Lecture Notes

from simple field theories to the standard model

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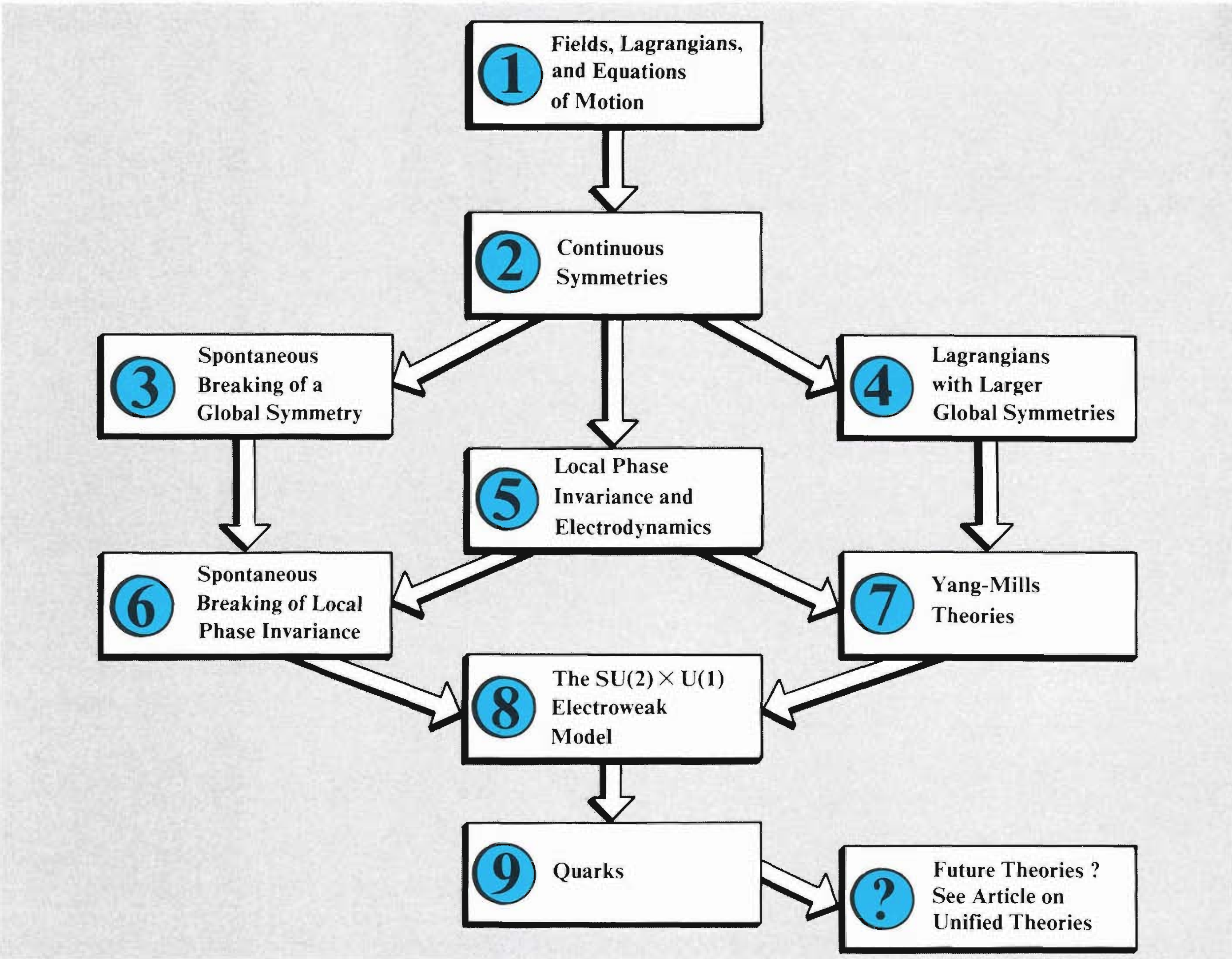
The standard model of electroweak and strong interactions consists of two relativistic quantum field theories, one to describe the strong interactions and one to describe the electromagnetic and weak interactions. This model, which incorporates all the known phenomenology of these fundamental interactions, describes spinless, spin-½, and spin-1 fields interacting with one another in a manner determined by its Lagrangian. The theory is relativistically invariant, so the mathematical form of the Lagrangian is unchanged by Lorentz transformations.

Although rather complicated in detail, the standard model Lagrangian is based on just two basic ideas beyond those necessary for a quantum field theory. One is the concept of local symmetry, which is encountered in its simplest form in electrodynamics. Local symmetry determines the form of the interaction between particles, or fields, that carry the charge associated with the symmetry (not necessarily the electric charge). The interaction is mediated by a spin-1 particle, the vector boson, or gauge particle. The second concept is spontaneous symmetry breaking, where the vacuum (the state with no particles) has a nonzero charge distribution. In the standard model the nonzero weak-interaction charge distribution of the vacuum is the source of most masses of the particles in the theory. These two basic ideas, local symmetry and spontaneous symmetry breaking, are exhibited by simple field theories. We begin these lecture notes with a Lagrangian for scalar fields and then, through the extensions and generalizations indicated by the arrows in the diagram below, build up the formalism needed to construct the standard model.



1 Fields, Lagrangians, and Equations of Motion

We begin this introduction to field theory with one of the simplest theories, a complex scalar field theory with independent fields $\varphi(x)$ and $\varphi^\dagger(x)$. ($\varphi^\dagger(x)$ is the complex conjugate of $\varphi(x)$ if $\varphi(x)$ is a classical field, and, if $\varphi(x)$ is generalized to a column vector or to a quantum field, $\varphi^\dagger(x)$ is the Hermitian conjugate of $\varphi(x)$.) Since $\varphi(x)$ is a complex function in classical field theory, it assigns a complex number to each four-dimensional point $x = (ct, \mathrm{x})$ of time and space. The symbol x denotes all four components. In quantum field theory $\varphi(x)$ is an operator that acts on a state vector in quantum-mechanical Hilbert space by adding or removing elementary particles localized around the space-time point x.

In this note we present the case in which $\varphi(x)$ and $\varphi^\dagger(x)$ correspond respectively to a spinless charged particle and its antiparticle of equal mass but opposite charge. The charge in this field theory is like electric charge, except it is not yet coupled to the electromagnetic field. (The word "charge" has a broader definition than just electric charge.) In Note 3 we show how this complex scalar field theory can describe a quite different particle spectrum: instead of a particle and its antiparticle of equal mass, it can describe a particle of zero mass and one of nonzero mass, each of which is its own antiparticle. Then the scalar theory exhibits the phenomenon called spontaneous symmetry breaking, which is important for the standard model.

A complex scalar theory can be defined by the Lagrangian density,

$$\mathscr{L}(\varphi,\partial_\mu\varphi,\varphi^\dagger,\partial_\mu\varphi^\dagger) = \partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2\,, \tag{1a}$$

where $\partial_\mu\varphi \equiv \partial\varphi/\partial x^\mu$. (Upper and lower indices are related by the metric tensor, a technical point not central to this discussion.) The Lagrangian itself is

$$\mathrm{L}(t_1,t_2) \equiv \int_{t_2}^{t_1} dt \int d^3\mathbf{x}\,\mathscr{L}\,. \tag{1b}$$

The first term in Eq. 1a is the kinetic energy of the fields $\varphi(x)$ and $\varphi^\dagger(x)$, and the last two terms are the negative of the potential energy. Terms quadratic in the fields, such as the $-m^2\varphi^\dagger\varphi$ term in Eq. 1a. are called mass terms. If $m^2 > 0$, then $\varphi(x)$ describes a spinless particle and $\varphi^\dagger(x)$ its antiparticle of identical mass. If $m^2 < 0$, the theory has spontaneous symmetry breaking.

