Q = 1$) of field configurations beginning and ending at the vacuum, each with finite energy. We then look for solutions of the field equations at $\eta = \frac{\pi}{2}$ with the given asymptotic conditions at ∞. Writing

$$\Phi(\eta = \frac{\pi}{2}, r, \theta, \varphi) = h(r)\Phi^{\infty}(\eta = \frac{\pi}{2}, \theta, \varphi) \tag{7.9}$$

and

$$A_{\theta,\varphi}(\eta = \frac{\pi}{2}, r, \theta, \varphi) = f(r) A^{\infty}_{\theta,\varphi}(\eta = \frac{\pi}{2}, \theta, \varphi), \tag{7.10}$$

together with the gauge condition (7.5) gives a finite energy ansatz for the sphaleron solution in the Weinberg-Salam theory. By substituting this ansatz into the field equations, Manton and Klinkhamer then showed that in the limit $\Theta_W \to 0$ a solution exists with an energy between 8 TeV and 14 TeV (depending on the unknown value of the Higgs mass), and that this solution persists in the full theory with finite Θ_W (although the ansatz must then be significantly more complicated, since spherical symmetry is no longer preserved). The sphaleron configuration, although not a simple analytic function as in our previous examples is easy to describe qualitatively in the limit of zero Weinberg angle. It is a spherically symetric configuration of non-Abelian magnetic field density concentrated in a region with a radius of order ${M_W}^{-1}$. Within this radius the magnetic field strength is of order $\frac{{M_W}^2}{g}$. Outside the field strength falls exponetially to zero. The Higgs field, in turn, has a zero at the origin, rises linearly at first, and then approaches its vacuum expectation value exponentially rapidly outside the central core region.

The importance of this work is that it established the existence and energy scale of the sphaleron solution in the electroweak theory, and made quite explicit the pendulum-like nature of the potential, separating the inequivalent degenerate vacuua of non-Abelian gauge theories. After Manton's work one now knows explicitly what the energy barrier between inequivalent vacua in the Weinberg-Salam theory is.

The suggestion that this sphaleron solution was crucial to estimating the rate for baryon and lepton number violation was subsequently emphasized by Kuzmin, Rubakov and Shaposhnikov[12]. The semiclassical calculation of the rate at finite temperature was carried out by Arnold and McLerran a few years later[13]. At temperatures of a few hundred GeV the sphaleron rate of B and L violating transitions far exceeds the instanton estimate of 't Hooft.

Lacking up to this point is a clear connection to the instanton analysis first carried out by 't Hooft, and in particular, precisely how perturbation theory breaks down at high temperatures and energies. Just recently a paper has appeared which addresses this issue and points the direction to its eventual clarification[14]. Most interesting is the suggestion that proton-proton collisions at center of mass energies in the 50-70 TeV range might be capable of producing observable baryon and lepton number violation in the laboratory. This energy scale corresponds to the that at which perturbation theory breaks down in the Weinberg-Salam theory, much as was suggested by the pendulum model when the excitation number $n \sim \alpha^{-1}$. If such an energy scale is ever achieved in the laboratory, it could provide dramatic and direct confirmation of the ideas reviewed in these lectures.

Perhaps more probable is the prospect of indirect verification. This could come if an extension of the standard model is found which makes use of the mechanism of baryon number violation described here to generate the observed baryon number asymmetry of the universe. One might then be able to explain the remarkable asymmetry between baryons and anti-baryons alluded to in the introduction, without recourse to grand unified speculations, but at far lower energy scales and within the framework of electroweak physics. The main obstacle in constucting such a model is the necessary introduction of a significant amount of

CP violation at the sphaleron scale, without disgreeing with the known very small amount of CP violation observed in the $K^0 - \bar{K}^0$ system. The predictions of such a model presumably could be tested in laboratory experiments at energies accessible to the SSC. It remains to be seen if any model satisfying the necessary conditions can be constructed, and the goal of explaining the baryon asymmetry of the universe by electroweak physics realized. This is one very interesting topic in an area in which there are still many possibilities for future research.

FIGURE CAPTIONS

Figure 1 : The triangle graph that gives rise to a singular contribution to the operator product $\bar{\psi}\gamma_5\psi$ in the presence of a background Abelian gauge field.
In the non-Abelian case this graph must be supplemented by graphs with three and four external gauge field lines in order to arrive at the gauge invariant divergence (1.3).

Figure 2 : The diangle graph which gives rise to the axial anomaly in the $O(3)$ model.
Fermion propagators are denoted by solid lines and scalar χ propagators by dashed lines.
It is the same graph as that in 1 + 1 dimensional QED with the role of the U(1) gauge field played by the A_μ defined by eq. (5.11).

