

Title:

Asymptotic Analysis of The Several Competitive Equations To Solve The Time-Dependent Neutron Transport Equation

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# ASYMPTOTIC ANALYSIS OF THE SEVERAL COMPETITIVE EQUATIONS TO SOLVE THE TIME-DEPENDENT NEUTRON TRANSPORT EQUATION

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## ABSTRACT

Using conventional diffusion limit analysis, we asymptotically compare three competitive time-dependent equations (the telegrapher's equation, the time-dependent Simplified  $P_2$  ( $SP_2$ ) equation, and the time-dependent Simplified Even-Parity (SEP) equation). The time-dependent  $SP_2$  equation contains higher order asymptotic approximations of the time-dependent transport equation than the other equations in a physical regime in which the time-dependent diffusion equation is the leading order approximation. In addition, we derive the multigroup *modified* time-dependent  $SP_2$  equation from the multigroup time-dependent transport equation by means of an asymptotic expansion in which the multigroup time-dependent diffusion equation is the leading order approximation. Numerical comparisons of the time-dependent diffusion, the telegrapher's, the time-dependent  $SP_2$ , and  $S_4$  solutions in 2-D X-Y geometry show that, in most cases, the  $SP_2$  solutions contain most of the transport corrections for the diffusion approximation.

## I. INTRODUCTION

A more or less complete model of the transport of neutrally-charged sub-atomic particles, as well as charged particles in some important physics regimes, is given by Boltzmann Equation.<sup>1</sup> The Discrete Ordinates ( $S_N$ ) method has been conventionally used to solve this equation numerically. The  $S_N$  method provides great accuracy but for many applications, especially time-dependent, multidimensional cases, such as reactor-core-disruptive problems, requires too much computational time. To overcome the high-cost of the conventional  $S_N$  equations, several competitive simplified equations, such as the telegrapher's equation<sup>2</sup> and the time-dependent Simplified  $P_2$  ( $SP_2$ ) equation<sup>3,4</sup> which could be solved with almost the same computational effort as the time-dependent diffusion equation, can be used to obtain relatively inexpensive, but approximate solutions to the transport solutions which are more accurate than the diffusion solutions.

An asymptotic analysis can be performed to show that, for an important class of problems, the diffusion equation is an asymptotic limit of the transport equation.<sup>5</sup> A recent paper by the authors<sup>6</sup> provides a broadened view of the result discussed above. It shows that the time-dependent  $SP_2$  equation has high-order asymptotic approximations of the time-dependent transport equation in a physical regime in which the conventional time-dependent diffusion equation is the leading-order approximation. In this paper, we compare the telegrapher's equation and the time-dependent  $SP_2$  equation using the conventional asymptotic approaches. As a result, we find that the time-dependent  $SP_2$  equation has the same

asymptotic approximations as the time-dependent transport equation up to the third order in a physical regime in which the time-dependent diffusion equation is the leading order approximation, whereas the telegrapher's equation has the same asymptotic approximations only up to the second order. This implies an advantage of the former over the latter.

We formulate the telegrapher's equation and the time-dependent SP<sub>2</sub> equation for one-group, 2-D X-Y geometry problems and compare numerical time-dependent diffusion, SP<sub>2</sub>, S<sub>4</sub>, and telegrapher's solutions in various classes of problems. As we have shown previously for slab geometry problems,<sup>4</sup> the results show that, in most cases, the time-dependent SP<sub>2</sub> solutions are significantly more accurate than the diffusion solutions and can be obtained with a very small fraction of the computational effort of an S<sub>4</sub> calculation. And even in the optically-thin regimes, in which both the time-dependent diffusion and the telegrapher's equation are no longer good approximations to the time-dependent transport equation, the time-dependent SP<sub>2</sub> solutions are quite close to the S<sub>4</sub> solutions.

In addition, as we have previously shown in a one-group problem,<sup>5</sup> we also derive here the multigroup *modified* time-dependent SP<sub>2</sub> equation from the multigroup time-dependent transport equation by means of an asymptotic expansion<sup>6</sup> in which the multigroup time-dependent diffusion equation is the leading order approximation.

According to the recent paper by Noh et al.,<sup>7</sup> in 2-D X-Y geometry, there is another competitive equation called the Simplified Even-Parity (SEP) equation which is much more computationally efficient than the even-parity equation. We derive here the time-dependent SEP equation in 2-D X-Y geometry and analyze this equation asymptotically. The results show that the time-dependent SEP equation also has the same asymptotic approximations as the time-dependent transport equation only up to the second order.