The equations of motion are derived from Eq. 1 by a variational method. Thus, let us change the fields and their derivatives by a small amount $\delta\varphi(x)$ and $\delta\partial_\mu\varphi(x) = \partial_\mu\delta\varphi(x)$. Then,

$$\delta\mathrm{L}(t_1,t_2) = \int_{t_2}^{t_2}\left[\frac{\partial\mathscr{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathscr{L}}{\partial\varphi^\dagger}\delta\varphi^\dagger + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\delta\varphi + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi^\dagger)}\partial_\mu\delta\varphi^\dagger\right] d^4x\,, \tag{2}$$

where the variation is defined with the restrictions $\delta\varphi(\mathbf{x},t_1) = \delta\varphi(\mathbf{x},t_2) = \delta\varphi^\dagger(\mathbf{x},t_1) = \delta\varphi^\dagger(\mathbf{x},t_2) = 0$, and $\delta\varphi(x)$ and $\delta\varphi^\dagger(x)$ are independent. The last two terms are integrated by parts, and the surface term is dropped since the integrand vanishes on the boundary. This procedure yields the Euler-Lagrange equations for $\varphi^\dagger(x)$,

$$\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi)}\right) - \frac{\partial\mathscr{L}}{\partial\varphi} = 0\,, \tag{3}$$

and for $\varphi(x)$. (The Euler-Lagrange equation for $\varphi(x)$ is like Eq. 3 except that $\varphi^\dagger$ replaces φ. There are two equations because $\delta\varphi(x)$ and $\delta\varphi^\dagger(x)$ are independent.) Substituting the Lagrangian density, Eq. 1a, into the Euler-Lagrange equations, we obtain the equations of motion,

$$\partial^\mu\partial_\mu\varphi + m^2\varphi + 2\lambda(\varphi^\dagger\varphi)\varphi = 0\,, \tag{4}$$

plus another equation of exactly the same form with $\varphi(x)$ and $\varphi^\dagger(x)$ exchanged.

This method for finding the equations of motion can be easily generalized to more fields and to fields with spin. For example, a field theory that is incorporated into the standard model is electrodynamics. Its list of fields includes particles that carry spin. The electromagnetic vector potential $A_\mu(x)$ describes a "vector" particle with a spin of 1 (in units of the quantum of action $\hbar = 1.0546 \times 10^{-27}$ erg second), and its four spin components are enumerated by the space-time vector index μ (= 0, 1, 2, 3, where 0 is the index for the time component and 1, 2, and 3 are the indices for the three space components). In electrodynamics only two of the four components of $A_\mu(x)$ are independent. The electron has a spin of ½, as does its antiparticle, the positron. Electrons and positrons of both spin projections, ±½, are described by a field $\psi(x)$, which is a column vector with four entries. Many calculations in electrodynamics are complicated by the spins of the fields.

There is a much more difficult generalization of the Lagrangian formalism: if there are constraints among the fields, the procedure yielding the Euler-Lagrange equations must be modified, since the field variations are not all independent. This technical problem complicates the formulation of electrodynamics and the standard model, especially when computing quantum corrections. Our examination of the theory is not so detailed as to require a solution of the constraint problem.

2 Continuous Symmetries

It is often possible to find sets of fields in the Lagrangian that can be rearranged or transformed in ways described below without changing the Lagrangian. The transformations that leave the Lagrangian unchanged (or invariant) are called symmetries. First, we will look at the form of such transformations, and then we will discuss implications of a symmetrical Lagrangian. In some cases symmetries imply the existence of conserved currents (such as the electromagnetic current) and conserved charges (such as the electric charge), which remain constant during elementary-particle collisions. The conservation of energy, momentum, angular momentum, and electric charge are all derived from the existence of symmetries.

Let us consider a continuous linear transformation on three real spinless fields $\varphi_i(x)$ (where $i = 1, 2, 3$) with $\varphi_i(x) = \varphi_i^\dagger(x)$. These three fields might correspond to the three pion states. As a matter of notation, $\varphi(x)$ is a column vector, where the top entry is $\varphi_1(x)$, the second entry is $\varphi_2(x)$, and the bottom entry is $\varphi_3(x)$. We write the linear transformation of the three fields in terms of a 3-by-3 matrix $U(\varepsilon)$, where

$$\varphi'(x') = U(\varepsilon)\varphi(x) , \tag{5a}$$

or in component notation

$$\varphi_i'(x') = U_{ij}(\varepsilon)\varphi_j(x) . \tag{5b}$$

The repeated index is summed from 1 to 3, and generalizations to different numbers or kinds of fields are obvious. The parameter ε is continuous, and as ε approaches zero, $U(\varepsilon)$ becomes the unit matrix. The dependence of x' on x and ε is discussed below. The continuous transformation $U(\varepsilon)$ is called linear since $\varphi_j(x)$ occurs linearly on the right-hand side of Eq. 5. (Nonlinear transformations also have an important role in particle physics, but this discussion of the standard model will primarily involve linear transformations except for the vector-boson fields, which have a slightly different transformation law, described in Note 5.) For N independent transformations, there will be a set of parameters ε_a, where the index a takes on values from 1 to N.

For these continuous transformations we can expand $\varphi'(x')$ in a Taylor series about $\varepsilon_a = 0$; by keeping only the leading term in the expansion, Eq. 5 can be rewritten in infinitesimal form as

$$\delta\varphi(x) \equiv \varphi'(x) - \varphi(x) = i\varepsilon^a T_a \varphi(x) , \tag{6a}$$

where T_a is the first term in the Taylor expansion,

$$i\varepsilon^a T_a = \varepsilon^a \left[\frac{\partial U(\varepsilon)}{\partial \varepsilon_a} \right]_{\varepsilon=0} - \delta x^\mu \partial_\mu , \tag{6b}$$

with $\delta x = x' - x$. The T_a are the "generators" of the symmetry transformations of $\varphi(x)$. (We note that $\delta\varphi(x)$ in Eq. 6a is a small symmetry transformation, not to be confused with the field variations $\delta\varphi$ in Eq. 2.)

The space-time point x' is, in general, a function of x. In the case where $x' = x$, Eq. 5 is called an internal transformation. Although our primary focus will be on internal transformations, space-time symmetries have many applications. For example, all theories we describe here have Poincaré symmetry, which means that these theories are invariant under transformations in which $x' = \Lambda x + b$, where Λ is a 4-by-4 matrix representing a Lorentz transformation that acts on a four-component column vector x consisting of time and the three space components, and b is the four-component column vector of the parameters of a space-time translation. A spinless field transforms under Poincaré transformations as $\varphi'(x') = \varphi(x)$ or $\delta\varphi = -b^\mu \partial_\mu \varphi(x)$. Upon solving Eq. 6b, we find the infinitesimal translation is represented by $i\partial_\mu$. The components of fields with spin are rearranged by Poincaré transformations according to a matrix that depends on both the ε's and the spin of the field.