Figure 3 : Geometrical representation of the parameterization of the sphere S^2 with center C at the origin, as defined by eq. (5.22). The circle S^1 is the intersection of the sphere with the plane, $x_2 \sin\eta + x_3 \cos\eta = -\cos\eta$, labeled by Σ_η and making dihedral angle, η with the plane, $x_3 = -1$, Σ_0. ξ is the azimuthal angle along this circle measured from V = (0, 0, -1) to the generic point, P.

REFERENCES

1. S. L. Adler, Phys. Rev. 177, 2426 (1969);
   J. S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969).

2. C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Lett. 63B, 334 (1976);
   R. Jackiw and C. Rebbi, Phys. Rev. Lett. 36, 1119; 37, 172 (1976).

3. J. S. Langer, Ann. Phys. 41, 108 (1967); 54, 258 (1969);
   A. Linde, Phys. Lett. 70B, 306 (1977); 92B, 119 (1980); 100B, 37 (1981);
   S. Coleman, Phys. Rev. D15, 2929 (1977); C. Callan and
   S. Coleman, Phys. Rev. D16, 1762 (1977).

4. I. Affleck, Phys. Rev. Lett. 46, 388 (1981).

5. E. Mottola and A. Wipf, Phys. Rev. D39, 588 (1989).

6. For a review and extensive references see V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Phys. Rep. 116, No. 3, 103 (1984); or A. M. Perelomov, Physica 4D, 1 (1981).
   A solution of the model is given by P. B. Wiegmann, Phys. Lett. 152B, 209 (1985).

7. A. M. Polyakov, Phys. Lett. 59B, 79 (1975).

8. K. Johnson, Phys. Lett. 5, 253 (1963).

9. A. N. Redlich and L. C. Wijewardhana, Phys. Rev. Lett. 54, 970 (1985);
   K. Tsokos, Phys. Lett. 157B, 413 (1985);
   A. Niemi and G. Semenoff, Phys. Rev. Lett. 54, 2166 (1985).

10. A. A. Belavin, A. M. Polyakov, A.S. Schwartz, and Y.S. Tyupkin, Phys. Lett. 59B, 85 (1975);
    G. 't Hooft, Phys. Rev. Lett. 37, 8 (1976) and Phys. Rev. D14, 3432 (1976), D18(E);
    D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. 53, No. 1, 43 (1981).

11. N. S. Manton, Phys. Rev. D28, 2019 (1983);
    F. R. Klinkhamer and N. S. Manton, Phys. Rev. D30, 2212 (1984).

12. V. A. Kuzmin, V. A. Rubakov, and M.E. Shaposhnikov, Phys. Lett. 155B, 36 (1985),
    V. A. Rubakov, Nucl. Phys. B256, 509 (1985);

13. P. Arnold and L. McLerran, Phys. Rev. D36, 581 (1987);ibid. D37, 1020 (1988).
    A. I. Bocharev and M. E. Shaposhnikov, Mod. Phys. Lett. A2, 417 (1987);
    M. E. Shaposhnikov, Nucl. Phys. B287, 757 (1987); B299, 797 (1988);
    A. Ringwald, Phys. Lett. 201B, 510 (1988).