An outline of the remainder of this paper is as follows. In Sec. II, we carry out an asymptotic analysis of the telegrapher's equation and the time-dependent SP<sub>2</sub> equation in general geometry. In Sec. III, we present numerical comparisons of the time-dependent diffusion, SP<sub>2</sub>, S<sub>4</sub>, and the telegrapher's solutions. In Sec. IV, we asymptotically derive the multigroup *modified* time-dependent SP<sub>2</sub> equation from the multigroup time-dependent transport equation. In Sec. V, we carry out an asymptotic analysis of the time-dependent SEP equation in 2-D X-Y geometry. In Sec. VI, we conclude with a summary and recommend for future work.

## II. ASYMPTOTIC ANALYSIS OF THE TELEGRAPHER'S EQUATION AND THE TIME-DEPENDENT SP<sub>2</sub> EQUATION IN GENERAL GEOMETRY

In this section, we shall consider the one-group time-dependent transport problem in multidimensional, multiplying medium with delayed neutrons and isotropic scattering. Multigroup problems require a more complicated asymptotic analysis that we will discuss in section IV.

The one-group, multidimensional time-dependent transport equation with delayed neutrons is

$$\frac{1}{v} \frac{\partial \psi}{\partial t} + \Omega \cdot \nabla \psi + \sigma_r \psi = \{(1-\beta)v\sigma_f + \sigma_s\} \phi + \lambda C + Q,$$

and

$$\frac{\partial C}{\partial t} = \beta v \sigma_f \phi - \lambda C,$$

where the notations are standard (see Ref. 1).

If we consider the asymptotic scaling,<sup>8</sup>

$$\sigma_0 \Rightarrow \frac{\sigma_0}{\epsilon}, \sigma_1 \Rightarrow \epsilon \sigma_1, \sigma_2 \Rightarrow \epsilon \sigma_2, \sigma_3 \Rightarrow \frac{\sigma_3}{\epsilon} - \epsilon \sigma_3, Q \Rightarrow \epsilon Q, C \Rightarrow \epsilon C, v \Rightarrow \frac{v}{\epsilon}$$

the scaling defined above has been known to be one in which the diffusion equation is an asymptotic limit of the transport equation as  $\epsilon \rightarrow 0$ .

We expand,

$$\psi = \psi_0 + \epsilon \psi_1 + \epsilon^2 \psi_2 + \dots, \phi = \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \dots, \text{ and } C = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \dots$$

Applying these to the time-dependent transport equation and equating the coefficients of different powers of  $\epsilon$ , we obtain the following equations:

$$O(\epsilon^{-1}) \quad \psi_0 = \phi_0,$$

$$O(1) \quad \psi_1 = \phi_1 - \frac{1}{\sigma_1} \Omega \cdot \nabla \phi_0,$$

$$O(\epsilon) \quad \frac{1}{v} \frac{\partial \phi_0}{\partial t} - D \nabla^2 \phi_0 + \sigma_0 \phi_0 = (1 - \beta) v \sigma_1 \phi_0 + \lambda C_0 + Q,$$

$$\text{and} \quad \frac{\partial C_0}{\partial t} = \beta v \sigma_1 \phi_0 - \lambda C_0, \quad \text{where } D = \frac{1}{3\sigma_1},$$

$$O(\epsilon^2) \quad \frac{1}{v} \frac{\partial \phi_1}{\partial t} - D \nabla^2 \phi_1 + \sigma_0 \phi_1 = (1 - \beta) v \sigma_1 \phi_1 + \lambda C_1,$$

$$\text{and} \quad \frac{\partial C_1}{\partial t} = \beta v \sigma_1 \phi_1 - \lambda C_1,$$

$$O(\epsilon^3) \quad \left( \frac{1}{v\sigma_1} \frac{\partial}{\partial t} - \frac{4}{15\sigma_1^2} \nabla^2 \right) D \nabla^2 \phi_0 + \frac{1}{v} \frac{\partial \phi_2}{\partial t} - D \nabla^2 \phi_2 + \sigma_0 \phi_2 = (1 - \beta) v \sigma_1 \phi_2 + \lambda C_2,$$

$$\text{and} \quad \frac{\partial C_2}{\partial t} = \beta v \sigma_1 \phi_2 - \lambda C_2.$$