We now restrict attention to internal transformations where the space-time point is unchanged; that is, $\delta x^\mu = 0$. If ε_a is an infinitesimal, arbitrary function of x, $\varepsilon_a(x)$, then Eqs. 5 and 6a are called local transformations. If the ε_a are restricted to being constants in space-time, then the transformation is called global.

Before beginning a lengthy development of the symmetries of various Lagrangians, we give examples in which each of these kinds of linear transformations are, indeed, symmetries of physical theories. An example of a global, internal symmetry is strong isospin, as discussed briefly in "Particle Physics and the Standard Model." (Actually, strong isospin is not an exact symmetry of Nature, but it is still a good example.) All theories we discuss here have global Lorentz invariance, which is a space-time symmetry. Electrodynamics has a local phase symmetry that is an internal symmetry. For a charged spinless field the infinitesimal form of a local phase transformation is $\delta\varphi(x) = i\varepsilon(x)\varphi(x)$ and $\delta\varphi^\dagger(x) = -i\varepsilon(x)\varphi^\dagger(x)$, where $\varphi(x)$ is a complex field. Larger sets of local internal symmetry transformations are fundamental in the standard model of the weak and strong interactions. Finally, Einstein's gravity makes essential use of local space-time Poincaré transformations. This complicated case is not discussed here. It is quite remarkable how many types of transformations like Eqs. 5 and 6 are basic in the formulation of physical theories.

Let us return to the column vector of three real fields $\varphi(x)$ and suppose we have a Lagrangian that is unchanged by Eqs. 5 and 6, where we now restrict our attention to internal transformations. (One such Lagrangian is Eq. 1a, where $\varphi(x)$ is now a column vector and $\varphi^\dagger(x)$ is its transpose.) Not only the Lagrangian, but the Lagrangian density, too, is unchanged by an internal symmetry transformation.

Let us consider the infinitesimal transformation (Eq. 6a) and calculate $\delta\mathscr{L}$ in two different ways. First of all, $\delta\mathscr{L} = 0$ if $\delta\varphi$ is a symmetry identified from the Lagrangian. Moreover, according to the rules of partial differentiation,

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\varphi_i}\,\delta\varphi_i + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_i)}\,\partial_\mu\delta\varphi_i\,. \tag{7}$$

Then, using the Euler-Lagrange equations (Eq. 3) for the first term and collecting terms, Eq. 7 can be written in an interesting way:

$$\delta\mathscr{L} = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_i)}\right)\delta\varphi_i\,. \tag{8}$$

The next step is to substitute Eq. 6a into Eq. 8. Thus, let us define the current $J^a_\mu(x)$ as

$$J^a_\mu(x) = i\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_i)}\,T^a_{ij}\varphi_j\,. \tag{9}$$

Then Eq. 8 plus the requirement that $\delta\varphi$ is a symmetry imply the continuity equation,

$$\partial^\mu J^a_\mu(x) = 0\,. \tag{10}$$

We can gain intuition about Eq. 10 from electrodynamics, since the electromagnetic current satisfies a continuity equation. It says that charge is neither created nor destroyed locally: the change in the charge density, $J_0(x)$, in a small region of space is just equal to the current $\mathbf{J}(x)$ flowing out of the region. Equation 10 generalizes this result of electrodynamics to other kinds of charges, and so $J^a_\mu(x)$ is called a current. In particle physics with its many continuous symmetries, we must be careful to identify which current we are talking about.

Although the analysis just performed is classical, the results are usually correct in the quantum theory derived from a classical Lagrangian. In some cases, however, quantum corrections contribute a nonzero term to the right-hand side of Eq. 10; these terms are called anomalies. For global symmetries these anomalies can improve the predictions from Lagrangians that have too much symmetry when compared with data because the anomaly wrecks the symmetry (it was never there in the quantum theory, even though the classical Lagrangian had the symmetry). However, for local symmetries anomalies are disastrous. A quantum field theory is locally symmetric only if its currents satisfy the continuity equation, Eq. 10. Otherwise local symmetry transformations simply change the theory. (Some care is needed to avoid this kind of anomaly in the standard model.) We now show that Eq. 10 can imply the existence of a conserved quantity called the global charge and defined by

$$Q^a(t) = \int d^3\mathbf{x}\, J^a_0(x)\,, \tag{11}$$

provided the integral over all space in Eq. 11 is well defined; that is, $J^a_0(x)$ must fall off rapidly enough as $|\mathbf{x}|$ approaches infinity that the integral is finite.

If $Q^a(t)$ is indeed a conserved quantity, then its value does not change in time, which means that its first time derivative is zero. We can compute the time derivative of $Q^a(t)$ with the aid of Eq. 10:

$$\frac{d}{dt}Q^a(t) = \int d^3\mathbf{x}\,\frac{\partial J^a_0(x)}{\partial t} = \int d^3\mathbf{x}\,\nabla\cdot\mathbf{J}^a(x) = \int \mathbf{J}^a\cdot d\mathbf{S} = 0\,. \tag{12}$$

The next to the last step is Gauss's theorem, which changes the volume integral of the divergence of a vector field into a surface integral. If $\mathbf{J}^a(x)$ falls off more rapidly than $1/|\mathbf{x}|^2$ as $|\mathbf{x}|$ becomes very large, then the surface integral must be zero. It is not a always true that $\mathbf{J}^a(x)$ falls off so rapidly, but when it does, $Q^a(t) = Q^a$ is a constant in time. One of the most important experimental tests of a Lagrangian is whether the conserved quantities it predicts are, indeed, conserved in elementary-particle interactions.

The Lagrangian for the complex scalar field defined by Eq. 1 has an internal global symmetry, so let us practice the above steps and identify the conserved current and charge. It is easily verified that the global phase transformation

$$\varphi'(x) = e^{i\varepsilon}\varphi(x) \tag{13}$$

leaves the Lagrangian density invariant. For example, the first term of Eq. 1 by itself is unchanged: $\partial_\mu\varphi^\dagger\partial^\mu\varphi$ becomes $\partial_\mu(e^{-i\varepsilon}\varphi^\dagger)\partial^\mu(e^{i\varepsilon}\varphi) = \partial_\mu\varphi^\dagger\partial^\mu\varphi$, where the last equality follows only if ε is constant in space-time. (The case of local phase transformations is treated in Note 5.) The next step is to write the infinitesimal form of Eq. 13 and substitute it into Eq. 9. The conserved current is

$$J_\mu(x) = i[(\partial_\mu\varphi^\dagger)\varphi - (\partial_\mu\varphi)\varphi^\dagger]\,, \tag{14}$$

where the sum in Eq. 9 over the fields $\varphi(x)$ and $\varphi^\dagger(x)$ is written out explicitly.

If $m^2 > 0$ in Eq. 1, then all the charge can be localized in space and time and made to vanish as the distance from the charge goes to infinity. The steps in Eq. 12 are then rigorous, and a conserved charge exists. The calculation was done here for classical fields, but the same results hold for quantum fields: the conservation law implied by Eq. 12 yields a conserved global charge equal to the number of φ particles minus the number of φ antiparticles. This number must remain constant in any interaction. (We will see in Note 3 that if $m^2 < 0$, the charge distribution is spread out over all space-time, so the global charge is no longer conserved even though the continuity equation remains valid.)

Identifying the transformations of the fields that leave the Lagrangian invariant not only satisfies our sense of symmetry but also leads to important predictions of the theory without solving the equations of motion. In Note 4 we will return to the example of three real scalar fields to introduce larger global symmetries, such as SU(2), that interrelate different fields.

