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Scale-dependent Darcy flows in composite media

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Abstract We develop probabilities and statistics for the parameters of Darcy flows through saturated porous media composed of units of different materials. Our probability model has two levels. On the local level, a porous medium is composed of disjoint, statistically homogeneous volumes (or blocks) each of which consists of a single type of material. On a larger scale, a porous medium is an arrangement of blocks whose extent and location are defined by uncertain boundaries. Using this two-scaled model, we derive general formulas for the probability distribution of hydraulic conductivity and its mean; then we develop a closed-form expression for mean head in one dimension. We express distributions and parameters in terms of mixtures of locally homogeneous block distributions weighted by large-scale block membership probabilities.

INTRODUCTION

According to Darcy's Law, water in a porous medium moves down pressure gradients influenced by the medium's permeability, or equivalently, by a hydraulic conductivity tensor. Darcy's Law has been used to predict groundwater flows across a very wide range of experimental scales; however, different scales generally require different parameterizations, even within the same site. This may be due to sampling across increasing levels of material heterogeneity as the volumes used to average parameters expand. This begs a couple of critical questions: Is there a simple scaling law for permeability and other hydraulic parameters? How can we average flow parameters when the scale of averaging includes significant material heterogeneity? We address the second question in this note by developing ensemble averages for flow parameters that include multiple scales of heterogeneity. In later papers we will relate our multiscale averages to the existence of simple scaling laws.

It has been hypothesized that effective flow parameters can be obtained by averaging local parameter values over representative elementary volumes (Bear, 1972). A representative elementary volume (REV) exists within a porous medium if there is a range of measurement over which averaged parameter values are approximately constant. Lately, the very existence of the traditional REV concept has been questioned (Neuman, 1994), because examinations of data indicate that permeabilities and other flow parameters vary over all scales (Neuman, 1994; Gelhar, 1993). The REV hypothesis is further complicated by our uncertain knowledge of the detailed structure of porous media. In fact, permeability and other parameters are usually observed at a relatively small number of locations in groundwater studies.

It has become common to quantify this uncertainty through probabilistic models in which permeability is represented as a stochastic process. Then the dependent flux, $\mathbf{q}(\mathbf{x})$, and head, $h(\mathbf{x})$, in Darcy's Law are also stochastic processes. In most studies conductivity, $K(\mathbf{x})$, is represented as the sum of a mean, $\overline{K}(\mathbf{x})$, and a random deviation, $K'(\mathbf{x})$, so that $K(\mathbf{x}) = \overline{K}(\mathbf{x}) + K'(\mathbf{x})$. Similarly, $h(\mathbf{x}) = \overline{h}(\mathbf{x}) + h'(\mathbf{x})$. In a steady-state flow regime, the averaged Darcy's Law becomes

$$\overline{\mathbf{q}}(\mathbf{x}) = -\overline{K}(\mathbf{x})\,\nabla\overline{h}(\mathbf{x}) + \overline{\mathbf{r}}(\mathbf{x}) \tag{1}$$

where $\overline{\mathbf{q}}(\mathbf{x})$ is the average flux. The averaged equation (1) consists of a deterministic mean part, $\overline{K}(\mathbf{x})\nabla\overline{h}(\mathbf{x})$, and a deterministic residual flux, $\overline{\mathbf{r}}(\mathbf{x}) = -\overline{K'(\mathbf{x})\nabla h'(\mathbf{x})}$. Solutions for $\overline{h}(\mathbf{x})$ require the statistics of $K(\mathbf{x})$, and in most cases, a method for closing an expansion of the residual flux. Usually $\overline{h}(\mathbf{x})$ is approximated through perturbation expansions based on σ_Y^2, the variance of (natural) log conductivity. This approach works well so long as σ_Y^2 is small.

Small σ_Y^2 is often a reasonable assumption within a volume, or block, composed of a single type of material since the properties of every point in a block of a single material have been generated by basically the same physical processes. This is frequently used to justify a further assumption, that the permeability field is statistically homogeneous within a block. The case is different at larger scales where permeability statistics are affected by variations among blocks of different materials, for instance, in stratified porous media. Here permeability variations can be expected to be large from one block to another.