#### A. The Telegrapher's Equation

The one-group, multidimensional telegrapher's equation with delayed neutrons is

$$\frac{1}{\sigma_1 v^2} \frac{\partial^2 \phi}{\partial t^2} + \frac{1}{v} \left( 1 + \frac{\sigma_0}{\sigma_1} \right) \frac{\partial \phi}{\partial t} - \frac{1}{\sigma_1 v} \frac{\partial Q}{\partial t} - D \nabla^2 \phi + \sigma_0 \phi = (1 - \beta) v \sigma_1 \phi + \lambda C + Q,$$

$$\text{and} \quad \frac{\partial C}{\partial t} = \beta v \sigma_1 \phi - \lambda C.$$

Applying the same scaling and expansion to these equations, we obtain the following equations:

$$O(\epsilon) \quad \frac{1}{v} \frac{\partial \phi_0}{\partial t} - D \nabla^2 \phi_0 + \sigma_a \phi_0 = (1-\beta) \nu \sigma_f \phi_0 + \lambda C_0 + Q,$$

and

$$\frac{\partial C_0}{\partial t} = \beta \nu \sigma_f \phi_0 - \lambda C_0,$$

$$O(\epsilon^2) \quad \frac{1}{v} \frac{\partial \phi_1}{\partial t} - D \nabla^2 \phi_1 + \sigma_a \phi_1 = (1-\beta) \nu \sigma_f \phi_1 + \lambda C_1,$$

and

$$\frac{\partial C_1}{\partial t} = \beta \nu \sigma_f \phi_1 - \lambda C_1,$$

$$O(\epsilon^3) \quad \frac{1}{\nu \sigma_f} \frac{\partial}{\partial t} (D \nabla^2 \phi_0) + \frac{1}{v} \frac{\partial \phi_2}{\partial t} - D \nabla^2 \phi_2 + \sigma_a \phi_2 = (1-\beta) \nu \sigma_f \phi_2 + \lambda C_2,$$

and

$$\frac{\partial C_2}{\partial t} = \beta \nu \sigma_f \phi_2 - \lambda C_2.$$

Comparing these asymptotic approximations with those of the time-dependent transport equation, we find that the telegrapher's equation has the same asymptotic approximations as the time-dependent transport equation up to the second order in a physical regime in which the time-dependent diffusion equation is the leading order approximation.

#### B. Time-Dependent SP<sub>3</sub> Equation

The one-group, multidimensional time-dependent SP<sub>3</sub> equations with delayed neutrons are

$$\frac{1}{v} \frac{\partial \psi_0}{\partial t} + \nabla \cdot \mathbf{v}_1 + \sigma_a \psi_0 = (1-\beta) \nu \sigma_f \psi_0 + \lambda C + Q,$$

$$\frac{1}{v} \frac{\partial \psi_1}{\partial t} + \frac{1}{3} \nabla \psi_0 + \frac{2}{3} \nabla \psi_1 + \sigma_a \psi_1 = 0,$$

$$\frac{1}{v} \frac{\partial \psi_2}{\partial t} + \frac{2}{5} \nabla \psi_1 + \sigma_a \psi_2 = 0,$$

and

$$\frac{\partial C}{\partial t} = \beta \nu \sigma_f \psi_0 - \lambda C.$$

Applying the same scaling and expansion to these equations, we obtain the following equations:

$$O(\epsilon) \quad \frac{1}{v} \frac{\partial \phi_0}{\partial t} - D \nabla^2 \phi_0 + \sigma_a \phi_0 = (1-\beta) \nu \sigma_f \phi_0 + \lambda C_0 + Q,$$

and

$$\frac{\partial C_0}{\partial t} = \beta \nu \sigma_f \phi_0 - \lambda C_0,$$

$$O(\epsilon^2) \quad \frac{1}{v} \frac{\partial \phi_1}{\partial t} - D \nabla^2 \phi_1 + \sigma_a \phi_1 = (1-\beta) \nu \sigma_f \phi_1 + \lambda C_1,$$

and 
$$\frac{\partial C_1}{\partial t} = \beta v \sigma_1 \phi_1 - \lambda C_1 .$$

$O(\epsilon^3)$  
$$\left( \frac{1}{v \sigma_1} \frac{\partial}{\partial t} - \frac{4}{15 \sigma_1^2} \nabla^2 \right) D \nabla^2 \phi_0 + \frac{1}{v} \frac{\partial \phi_1}{\partial t} - D \nabla^2 \phi_1 + \sigma_s \phi_1 = (1 - \beta) v \sigma_1 \phi_1 + \lambda C_1$$

and 
$$\frac{\partial C_2}{\partial t} = \beta v \sigma_2 \phi_2 - \lambda C_2 .$$

Thus, the time-dependent SP<sub>2</sub> equation has the same asymptotic approximations as the time-dependent transport equation up to the third order in a physical regime in which the time-dependent diffusion equation is the leading order approximation. Conclusively, we find that the time-dependent SP<sub>2</sub> equation contains higher-order asymptotic corrections to the time-dependent diffusion equation than the telegrapher's equation. Thus, one might expect the higher accuracy from the time-dependent SP<sub>2</sub> equation, especially in diffusive regimes.