3 Spontaneous Breaking of a Global Symmetry

It is possible for the vacuum or ground state of a physical system to have less symmetry than the Lagrangian. This possibility is called spontaneous symmetry breaking, and it plays an important role in the standard model. The simplest example is the complex scalar field theory of Eq. 1a with $m^2 < 0$.

In order to identify the classical fields with particles in the quantum theory, the classical field must approach zero as the number of particles in the corresponding quantum-mechanical state approaches zero. Thus the quantum-mechanical vacuum (the state with no particles) corresponds to the classical solution $\varphi(x) = 0$. This might seem automatic, but it is not. Symmetry arguments do not necessarily imply that $\varphi(x) = 0$ is the lowest energy state of the system. However, if we rewrite $\varphi(x)$ as a function of new fields that do vanish for the lowest energy state, then the new fields may be directly identified with particles. Although this prescription is simple, its justification and analysis of its limitations require extensive use of the details of quantum field theory.

The energy of the complex scalar theory is the sum of kinetic and potential energies of the $\varphi(x)$ and $\varphi^\dagger(x)$ fields, so the energy density is

$$\mathcal{H} = \partial^\mu\varphi^\dagger\partial_\mu\varphi + m^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2\,, \tag{15}$$

with $\lambda > 0$. Note that $\partial^\mu\varphi^\dagger\partial_\mu\varphi$ is nonnegative and is zero if φ is a constant. For $\varphi = 0$, $\mathcal{H} = 0$. However, if $m^2 < 0$, then there are nonzero values of $\varphi(x)$ for which $\mathcal{H} < 0$. Thus, there is a nonzero field configuration with lowest energy. A graph of $\mathcal{H}$ as a function of $|\varphi|$ is shown in Fig. 1. In this example $\mathcal{H}$ is at its lowest value when both the kinetic and potential energies ($V = m^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2$) are at their lowest values. Thus, the vacuum solution for $\varphi(x)$ is found by solving the equation $\partial V/\partial\varphi = 0$, or

$$\varphi^\dagger(x)\varphi(x) = -\frac{m^2}{2\lambda} = \frac{1}{2}|\varphi_0|^2 > 0\,. \tag{16}$$

Next we find new fields that vanish when Eq. 16 is satisfied. For example, we can set

$$\varphi(x) = \frac{1}{\sqrt{2}}\left[\rho(x) + \varphi_0\right]\exp\left[i\pi(x)/\varphi_0\right], \tag{17}$$

where the real fields $\rho(x)$ and $\pi(x)$ are zero when the system is in the lowest energy state. Thus $\rho(x)$ and $\pi(x)$ may be associated with particles. Note, however, that φ_0 is not completely specified; it may lie at any point on the circle in field space defined by Eq. 16, as shown in Fig. 2.

Suppose φ_0 is real and given by

$$\varphi_0 = (-m^2/\lambda)^{1/2}\,. \tag{18}$$

Then the Lagrangian is still invariant under the phase transformations in Eq. 13, but the choice of the vacuum field solution is changed by the phase transformation. Thus, the vacuum solution is not invariant under the phase transformations, so the phase symmetry is spontaneously broken. The symmetry of the Lagrangian is *not* a symmetry of the vacuum. (For $m^2 > 0$ in Eq. 1, the vacuum and the Lagrangian both have the phase symmetry.)

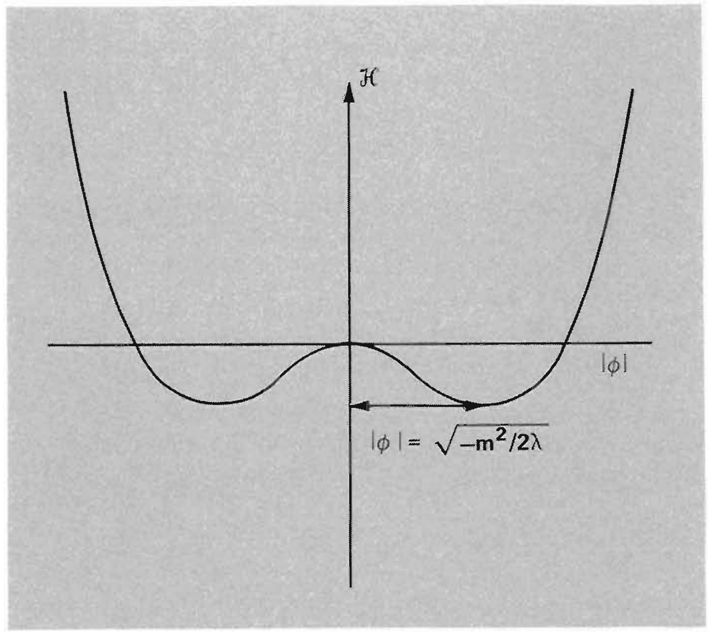


Fig. 1. The Hamiltonian $\mathcal{H}$ defined by Eq. 15 has minima at nonzero values of the field φ.

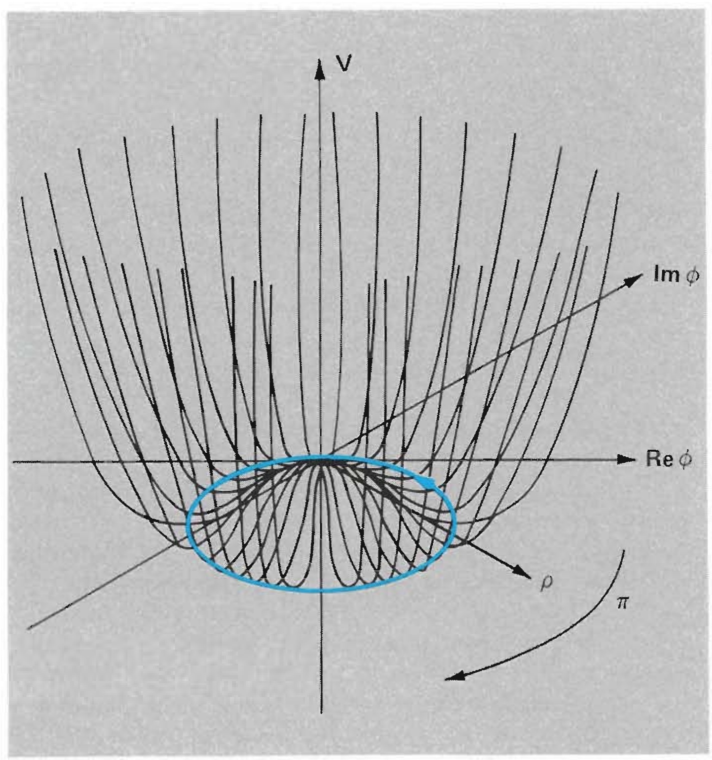


Fig. 2. The blue curve is the location of the minimum of V in the field space φ.