Uncertainty arises from two distinct sources in this view of composite media: small scale within-block variations and large scale across-block variations. Each block corresponds to a diffferent ensemble of porous media by definition. Thus, permeability fields are statistically heterogeneous when viewed across blocks. Solutions for $\overline{h}(\mathbf{x})$ require an expression of $\overline{K}(\mathbf{x})$ that reflects heterogeneity at the larger, across-block scale. Furthermore, closure approximations for the residual flux must accomodate the large variances that are due to across-block variability. Across-block variation adds the spatial extent and arrangement of the blocks themselves as a new element of randomness in the analysis of the stochastic Darcy model. We show that the permeability and head random fields can be averaged by dealing with each scale explicitly and more or less separately.

The essence of our note is that the permeability and head fields can be averaged by using large-scale probabilities of across-block geometry to weight small-scale within-block probabilities. As a result, we can apply the convenient properties of small within-block σ_Y^2 to perturbation expansions, while at the same time we allow statistical non-homogeneity due to random variations in the large-scale geometry of blocks. Suppose the usual assumptions of stochastic hydrogeology hold: the statistics of the block structure (for instance, the stratigraphy) of a porous medium are known, as are the distributions of permeability within individual blocks (e.g., strata). The result of averaging is a mixture of small-scale probability distributions that leads to a straightforward expression for the critical parameter, $\overline{K}(\mathbf{x})$. Next assume, as usual, that σ_Y^2 is small within blocks. Then our approach also extends the range of perturbation analysis to many heterogeneous domains, although we do not explicitly show that here. It substitutes the relatively tractable problem of determining the spatial distribution of disjoint blocks of homogeneous material for the difficult problem of dealing with large perturbation variances.

Our approach is similar in its goals to the Boolean algorithms used in geostatistical simulations of heterogeneous random fields (Deutsch & Journel, 1992); however, the methods and results are completely different. We give explicit expressions for the univariate probability density of $K(\mathbf{x})$ and of derived quantities like $\overline{K}(\mathbf{x})$ and the moments of $h(\mathbf{x})$ in the next two sections. These expressions can be examined qualitatively to understand the general behavior of the averaged flow system, which of course, is not possible with a simulation-based approach. Simulations are generally used to estimate moments of $h(\mathbf{x})$, thereby requiring large numbers of Monte Carlo trials. On the other hand, analytical methods are computationally efficient. Expressions for the moments of $h(\mathbf{x})$ arise directly from (1) and related equations.

Multi-scale averaging provides a natural framework for assimilating the results of different methods of aquifer characterization. First, the multi-scale method includes the kinds of spatially distributed material heterogeneities that are the result of most characterization studies; second, error models for characterization techniques can be explicitly included in models of random block boundaries; and third, the outputs of different characterizations can be combined using standard techniques like Bayesian updating since the multi-scale model is probabilistic. This is different from the ordinary approach in stochastic hydrology where observations of permeability are lumped together without regard to material distribution and are then used to estimate statistics for an equivalent continuum. In most cases this results in estimates of $\overline{K}(\mathbf{x})$ that are very coarse and estimates of σ_Y^2 that are very large.

RE-SCALED PERMEABILITY

In general, a porous medium contains blocks of many types of material, each material type having its own (multivariate) permeability random field determined by a probability density. Here we consider porous media composed of only two types of material for simplicity, but extensions to multiple materials are obvious. A point, $\mathbf{x}$, of the medium lies in material M_1 with probability $P(\mathbf{x} \in M_1)$ and in material M_2 with probability $P(\mathbf{x} \in M_2) = 1 - P(\mathbf{x} \in M_1)$. We suppose that β is a random boundary separating the medium into blocks of type M_1 and M_2 according to probability density function $p(\beta)$. Also, conductivity, $K(\mathbf{x})$, has probability density functions $p_1(K(\mathbf{x}))$ or $p_2(K(\mathbf{x}))$ respectively within the blocks. Usually, $p_i(K(\mathbf{x}))$ is log normal. We drop the dependence of K on $\mathbf{x}$ below except where it is needed for clarity. The joint density, $p(K,\beta)$, defines the multiscale within- and across-block process.