### III. NUMERICAL CALCULATIONS AND RESULTS

First, differencing the time variable in the time-dependent SP<sub>2</sub> equations using a fully-implicit scheme and formulating an equation with the scalar flux,  $\psi_0 (= \phi)$ , only, we obtain;

$$-\nabla \cdot D' \nabla \left\{ \phi^{n+1} + \frac{4}{5 \sigma_s'} (\sigma_s' \phi^{n+1} - Q) \right\} + \sigma_a' \phi^{n+1} = S^n, \quad (1)$$

where 
$$\sigma_s' = \sigma_s + \frac{1}{v \Delta t}, \quad \sigma_a' = \sigma_a + \frac{1}{v \Delta t}, \quad D' = \frac{1}{3 \sigma_s'}$$

and 
$$S^n = Q + \frac{1}{v \Delta t} \phi^n - \nabla \cdot D' \nabla \left( \frac{1}{v \Delta t} \phi^n - \frac{5}{2} \psi_1^n \right) - \frac{1}{\sigma_s'} \nabla \cdot \left( \frac{1}{v \Delta t} \psi_1^n \right).$$

Equation (1) is in the form of a conventional diffusion equation. Spatially differencing this equation in 2-D X-Y geometry, we obtain the matrix equation  $A \cdot \phi^{n+1} = S^n$  at each time step,  $n$ , where  $A$  is a five-diagonal symmetric matrix. Since these equations are very similar to those obtained from differencing the time-dependent diffusion equation, solutions are obtained with almost the same computational effort and with a very small fraction of the time needed for the time-dependent S<sub>2</sub> solutions.

We have compared the time-dependent diffusion, the telegrapher's, the time-dependent SP<sub>2</sub>, and S<sub>2</sub> solutions in various classes of 2-D X-Y geometry problems with and without delayed neutrons and the numerical results show that, in most cases, the time-dependent SP<sub>2</sub> solutions are significantly more accurate than the conventional time-dependent diffusion and the telegrapher's solutions and can be obtained with a very small fraction of the computational effort of an S<sub>2</sub> calculation (see Table 1). And even in the optically-thin regimes, in which both the time-dependent diffusion and the telegrapher's equation are no longer good approximations to the time-dependent transport equation, the time-dependent SP<sub>2</sub> solutions are quite close to the S<sub>2</sub> solutions.

The configuration of a sample problem is shown in Fig. 1. In Fig. 1,  $\sigma$  is the total cross section with a unit of [1/cm],  $c$  is the scattering ratio defined as  $\sigma_s/\sigma$ , and  $Q$  is the internal source with a unit of [1/cm<sup>3</sup>/sec]. The left and bottom boundaries of the system are reflective and the right and top

boundaries are vacuum. We start with zero initial fluxes everywhere and compare the time-dependent variations of the scalar fluxes which are calculated by the time-dependent diffusion, SP<sub>2</sub>, S<sub>4</sub>, and the telegrapher's equations at point 1 and 2.