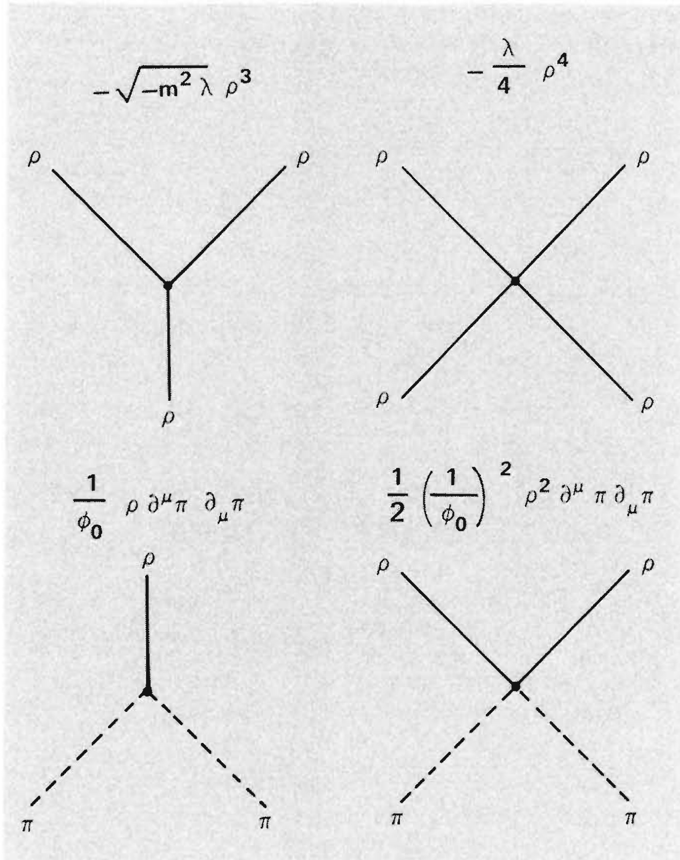


Fig. 3. A graphic representation of the last four terms of Eq. 20, the interaction terms. Solid lines denote the ρ field and dotted lines the π field. The interaction of three $\rho(x)$ fields at a single point is shown as three solid lines emanating from a single point. In perturbation theory this so-called vertex represents the lowest order quantum-mechanical amplitude for one particle to turn into two. All possible configurations of these vertices represent the quantum-mechanical amplitudes defined by the theory.

We now rewrite the Lagrangian in terms of the particle fields $\rho(x)$ and $\pi(x)$ by substituting Eq. 17 into Eq. 1. The Lagrangian becomes

$$\mathscr{L} = \frac{1}{2}\,\partial^\mu\rho\partial_\mu\rho + \frac{1}{2}(1+\rho/\varphi_0)^2\partial^\mu\pi\partial_\mu\pi - \frac{m^2}{2}(\rho+\varphi_0)^2 - \frac{\lambda}{4}(\rho+\varphi_0)^4\,. \tag{19}$$

To estimate the masses associated with the particle fields $\rho(x)$ and $\pi(x)$, we substitute Eq. 18 for the constant φ_0 and expand $\mathscr{L}$ in powers of the fields $\pi(x)$ and $\rho(x)$, obtaining

$$\mathscr{L} = \frac{1}{2}\,\partial^\mu\rho\partial_\mu\rho + \frac{1}{2}\,\partial^\mu\pi\partial_\mu\pi + \frac{m^4}{4\lambda} + m^2\rho^2 - (-\lambda m^2)^{1/2}\rho^3 - \frac{\lambda}{4}\rho^4 + \frac{1}{\varphi_0}\rho\partial^\mu\pi\partial_\mu\pi + \frac{1}{2\varphi_0^2}\,\rho^2\partial^\mu\pi\partial_\mu\pi\,. \tag{20}$$

This Lagrangian has the following features.

- The fields $\rho(x)$ and $\pi(x)$ have standard kinetic energy terms.
- Since $m^2 < 0$, the term $m^2\rho^2$ can be interpreted as the mass term for the $\rho(x)$ field. The $\rho(x)$ field thus describes a particle with mass-squared equal to $|m^2|$, *not* $-|m^2|$.
- The $\pi(x)$ field has no mass term. (This is obvious from Fig. 2, which shows that $\mathscr{L}(\rho,\pi)$ has no curvature (that is, $\partial^2\mathscr{L}/\partial\pi^2 = 0$) in the $\pi(x)$ direction.) Thus, $\pi(x)$ corresponds to a massless particle. This result is unchanged when all the quantum effects are included.
- The phase symmetry is hidden in $\mathscr{L}$ when it is written in terms of $\rho(x)$ and $\pi(x)$. Nevertheless, $\mathscr{L}$ has phase symmetry, as is proved by working backward from Eq. 20 to Eq. 16 to recover Eq. 1a.
- In theories without gravity, the constant term $V \propto m^4/\lambda$ can be ignored, since a constant overall energy level is not measurable. The situation is much more complicated for gravitational theories, where terms of this type contribute to the vacuum energy-momentum tensor and, by Einstein's equations, modify the geometry of space-time.
- The ρ field interacts with the π field only through derivatives of π. The interaction terms in Eq. 20 may be pictured as in Fig. 3.

Although this model might appear to be an idle curiosity, it is an example of a very general result known as Goldstone's theorem. This theorem states that in any field theory there is a zero-mass spinless particle for each independent global continuous symmetry of the Lagrangian that is spontaneously broken. The zero-mass particle is called a Goldstone boson. (This general result does not apply to local symmetries, as we shall see.)

There has been one very important physical application of spontaneously broken global symmetries in particle physics, namely, theories of pion dynamics. The pion has a surprisingly small mass compared to a nucleon, so it might be understood as a zero-mass particle resulting from spontaneous symmetry breaking of a global symmetry. Since the pion mass is not exactly zero, there must also be some small but explicit terms in the Lagrangian that violate the global symmetry. The feature of pion dynamics that justifies this procedure is that the interactions of pions with nucleons and other pions are similar to the interactions (see Fig. 3) of the $\pi(x)$ field with the $\rho(x)$ field and with itself in the Lagrangian of Eq. 20. Since the pion has three (electric) charge states, it must be associated with a larger global symmetry than the phase symmetry, one where three independent symmetries are spontaneously broken. The usual choice of symmetry is global SU(2) $\times$ SU(2) spontaneously broken to the SU(2) of the strong-interaction isospin symmetry (see Note 4 for a discussion of SU(2)). This description accounts reasonably well for low-energy pion physics.

Perhaps we should note that only spinless fields can acquire a vacuum value. Fields carrying spin are not invariant under Lorentz transformations, so if they acquire a vacuum value, Lorentz invariance will be spontaneously broken, in disagreement with experiment. Spinless particles trigger the spontaneous symmetry breaking in the standard model.

4 Lagrangians with Larger Global Symmetries

In a theory with a single complex scalar field the phase transformation in Eq. 13 defines the "largest" possible internal symmetry since the only possible symmetries must relate $\varphi(x)$ to itself. Here we will discuss global symmetries that interrelate different fields and group them together into "symmetry multiplets." Strong isospin, an approximate symmetry of the observed strongly interacting particles, is an example. It groups the neutron and the proton into an isospin doublet, reflecting the fact that the neutron and proton have nearly the same mass and share many similarities in the way that they interact with other particles. Similar comments hold for the three pion states (π^+, π^0, and π^-), which form an isospin triplet.

We will derive the structure of strong isospin symmetry by examining the invariance of a specific Lagrangian for the three real scalar fields $\varphi_i(x)$ already described in Note 2. (Although these fields could describe the pions, the Lagrangian will be chosen for simplicity, not for its capability to describe pion interactions.)

We are about to discover a symmetry by deriving it from a Lagrangian; however, in particle physics the symmetries are often discovered from phenomenology. Moreover, since there can be many Lagrangians with the same symmetry, the predictions following from the symmetry are viewed as more general than the predictions of a specific Lagrangian with the symmetry. Consequently, it becomes important to abstract from specific Lagrangians the general features of a symmetry; see the comments later in this note.