Our goal, the rescaled permeability density, $p(K)$, is the marginal of $p(K,\beta) = p(K \mid \beta)p(\beta)$. Since the conditional probability fixes the boundary, we can use indicator functions to rewrite $p(K \mid \beta) = I_1(x;\beta)p_1(K) + I_2(x;\beta)p_2(K)$. The indicator function $I_i(x;\beta) = 1$ if $\mathbf{x}$ is in a block of type M_i and is 0 otherwise. Thus,

$$\begin{aligned} p(K) &= \int [I_1(x;\beta)\, p_1(K)\, p(\beta) + I_2(x;\beta)\, p_2(K)\, p(\beta)]\, d\beta \\ &= p_1(K)\, P(\mathbf{x} \in M_1) + p_2(K)\, P(\mathbf{x} \in M_2) \qquad (2) \end{aligned}$$

since $\int I_i(\mathbf{x};\beta)\, p(\beta)\, d\beta = P(\mathbf{x} \in M_i)$; *i.e.*, since the probability that $\mathbf{x}$ is an element of the ith block, is just the measure of all β for which $\mathbf{x}$ falls in block M_i. In other words, (2) states that $p(K)$ is the weighted sum of the within-block densities, p_1 and p_2 where the weighting function is the probability of block membership for a point. When the point $\mathbf{x}$ is deep within a block of type M_i, $P(\mathbf{x} \in M_i) \approx 1$ and $p(K) \approx p_i(K)$. As $\mathbf{x}$ approaches β, $p(K)$ approaches the average of p_1 and p_2. In an unbounded one-dimensional porous medium where a single block of material M_1 is below a single block of material M_2, $P(x \in M_1) = \int_x^\infty p(\beta)\, d\beta$ and $P(x \in M_2) = \int_{-\infty}^x p(\beta)\, d\beta$. It is obvious from (2) that the ensemble average of $K(x)$ must be a weighted sum of mean permeabilities in the two material types, $\overline{K}(x) = \overline{K_1}\, P(x \in M_1) + \overline{K_2}\, P(x \in M_2)$.

CASE STUDY: ONE-DIMENSIONAL FLOW

We use steady-state flow in a bounded one- dimensional porous medium to illustrate averaging (1). In this simple case we can derive closed-form expressions for mean hydraulic head, $\overline{h}(\mathbf{x})$, and the derivative of $\overline{h}(x)$, which is the reciprocal of effective conductivity. Both expressions depend on the variance of the location, β, of the boundary between material types. We are currently investigating similar questions in more complicated, higher dimensional media.

Since the one-dimensional medium is bounded, we may as well suppose it is $[0,1]$. The medium consists of two materials with random hydraulic conductivities $K_1(x)$ ($0 < x < \beta$) and $K_2(x)$ ($\beta < x < 1$), where β is the contact point between the two materials. The exact position of the point of contact is not known; instead, we assume it is a truncated normally distributed random variable with mean $\overline{\beta}$ and variance σ_β^2. The hydraulic conductivities are treated as log-normal random fields.

For one-dimensional steady state flow, continuity of the flux in Darcy's Law implies that

$$\frac{d}{dx}\left[K(x)\frac{dh(x)}{dx}\right] = 0 \qquad x \in (0,1). \tag{3}$$

Then specifying constant flux q_0 at the boundary $x = 0$, and zero hydraulic head at the boundary $x = 1$, leads to the random head distribution

$$h(x) = q_0 \mathcal{H}(\beta - x)\left[\int_x^\beta \frac{ds}{K_1(s)} + \int_\beta^1 \frac{ds}{K_2(s)}\right] + q_0 \mathcal{H}(x-\beta)\int_x^1 \frac{ds}{K_2(s)}. \tag{4}$$