14. A. Ringwald, DESY preprint 89/074, June, 1989.

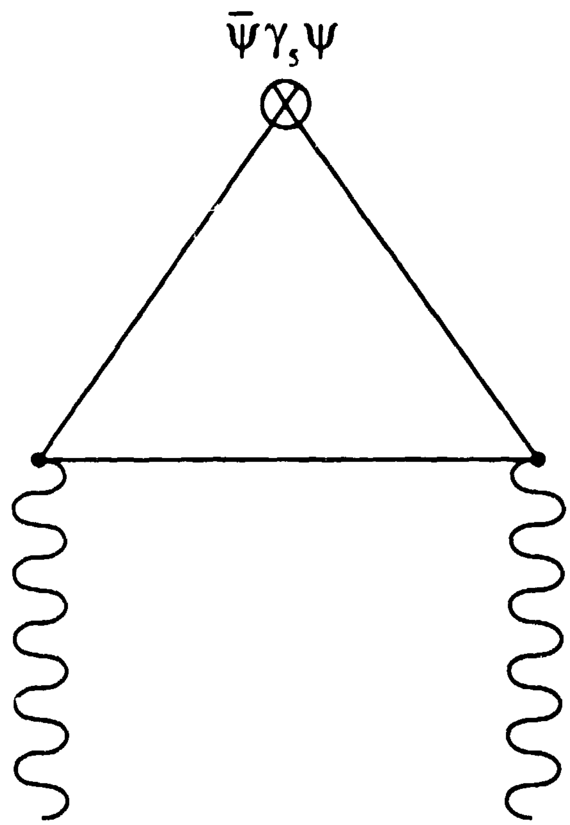

Figure 1

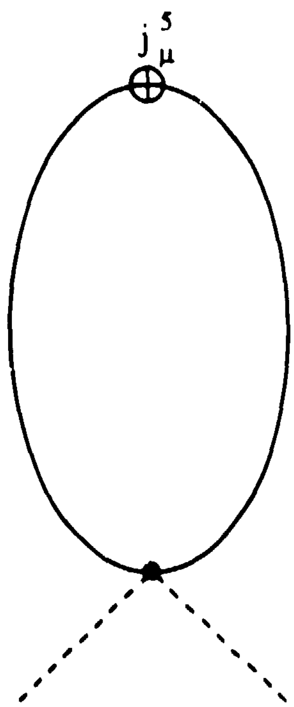

Figure 2

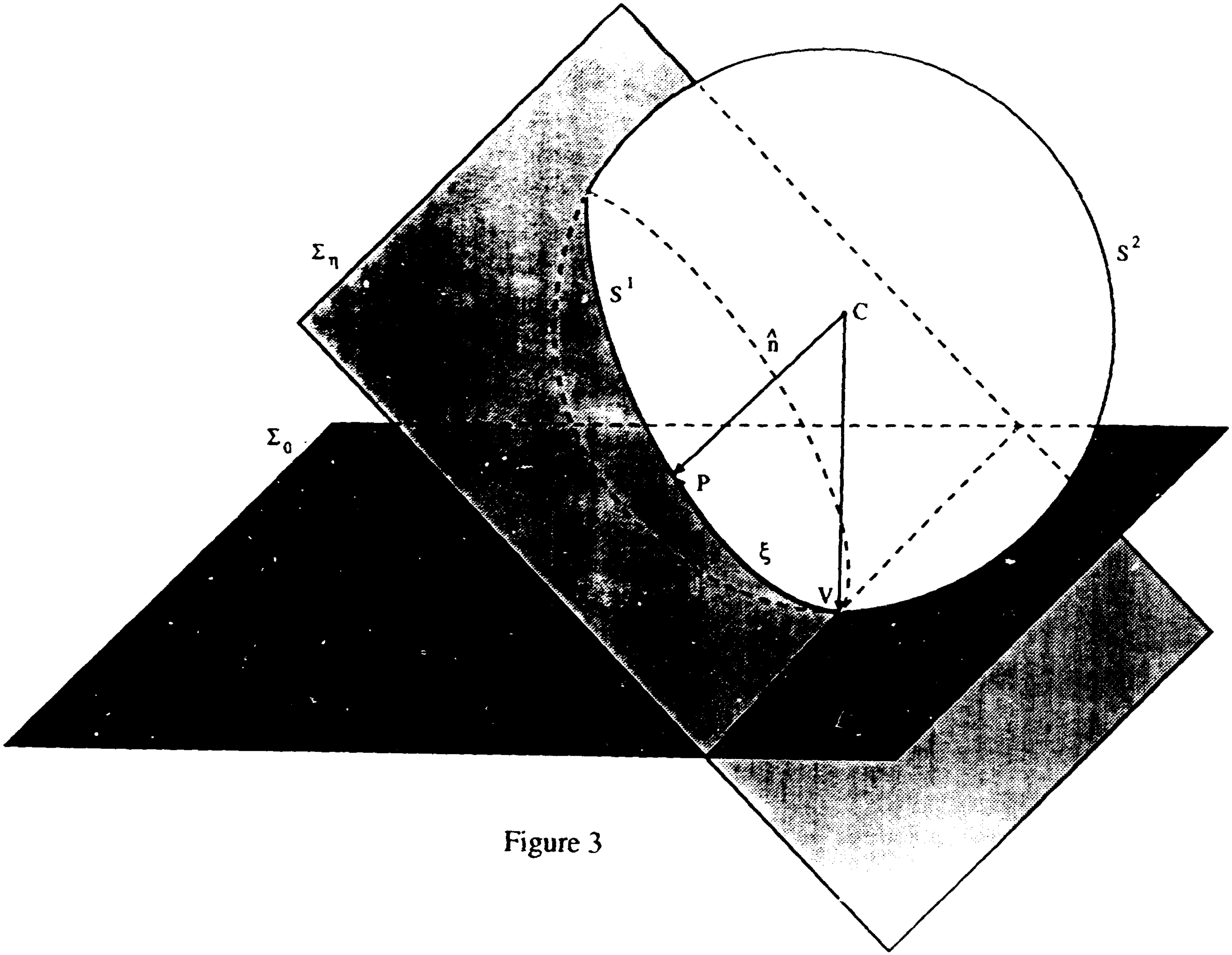

Figure 3