A general linear transformation law for the three real fields can be written

$$\varphi_i'(x) = [\exp(i\varepsilon^a T_a)]_{ij}\varphi_j(x)\,, \tag{21}$$

where the sum on j runs from 1 to 3. One reason for choosing this form of $U(\varepsilon)$ is that it explicitly approaches the identity as ε approaches zero.

To identify the generators T_a with matrix elements $(T_a)_{ij}$, we use a specific Lagrangian,

$$\mathscr{L} = \frac{1}{2}\partial^\mu\varphi_i\partial_\mu\varphi_i - \frac{1}{2}\,m^2\varphi_i\varphi_i - \lambda\,(\varphi_i\varphi_i)^2\,. \tag{22}$$

Let us place primes on the fields in Eq. 22 and substitute Eq. 21 into it. Then $\mathscr{L}$ written in terms of the new $\varphi(x)$ is exactly the same as Eq. 22 if

$$[\exp(i\varepsilon^a T_a)]_{ij}\,[\exp(i\varepsilon^b T_b)]_{ik} = \delta_{jk}\,, \tag{23}$$

where δ_{jk} are the matrix elements of the 3-by-3 identity matrix. (In the notation of Eq. 5a, Eq. 23 is $U(\varepsilon)U^T(\varepsilon) = I$.) Equation 23 can be expanded in ε_a, and the linear term then requires that T_a be an antisymmetric matrix. Moreover, $\exp(i\varepsilon^a T_a)$ must be a real matrix so that $\varphi(x)$ remains real after the transformation. This implies that all elements of the T_a are imaginary. These constraints are solved by the three imaginary antisymmetric 3-by-3 matrices with elements

$$(T_a)_{ij} = -i\varepsilon_{aij}\,, \tag{24}$$

where $\varepsilon_{123} = +1$ and ε_{abc} is antisymmetric under the interchange of any two indices (for example, $\varepsilon_{321} = -1$). (It is a coincidence in this example that the number of fields is equal to the number of independent symmetry generators. Also, the parameter ε_a with one index should not be confused with the tensor ε_{abc} with three indices.)

The conditions on $U(\varepsilon)$ imply that it is an orthogonal matrix; 3-by-3 orthogonal matrices can also describe rotations in three spatial dimensions. Thus, the three components of φ_i transform in the same way under isospin rotation as a spatial vector $\mathbf{x}$ transforms under a rotation. Since the rotational symmetry is SU(2), so is the isospin symmetry. (Thus "isospin" is like spin.) The T_a matrices satisfy the SU(2) commutation relations

$$[T_a, T_b] \equiv T_a T_b - T_b T_a = i\varepsilon_{abc} T_c \,. \tag{25}$$

Although the explicit matrices of Eq. 24 satisfy this relation, the T_a can be generalized to be quantum-mechanical operators. In the example of Eqs. 21 and 22, the isospin multiplet has three fields. Drawing on angular momentum theory, we can learn other possibilities for isospin multiplets. Spin-J multiplets (or representations) have $2J + 1$ components, where J can be any nonnegative integer or half integer. Thus, multiplets with isospin of ½ have two fields (for example, neutron and proton) and isospin-3/2 multiplets have four fields (for example, the Δ^{++}, Δ^{+}, Δ^{0}, and Δ^{-} baryons of mass $\sim 1232\ \mathrm{GeV}/c^2$).

The basic structure of all continuous symmetries of the standard model is completely analogous to the example just developed. In fact, part of the weak symmetry is called weak isospin, since it also has the same mathematical structure as strong isospin and angular momentum. Since there are many different applications to particle theory of given symmetries, it is often useful to know about symmetries and their multiplets. This mathematical endeavor is called group theory, and the results of group theory are often helpful in recognizing patterns in experimental data.

Continuous symmetries are defined by the algebraic properties of their generators. Group transformations can always be written in the form of Eq. 21. Thus, if Q_a ($a = 1, \ldots, N$) are the generators of a symmetry, then they satisfy commutation relations analogous to Eq. 25:

$$[Q_a, Q_b] = i f_{abc} Q_c \,, \tag{26}$$

where the constants f_{abc} are called the structure constants of the Lie algebra. The structure constants are determined by the multiplication rules for the symmetry operations, $U(\varepsilon_1)U(\varepsilon_2) = U(\varepsilon_3)$, where ε_3 depends on ε_1 and ε_2. Equation 26 is a basic relation in defining a Lie algebra, and Eq. 21 is an example of a Lie group operation. The Q_a, which generate the symmetry, are determined by the "group" structure. The focus on the generators often simplifies the study of Lie groups. The generators Q_a are quantum-mechanical operators. The $(T_a)_{ij}$ of Eqs. 24 and 25 are matrix elements of Q_a for some symmetry multiplet of the symmetry.

The general problem of finding all the ways of constructing equations like Eq. 25 and Eq. 26 is the central problem of Lie-group theory. First, one must find all sets of f_{abc}. This is the problem of finding all the Lie algebras and was solved many years ago. The second problem is, given the Lie algebra, to find all the matrices that represent the generators. This is the problem of finding all the representations (or multiplets) of a Lie algebra and is also solved in general, at least when the range of values of each ε_a is finite. Lie group theory thus offers an orderly approach to the classification of a huge number of theories.

Once a symmetry of the Lagrangian is identified, then sets of n fields are assigned to n-dimensional representations of the symmetry group, and the currents and charges are analyzed just as in Note 2. For instance, in our example with three real scalar fields and the Lagrangian of Eq. 22, the currents are

$$J^a_\mu(x) = \varepsilon^{aij}(\partial_\mu \varphi_i)\varphi_j \tag{27}$$

and, if $m^2 > 0$, the global symmetry charge is

$$Q^a = \int d^3\mathbf{x}\, \varepsilon^{aij} \frac{\partial \varphi_i}{\partial t}\, \varphi_j \,, \tag{28}$$

where the quantum-mechanical charges Q_a satisfy the commutation relations

$$[Q_a, Q_b] = i\varepsilon_{abc} Q_c \,. \tag{29}$$

(The derivation of Eq. 29 from Eq. 28 requires the canonical commutation relations of the quantum $\varphi_i(x)$ fields.)

The three-parameter group SU(2) has just been presented in some detail. Another group of great importance to the standard model is SU(3), which is the group of 3-by-3 unitary matrices with unit determinant. The inverse of a unitary matrix U is $U^\dagger$, so $U^\dagger U = I$. There are eight parameters and eight generators that satisfy Eq. 26 with the structure constants of SU(3). The low-dimensional representations of SU(3) have 1, 3, 6, 8, 10, . . . fields, and the different representations are referred to as $\mathbf{1}$, $\mathbf{3}$, $\bar{\mathbf{3}}$, $\mathbf{6}$, $\bar{\mathbf{6}}$, $\mathbf{8}$, $\mathbf{10}$, $\overline{\mathbf{10}}$, and so on.

5 Local Phase Invariance and Electrodynamics

The theories that make up the standard model are all based on the principle of local symmetry. The simplest example of a local symmetry is the extension of the global phase invariance discussed at the end of Note 2 to local phase invariance. As we will derive below, the requirement that a theory be invariant under local phase transformations implies the existence of a gauge field in the theory that mediates or carries the "force" between the matter fields. For electrodynamics the gauge field is the electromagnetic vector potential $A_\mu(x)$ and its quantum particle is the massless photon. In addition, in the standard model the gauge fields mediating the strong interactions between the quarks are the massless gluon fields and the gauge fields mediating the weak interactions are the fields for the massive Z^0 and $W^\pm$ weak bosons.