The Heaviside function, $\mathcal{H}(\beta - x)$, is the one-dimensional equivalent of the indicator functions used in the previous section. The ensemble mean of $h(x)$ is just the expected value with respect to the joint probability density, $p(K,\beta)$. Rewriting $p(K,\beta) = p(K \mid \beta)\,p(\beta)$ as before, yields, after some additional manipulations,

$$\begin{aligned}
\frac{\overline{h}(x)}{q_0} &= \frac{\sqrt{2}\sigma_\beta}{\sqrt{\pi}\mathcal{W}}\left[\frac{1}{K_{H_1}} - \frac{1}{K_{H_2}}\right]\left[\exp\left(-\frac{\{x-\overline{\beta}\}^2}{2\sigma_\beta^2}\right) - \exp\left(-\frac{\{1-\overline{\beta}\}^2}{2\sigma_\beta^2}\right)\right] \\
&- \frac{\overline{\beta}-x}{\mathcal{W}}\left[\frac{1}{K_{H_1}} - \frac{1}{K_{H_2}}\right]\operatorname{erf}\left(\frac{x-\overline{\beta}}{\sqrt{2}\sigma_\beta}\right) + \frac{1}{\mathcal{W}}\left[\frac{\overline{\beta}-x}{K_{H_1}} + \frac{1-\overline{\beta}}{K_{H_2}}\right]\operatorname{erf}\left(\frac{1-\overline{\beta}}{\sqrt{2}\sigma_\beta}\right) \\
&+ \frac{1}{\mathcal{W}}\frac{1-x}{K_{H_2}}\operatorname{erf}\left(\frac{\overline{\beta}}{\sqrt{2}\sigma_\beta}\right)
\end{aligned} \tag{5}$$

where K_{H_i} are the geometric means of hydraulic conductivities of M_i materials and

$$\mathcal{W} = \operatorname{erf}\left(\frac{1-\overline{\beta}}{\sqrt{2}\sigma_\beta}\right) + \operatorname{erf}\left(\frac{\overline{\beta}}{\sqrt{2}\sigma_\beta}\right). \tag{6}$$

If the medium consists of a single material, (5) reduces to a well known relation

$$\overline{h}(x) = \frac{q_0}{K_H}(1-x) \tag{7}$$

where $K_{H_1} = K_{H_2} \equiv K_H$. If the location of the contact between the two materials is known precisely, then $\sigma_\beta^2 = 0$ and $\overline{\beta} = \beta$. Then $\mathcal{W} = 2\operatorname{erf}(\infty) = 1$, and (5) becomes

$$\overline{h}(x) = q_0 \begin{cases} \frac{\beta-x}{K_{H_1}} + \frac{1-\beta}{K_{H_2}} & 0 < x < \beta \\ \frac{1-x}{K_{H_2}} & \beta < x < 1. \end{cases} \tag{8}$$

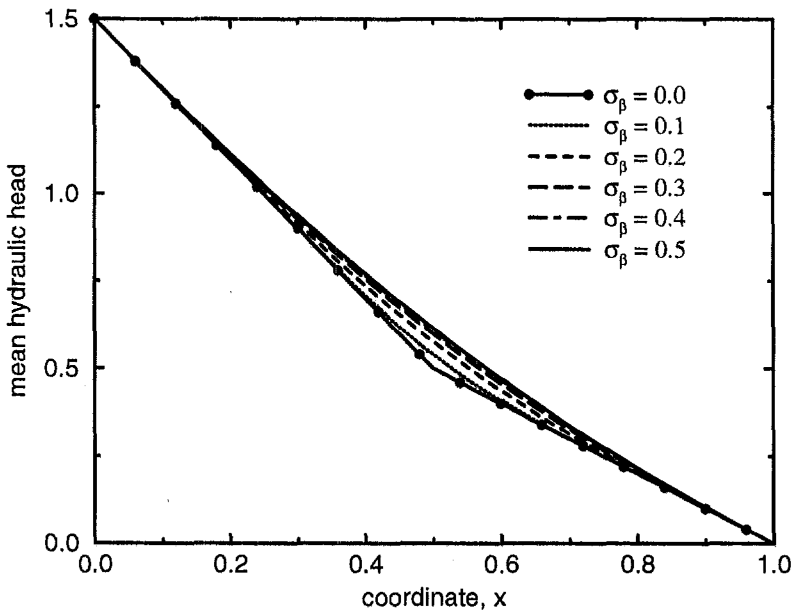

Fig. 1 Mean hydraulic head distribution for several values of the standard diviation σ_β.