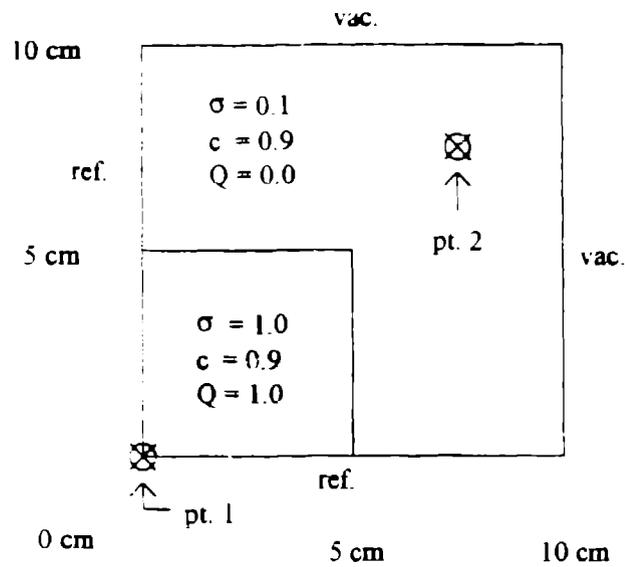


Fig. 1 Sample Problem

The results are shown in Fig. 2 and Fig. 3. At each point, the SP<sub>2</sub> solutions are much closer to the S<sub>4</sub> solutions than the diffusion and the telegrapher's solutions. Moreover, as shown in Fig. 3, SP<sub>2</sub> solutions are quite close to the S<sub>4</sub> solutions even in the optically-thin regimes in which both the diffusion and the telegrapher's equation are no longer good approximations.

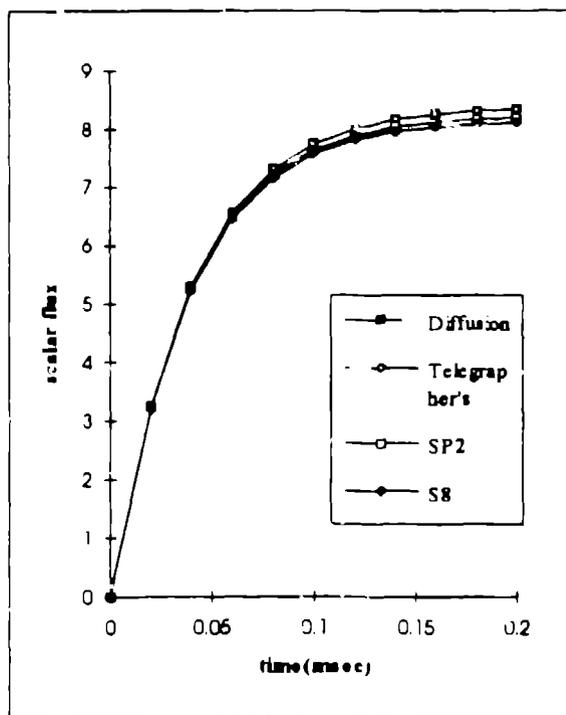


Fig. 2 Scalar Flux Variation at Pt. 1

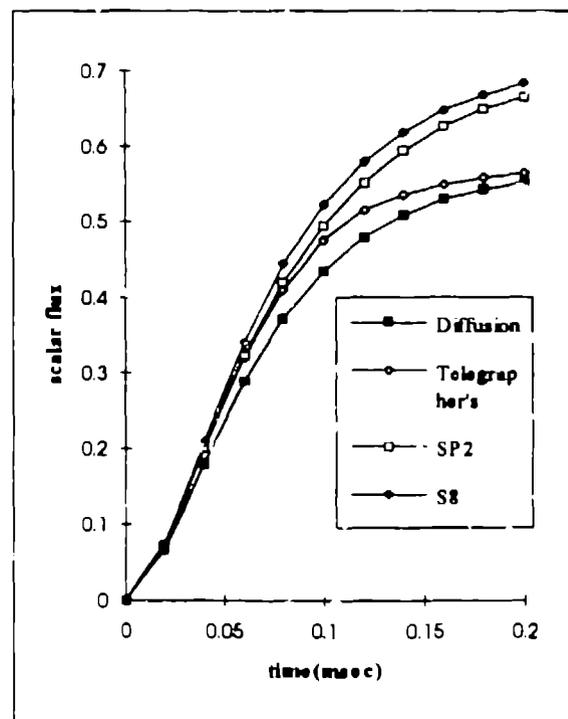


Fig. 3 Scalar Flux Variation at Pt. 2

Equation	Diffusion	Telegrapher's	SP2	S8
CPU time (sec)	0.235	0.264	0.291	6.138

Table 1 Elapsed CPU Time for Each Equation

#### IV. ASYMPTOTIC DERIVATION OF THE MULTIGROUP MODIFIED TIME-DEPENDENT SP<sub>2</sub> EQUATION

As we have shown in Section II, delayed neutrons do not alter the results of the asymptotic analyses and thus, for simplicity, we shall not consider delayed neutrons in this section ( $\sigma_d$  can be included in  $\sigma_s$ ). For the same reason, we only present here the asymptotic analysis for planar geometry, however, we have performed the multidimensional analysis and obtained the same results.