To illustrate these principles we extend the global phase invariance of the Lagrangian of Eq. 1 to a theory that has local phase invariance. Thus, we require $\mathscr{L}$ to have the same form for $\varphi'(x)$ and $\varphi(x)$, where the local phase transformation is defined by

$$\varphi'(x) = e^{i\varepsilon(x)}\varphi(x) . \tag{30}$$

The potential energy,

$$V(\varphi,\varphi^\dagger) = m^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2 , \tag{31}$$

already has this symmetry, but the kinetic energy, $\partial^\mu\varphi^\dagger\partial_\mu\varphi$, clearly does not, since

$$\partial_\mu\varphi'(x) = e^{i\varepsilon(x)}\left[\partial_\mu\varphi + i(\partial_\mu\varepsilon)\varphi\right] . \tag{32}$$

$\mathscr{L}$ does not have local phase invariance if the Lagrangian of the transformed fields depends on $\varepsilon(x)$ or its derivatives. The way to eliminate the $\partial_\mu\varepsilon$ dependence is to add a new field $A_\mu(x)$ called the gauge field and then require the local symmetry transformation law for this new field to cancel the $\partial_\mu\varepsilon$ term in Eq. 32. The gauge field can be added by generalizing the derivative ∂_μ to D_μ, where

$$D_\mu = \partial_\mu - ieA_\mu(x) . \tag{33}$$

This is just the minimal-coupling procedure of electrodynamics. We can then make a kinetic energy term of the form $(D^\mu\varphi)^\dagger(D_\mu\varphi)$ if we require that

$$D'_\mu\varphi'(x) = e^{i\varepsilon(x)}D_\mu\varphi(x) . \tag{34}$$

When written out with Eq. 33, Eq. 34 becomes an equation for $A'_\mu(x)$ in terms of $A_\mu(x)$, which is easily solved to give

$$A'_\mu(x) = A_\mu(x) + \frac{1}{e}\,\partial_\mu\varepsilon(x) . \tag{35}$$

Equation 35 prescribes how the gauge field transforms under the local phase symmetry.

Thus the first step to modifying Eq. 1 to be a theory with local phase invariance is simply to replace ∂_μ by D_μ in $\mathscr{L}$. (A slightly generalized form of this trick is used in the construction of all the theories in the standard model.) With this procedure the dominant interaction of the gauge field $A^\mu(x)$ with the matter field φ is in the form of a current times the gauge field, $eJ^\mu A_\mu$, where J_μ is the current defined in Eq. 14.

6 Spontaneous Breaking of Local Phase Invariance

We now show that spontaneous breaking of local symmetry implies that the associated vector boson has a mass, in spite of the fact that $A^\mu A_\mu$ by itself is not locally phase invariant. Much of the calculation in Note 3 can be translated to the Lagrangian of Eq. 38. In fact, the calculation is identical from Eq. 16 to Eq. 18, so the first new step is to substitute Eq. 17 into Eq. 38. The only significantly new part of the calculation is replacing $\partial^\mu\varphi^\dagger\partial_\mu\varphi$ by $(D^\mu\varphi)^\dagger(D_\mu\varphi)$. However, instead of simply substituting Eq. 17 for φ and computing $(D^\mu\varphi)^\dagger(D_\mu\varphi)$ directly, it is convenient to make a local phase transformation first:

$$\varphi'(x) = \frac{1}{\sqrt{2}}[\rho(x) + \varphi_0]\exp[i\pi(x)/\varphi_0] , \tag{41}$$

where $\varphi(x) = [\rho(x) + \varphi_0]/\sqrt{2}$. (The local phase invariance permits us to remove the phase of $\varphi(x)$ at every space-time point.) We emphasize the difference between Eqs. 17 and 41: Eq. 17 defines the $\rho(x)$ and $\pi(x)$ fields; Eq. 41 is a local phase transformation of $\varphi(x)$ by angle $\pi(x)$. Don't be fooled by the formal similarity of the two equations. Thus, we may write Eq. 38 in terms of $\varphi(x) = [\rho(x) + \varphi_0]/\sqrt{2}$ and obtain

This leaves a problem. If we simply replace $\partial_\mu\varphi$ by $D_\mu\varphi$ in the Lagrangian and then derive the equations of motion for A_μ, we find that A_μ is proportional to the current J_μ. The A_μ field equation has no space-time derivatives and therefore $A_\mu(x)$ does not propagate. If we want A_μ to correspond to the electromagnetic field potential, we must add a kinetic energy term for it to $\mathscr{L}$.

The problem then is to find a locally phase invariant kinetic energy term for $A_\mu(x)$. Note that the combination of covariant derivatives $D_\mu D_\nu - D_\nu D_\mu$, when acting on any function, contains no derivatives of the function. We define the electromagnetic field tensor of electrodynamics as

$$F_{\mu\nu} \equiv \frac{i}{e}[D_\mu , D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{36}$$

It contains derivatives of A_μ. Its transformation law under the local symmetry is

$$F'_{\mu\nu} = F_{\mu\nu} . \tag{37}$$

Thus, it is completely trivial to write down a term that is quadratic in the derivatives of A_μ, which would be an appropriate kinetic energy term. A fully phase invariant generalization of Eq. 1a is

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu}F_{\mu\nu} + (D^\mu\varphi)^\dagger (D_\mu\varphi) - m^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2 . \tag{38}$$

We should emphasize that $\mathscr{L}$ has no mass term for $A_\mu(x)$. Thus, when the fields correspond directly to the particles in Eq. 38, the vector particles described by $A_\mu(x)$ are massless. In fact, $A^\mu A_\mu$ is not invariant under the gauge transformation in Eq. 35, so it is not obvious how the A_μ field can acquire a mass if the theory does have local phase invariance. In Note 6 we will show how the gauge field becomes massive through spontaneous symmetry breaking. This is the key to understanding the electroweak theory.

We now rediscover the Lagrangian of electrodynamics for the interaction of electrons and photons following the same procedure that we used for the complex scalar field. We begin with the kinetic energy term for a Dirac field of the electron ψ, replace ∂_μ by D_μ defined in Eq. 33, and then add $-\,{}^1\!/\!_4 F^{\mu\nu}F_{\mu\nu}$, where $F^{\mu\nu}$ is defined in Eq. 36. The Lagrangian for a free Dirac field is

$$\mathscr{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi , \tag{39}$$

where γ^μ are the four Dirac γ matrices and $\bar{\psi} = \psi^\dagger\gamma_0$. Straightening out the definition of the γ^μ matrices and the components of ψ is the problem of describing a spin-½ particle in a theory with Lorentz invariance. We leave the details of the Dirac theory to textbooks, but note that we will use some of these details when we finally write down the interactions of the quarks and leptons. The interaction of the electron field ψ with the electromagnetic field follows by replacing ∂_μ by D_μ. The electrodynamic Lagrangian is

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi , \tag{40a}$$

where the interaction term in $i\bar{\psi}\gamma^\mu D_\mu\psi$ has the form

$$\mathscr{L}_{\text{interaction}} = e\bar{\psi}\gamma_\mu\psi A^\mu = eJ^{\text{em}}_\mu A^\mu , \tag{40b}$$

where $J^{\text{em}}_\mu = \bar{\psi}\gamma_\mu\psi$ is the electromagnetic current of the electron. What is amazing about the standard model is that all the electroweak and strong interactions between fermions and vector bosons are similar in form to Eq. 40b, and much phenomenology can be understood in terms of such interaction terms as long as we can approximate the quantum fields with the classical solutions.