The derivative (or gradient in higher dimensions) of the right-hand side of (5) is the reciprocal of the effective conductivity for this problem,

$$\begin{aligned}\frac{1}{q_0}\frac{d\bar{h}(x)}{dx} &= -\frac{1}{\mathcal{W}K_{H_2}}\operatorname{erf}\left(\frac{\bar{\beta}}{\sqrt{2}\sigma_\beta}\right)+\frac{1}{\mathcal{W}}\left[\frac{1}{K_{H_1}}-\frac{1}{K_{H_2}}\right]\operatorname{erf}\left(\frac{x-\bar{\beta}}{\sqrt{2}\sigma_\beta}\right)\\ &- \frac{1}{\mathcal{W}K_{H_1}}\operatorname{erf}\left(\frac{1-\bar{\beta}}{\sqrt{2}\sigma_\beta}\right). \qquad (9)\end{aligned}$$

Uncertainty in the location of the contact affects both mean head, $\bar{h}(x)$, and its derivative. The magnitude of σ_β, the standard deviation of β, is a measure of location uncertainty. Supposing that $\bar{\beta} = 1/2$ and considering $\bar{h}(x)$ first, we see that large σ_β leads to an almost linear trend from one boundary value to the other (Figure 1). This is to be expected, since in this case we are basically not sure whether there is one material or two; hence, $P(x \in M_1) \approx P(x \in M_2)$. In the deterministic case ($\sigma_\beta = 0$), mean head $\bar{h}(x)$ exhibits typical behavior: linear trends in each material and continuity at the boundary, but with a change in slope. Other values of σ_β induce intermediate behavior with the greatest effect near the location of the expected contact. Mean hydraulic gradient, the reciprocal of effective conductivity, is similarly affected (Figure 2). In the deterministic boundary case ($\sigma_\beta = 0$) there is a jump at the boundary, just as there should be. The conductivity random fields are known to be different on each side of the known boundary. Large σ_β, on the other hand, shows an influence of location uncertainty throughout the domain with a nearly linear trend in the gradient between the fixed boundary points. Of course, intermediate values of σ_β cause mean head gradients to fall between these two extremes.

It remains to investigate similar effects in two- and three-dimensional media with more realistic block geometries and for higher order moments of both K and h.

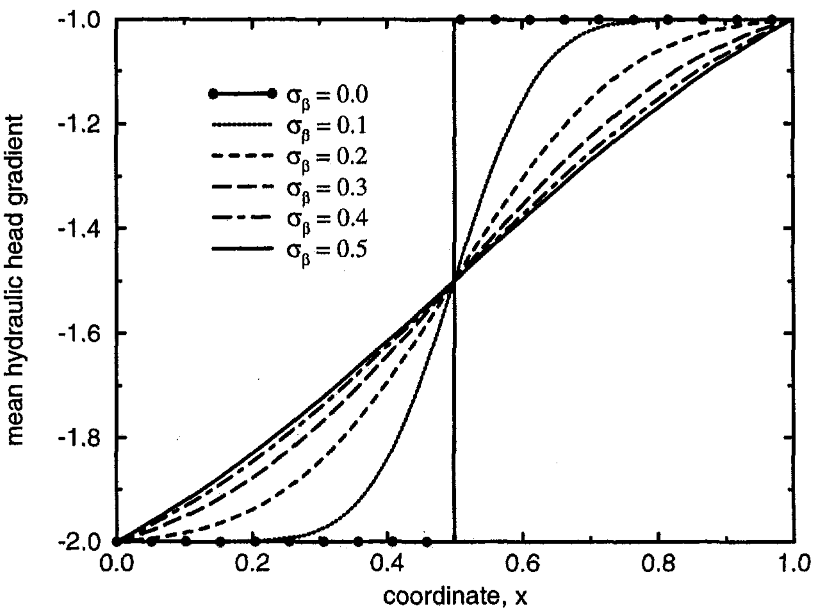

Fig. 2 Mean head gradient distribution for several values of the standard diviation σ_β.

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