The multigroup, planar geometry time-dependent transport equation with isotropic scattering is<sup>4</sup>

$$\frac{1}{v} \frac{\partial}{\partial t} \psi(x, \mu, t) + \mu \frac{\partial \psi}{\partial x} + \sigma_t \psi = \frac{\sigma_s}{2} \int_{-1}^1 \psi(x, \mu', t) d\mu' + \frac{1}{2} Q(x, t), \quad (2)$$

where

- $\psi(x, \mu, t)$  = angular flux (a G vector)
- $Q(x, t)$  = source (a G vector)
- $\sigma_t$  = total cross section (a G×G diagonal matrix)
- $\sigma_s$  = scattering cross section (a G×G matrix)
- G = number of energy groups.

Integrating Eq. (2) over the whole angle,  $\frac{1}{2} \int_{-1}^1 Eq.(2) d\mu$ , and introducing

$$\phi(x, t) = \int_{-1}^1 \psi(x, \mu, t) d\mu = \text{scalar flux (a G vector)},$$

we obtain

$$\frac{1}{v} \frac{\partial}{\partial t} \phi(x, t) + \frac{\partial}{\partial x} \int_{-1}^1 \mu' \psi(x, \mu', t) d\mu' + (\sigma_t - \sigma_s) \phi = Q(x, t). \quad (3)$$

And from Eq. (2) and Eq. (3), we also obtain

$$\frac{1}{v} \frac{\partial}{\partial t} \psi(x, \mu, t) + \mu \frac{\partial \psi}{\partial x} + \sigma_t \psi = \frac{1}{2} \left\{ \frac{1}{v} \frac{\partial}{\partial t} \phi(x, t) + \sigma_t \phi + \frac{\partial}{\partial x} \int_{-1}^1 \mu' \psi(x, \mu', t) d\mu' \right\}. \quad (4)$$

Introducing into Eq. (3) and Eq. (4) the asymptotic scaling:

$$\sigma_t \Rightarrow \frac{\sigma_t}{\epsilon}, \quad \sigma_s \Rightarrow \frac{\sigma_s}{\epsilon}, \quad Q \Rightarrow \epsilon Q, \quad v \Rightarrow \frac{v}{\epsilon},$$

we obtain

$$\frac{\varepsilon}{v} \frac{\partial}{\partial t} \phi(x, t) + \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \mu' \psi(x, \mu', t) d\mu' + \frac{1}{\varepsilon} (\sigma_+ - \sigma_-) \phi = \varepsilon Q(x, t), \quad (5)$$

and

$$\frac{\varepsilon}{v} \frac{\partial}{\partial t} \psi(x, \mu, t) + \mu \frac{\partial \psi}{\partial x} + \frac{\sigma_-}{\varepsilon} \psi = \frac{1}{2} \left\{ \frac{\varepsilon}{v} \frac{\partial}{\partial t} \phi(x, t) + \frac{\sigma_+}{\varepsilon} \phi + \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \mu' \psi(x, \mu', t) d\mu' \right\}. \quad (6)$$

And we introduce into Eq. (6) the expansion:

$$\psi = \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \dots, \quad (7)$$

and, equate the coefficients of different powers of  $\varepsilon$ , we obtain the following equations:

$$O(\varepsilon^{-1}) \quad \psi_0 = \frac{1}{2} \phi,$$

$$O(1) \quad \psi_1 = (-\mu) \left( \frac{1}{\sigma_+} \frac{\partial}{\partial x} \right) \frac{\phi}{2},$$

$$O(\varepsilon) \quad \psi_2 = (\mu^2 - \frac{1}{3}) \left( \frac{1}{\sigma_+} \frac{\partial}{\partial x} \right)^2 \frac{\phi}{2},$$

$$O(\varepsilon^2) \quad \psi_3 = \frac{1}{v\sigma_+} \frac{\partial}{\partial t} \left( \frac{\mu}{\sigma_+} \frac{\partial}{\partial x} \right) \frac{\phi}{2} + (\frac{\mu}{3} - \mu') \left( \frac{1}{\sigma_+} \frac{\partial}{\partial x} \right)^3 \frac{\phi}{2},$$

$$O(\varepsilon^3) \quad \psi_4 = -\frac{1}{v\sigma_+} \frac{\partial}{\partial t} (\mu^2 - \frac{1}{3}) \left( \frac{1}{\sigma_+} \frac{\partial}{\partial x} \right)^2 \frac{\phi}{2} + (\mu^4 - \frac{\mu^2}{3} - \frac{4}{45}) \left( \frac{1}{\sigma_+} \frac{\partial}{\partial x} \right)^4 \frac{\phi}{2},$$

and so on. Introducing these moments into Eq.(7), we obtain an asymptotic expansion for  $\psi$  in terms of  $\phi$ . Applying this expansion into Eq. (5), we finally obtain the equation for  $\phi$  as follows;

$$\frac{\varepsilon}{v} \frac{\partial}{\partial t} \phi(x, t) - \frac{\partial}{\partial x} \left[ \frac{\varepsilon}{3} \left( \frac{1}{\sigma_+} \frac{\partial}{\partial x} \right) - \frac{\varepsilon^2}{3} \left( \frac{1}{\sigma_+} \frac{\partial}{\partial x} \right) \left\{ \frac{1}{v\sigma_+} \frac{\partial}{\partial t} - \frac{4}{15} \left( \frac{1}{\sigma_+} \frac{\partial}{\partial x} \right)^2 \right\} + O(\varepsilon^3) \right] \phi + \frac{1}{\varepsilon} (\sigma_+ - \sigma_-) \phi = \varepsilon Q(x, t). \quad (8)$$

If we delete the terms of  $O(\varepsilon^3)$  and higher in Eq. (8) and reapply the scaling reversely, we obtain,

$$\frac{1}{v} \frac{\partial}{\partial t} \phi(x, t) - \frac{\partial}{\partial x} \frac{1}{3\sigma_+} \frac{\partial}{\partial x} \phi + (\sigma_+ - \sigma_-) \phi = Q(x, t). \quad (9)$$

These are the conventional multigroup time-dependent diffusion equations.

If we delete the terms of  $O(\varepsilon^3)$  and higher in Eq. (8) and introduce Eq. (9) into the resultant equation, we obtain,

$$\left(\frac{1}{v\sigma} \frac{\partial}{\partial t} - \frac{4}{\partial x} \frac{1}{15\sigma} \frac{\partial}{\partial x} \frac{1}{\sigma}\right) \left(-\frac{1}{v} \frac{\partial}{\partial t} \phi + (\sigma_+ - \sigma_-) \phi - Q\right) + \frac{1}{v} \frac{\partial}{\partial t} \phi(x,t) - \frac{\partial}{\partial x} \frac{1}{3\sigma} \frac{\partial}{\partial x} \phi + (\sigma_+ - \sigma_-) \phi = Q(x,t). \quad (10)$$

These are the multigroup *modified* time-dependent SP<sub>3</sub> equations<sup>4</sup> which could be obtained by neglecting the time derivative term of the second moment of angular flux,  $\partial v_2 / \partial t$ , in the multigroup time-dependent SP<sub>3</sub> equations. We carry out the same asymptotic analysis on this equation as we do in Sec. II, and the results show that the *modified* time-dependent SP<sub>3</sub> equation has also the same asymptotic approximations as the time-dependent transport equation up to the third order in a physical regime in which the time-dependent diffusion equation is the leading order approximation. The advantages of using *modified* time-dependent SP<sub>3</sub> equation for the one-group case are discussed in Ref. 6.

## V. ASYMPTOTIC ANALYSIS OF THE TIME-DEPENDENT SEP EQUATION IN 2-D X-Y GEOMETRY

In the recent paper by Noh et al.,<sup>5</sup> the authors introduced the Simplified Even-Parity (SEP) equation which could be obtained from the even-parity equation using the assumption,  $\chi(\mu, \eta) = \chi(\mu, -\eta)$ , in 2-D X-Y geometry ( $\chi$  is even-parity flux). With the resulting elimination of the cross-derivative terms of the even-parity equation, SEP equation is much more computationally efficient than the original even-parity equation. All the work in that paper deals with the steady-state case.

In this paper, we derive the time-dependent SEP equation from the time-dependent transport equation using the same assumption. This derivation is straightforward, and we will not show it here for brevity.

The time-dependent SEP equation in 2-D X-Y geometry is

$$\frac{1}{\sigma_+ v^2} \frac{\partial^2 \chi}{\partial t^2} + \frac{2}{v} \frac{\partial \chi}{\partial t} - \frac{\mu^2}{\sigma_+} \frac{\partial^2 \chi}{\partial x^2} - \frac{\eta^2}{\sigma_+} \frac{\partial^2 \chi}{\partial y^2} + \sigma_+ \chi = \left(\frac{1}{v\sigma_+} \frac{\partial}{\partial t} + 1\right) (\sigma_+ \phi + Q). \quad (11)$$

Applying the same scaling and expansion to this equation as in Sec. II, we obtain the following equations;

$$O(\epsilon) \quad \frac{1}{v} \frac{\partial \phi_0}{\partial t} - D \nabla^2 \phi_0 + \sigma_+ \phi_0 = Q,$$

$$O(\epsilon^2) \quad \frac{1}{v} \frac{\partial \phi_1}{\partial t} - D \nabla^2 \phi_1 + \sigma_+ \phi_1 = 0,$$

$$O(\epsilon^3) \quad \left(\frac{1}{v\sigma_+} \frac{\partial}{\partial t} - \frac{4}{15\sigma_+^2} \nabla^4\right) D \nabla^2 \phi_0 + \frac{4}{15\sigma_+^2} \frac{\partial^4 \phi_0}{\partial x^2 \partial y^2} + \frac{1}{v} \frac{\partial \phi_1}{\partial t} - D \nabla^2 \phi_1 + \sigma_+ \phi_1 = 0.$$

Comparing these asymptotic approximations with those of the time-dependent transport equation in Sec. II, we find that the time-dependent SEP equation also has the same asymptotic approximations as the time-dependent transport equation only up to the second order in a physical regime in which the time-dependent diffusion equation is the leading order approximation.

In addition, we derive here the *modified* time-dependent SEP equation which could be obtained by neglecting the time derivative term of the odd-parity flux,  $\partial \beta / \partial t$ , in the time-dependent transport equation. The *modified* time-dependent SEP equation in 2-D X-Y geometry is

$$\frac{1}{v} \frac{\partial \chi}{\partial t} - \frac{\mu^2}{\sigma_t} \frac{\partial^2 \chi}{\partial x^2} - \frac{\eta^2}{\sigma_t} \frac{\partial^2 \chi}{\partial y^2} + \sigma_a \chi = \sigma_s \phi + Q. \quad (12)$$

Compared with Eq. (11), Eq. (12) contains only the first order time derivative term of  $\chi$ . Accordingly, solutions can be obtained more computationally efficiently than the original time-dependent SEP equation. We carry out the same asymptotic analysis on this equation and the results are as follows:

$$O(\epsilon) \quad \frac{1}{v} \frac{\partial \phi_0}{\partial t} - D \nabla^2 \phi_0 + \sigma_a \phi_0 = Q.$$

$$O(\epsilon^2) \quad \frac{1}{v} \frac{\partial \phi_1}{\partial t} - D \nabla^2 \phi_1 + \sigma_a \phi_1 = 0.$$

$$O(\epsilon^3) \quad \left( -\frac{4}{15\sigma_t^2} \nabla^2 \right) D \nabla^2 \phi_0 + \frac{4}{15\sigma_t^2} \frac{\partial^2 \phi_0}{\partial x^2 \partial y^2} + \frac{1}{v} \frac{\partial \phi_2}{\partial t} - D \nabla^2 \phi_2 + \sigma_a \phi_2 = 0.$$

Thus, the time-dependent *modified* SEP equation also has the same asymptotic approximations as the time-dependent transport equation up to the second order.

## VI. CONCLUSIONS

We have shown that the time-dependent SP<sub>2</sub> equation contains higher order asymptotic approximations to the time-dependent transport equation than the other competitive time-dependent equations in a physical regime in which the time-dependent diffusion equation is the leading order approximation. And numerical results show that, in most of cases in 2-D X-Y geometry, the time-dependent SP<sub>2</sub> solutions are a lot more accurate than the time-dependent diffusion and the telegrapher's solutions. In addition, we derive the multigroup *modified* time-dependent SP<sub>2</sub> equation from the multigroup time-dependent transport equation by an asymptotic expansion. These results imply that, in many problems in which the conventional time-dependent diffusion equation is not sufficiently accurate, the time-dependent SP<sub>2</sub> equation could give a lot more accurate solutions than the time-dependent diffusion equation with almost the same computational efforts. A numerical test for the time-dependent SP<sub>2</sub> equation in more general cases, such as in 2-D R-Z geometry, with a multigroup energy treatment, is being carried out by the authors to validate the advantages of the time-dependent SP<sub>2</sub> equation in more realistic problems. Those conclusions and future work mentioned above for the time-dependent SP<sub>2</sub> equation are also applicable to the *modified* time-dependent SP<sub>2</sub> equation.

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