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial^\mu\rho\partial_\mu\rho + \frac{e^2}{2}(\rho + \varphi_0)^2 A^\mu A_\mu$$
$$-\frac{m^2}{2}(\rho + \varphi_0)^2 - \frac{\lambda}{4}(\rho + \varphi_0)^4 . \tag{42}$$

(At the expense of a little algebra, the calculation can be done the other way. First substitute Eq. 17 for φ in Eq. 38. One then finds an $A^\mu\partial_\mu\pi$ term in $\mathscr{L}$ that can be removed using the local phase transformation $A'_\mu = A_\mu - [1/(e\varphi_0)]\partial_\mu\pi$, $\rho' = \rho$, and $\pi' = 0$. Equation 42 then follows, although this method requires some effort. Thus, a reason for doing the calculation in the order of Eq. 41 is that the algebra gets messy rather quickly if the local symmetry is not used early in the calculation of the electroweak case. However, in principle it makes little difference.)

The Lagrangian in Eq. 42 is an amazing result: the π field has vanished from $\mathscr{L}$ altogether (according to Eq. 41, it was simply a gauge artifact), and there is a term $\tfrac{1}{2}e^2\varphi_0^2 A^\mu A_\mu$ in $\mathscr{L}$, which is a mass term for the vector particle. Thus, the massless particle of the global case has become the longitudinal mode of a massive vector particle, and there is only one scalar particle ρ left in the theory. In somewhat more picturesque language the vector boson has eaten the Goldstone boson and become heavy from the feast. However, the existence of the vector boson mass terms should not be understood in isolation: the phase invariance of Eq. 42 determines the form of the interaction of the massive A_μ field with the ρ field.

This calculation makes it clear that it can be tricky to derive the spectrum of a theory with local symmetry and spontaneous symmetry breaking. Theoretical physicists have taken great care to confirm that this interpretation is correct and that it generalizes to the full quantum field theory.

7 Yang-Mills Theories

The standard model possesses symmetries of the type described in Note 4, except that they are local. Thus, we need to carry out the calculations of Note 5 for Lie-group symmetries. As the reader might expect, this requires replacing $\varepsilon(x)$ of Eq. 13 by a matrix or, equivalently, the matrix of Eq. 21 by a matrix function of x, $\varepsilon^a(x)T_a$. The Yang-Mills Lagrangian can be derived by mimicking with matrix functions Eqs. 34 to 38.

The internal, local transformation of the φ field (φ is a column vector with components φ_i, where i runs from 1 to n) is

$$\varphi'(x) = e^{i\varepsilon(x)}\varphi(x) , \tag{43}$$

which is formally identical to Eq. 30, except that $\varepsilon(x)$ is now an n-by-n matrix. Thus,

$$\varepsilon(x) = \varepsilon^a(x)T_a , \tag{44}$$

where the sum on a is over the N independent symmetries. Equation 43 is a symmetry of the potential energy

$$V = \mu^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2 , \tag{45}$$

if $\varepsilon(x)$ in Eq. 44 is a Hermitian matrix (that is, if $T_a = T_a^\dagger$ and the $\varepsilon^a(x)$ are real functions). The kinetic energy $(\partial^\mu\varphi)^\dagger(\partial_\mu\varphi)$ can be made phase invariant by extending ∂_μ to D_μ, analogous to Eq. 33 for electrodynamics:

$$D_\mu = \partial_\mu - ieA_\mu , \tag{46a}$$

where

$$A_\mu = A_\mu^a T_a , \tag{46b}$$

so that A_μ is an n-by-n matrix that acts on the φ vector. Just as for Eq. 35, the transformation properties of A_μ are derived from the equation

$$D'_\mu\varphi'(x) = e^{i\varepsilon(x)}\, D_\mu\varphi(x) . \tag{47}$$

After some matrix manipulation one finds the solution of Eq. 47 for $A'_\mu(x)$ in terms of $A_\mu(x)$ to be

$$A'_\mu(x) = e^{i\varepsilon(x)}\, A_\mu(x)e^{-i\varepsilon(x)} - \frac{1}{e}\,\partial_\mu\varepsilon(x) , \tag{48}$$

where $e^{-i\varepsilon(x)}$ is the inverse of the matrix $e^{i\varepsilon(x)}$. With these requirements, it is easily seen that $(D^\mu\varphi)^\dagger(D_\mu\varphi)$ is invariant under the group of local transformations.

The calculation of the field tensor is formally identical to Eq. 36, except we must take into account that $A_\mu(x)$ is a matrix. Thus, we define a matrix $F_{\mu\nu}$ field tensor as

$$F_{\mu\nu} \equiv \frac{i}{e}\left[D_\mu , D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu - ie\left[A_\mu , A_\nu\right] . \tag{49}$$

There is a field tensor for each group generator, and some further matrix manipulation plus Eq. 26 gives the components,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + ef^{abc}A_{\mu b}A_{\nu c} . \tag{50}$$

The transformation law for the matrix $F_{\mu\nu}$ is

$$F'_{\mu\nu} = e^{i\varepsilon(x)}F_{\mu\nu}e^{-i\varepsilon(x)} . \tag{51}$$

Thus, we can write down a kinetic energy term in analogy to electrodynamics:

$$\mathscr{L}_{\text{kinetic energy}} = -\frac{1}{4}\, F_{\mu\nu}^a F_a^{\mu\nu} . \tag{52}$$

The locally invariant Yang-Mills Lagrangian for spinless fields coupled to the vector bosons is

$$\mathscr{L} = -\frac{1}{4}\, F_{\mu\nu}^a F_a^{\mu\nu} + (D^\mu\varphi)^\dagger(D_\mu\varphi) - \mu^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2 . \tag{53}$$

Just as in electrodynamics, we can add fermions to the theory in the form

$$\mathscr{L}_{\text{fermion}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi , \tag{54}$$

where D_μ is defined in Eq. 46 and ψ is a column vector with n_f entries (n_f = number of fermions). The matrices T_a in D_μ for the fermion covariant derivative are usually different from the matrices for the spinless fields, since there is no requirement that φ and ψ need to belong to the same representation of the group. It is, of course, necessary for the sets of T_a matrices to satisfy the commutation relations of Eq. 26 with the same set of structure constants.

We will not look at the general case of spontaneous symmetry breaking in a Yang-Mills theory, which is a messy problem mathematically. There is spontaneous symmetry breaking in the electroweak sector of the standard model, and we will work out the steps analogous to Eqs. 41 and 42 for this particular case in the next Note.

8 The SU(2) × U(1) Electroweak Model

The main emphasis in these Notes has been on developing just those aspects of Lagrangian field theory that are needed for the standard model. We have now come to the crucial step: finding a Lagrangian that describes the electroweak interactions. It is rather difficult to be systematic. The historical approach would be complicated by the rather late discovery of the weak neutral currents, and a purely phenomenological development is not yet totally logical because there are important aspects of the standard model that have not yet been tested experimentally. (The most important of these are the details of the spontaneous symmetry breaking.) Although we will write down the answer without excessive explanation, the reader should not forget the critical role that experimental data played in the development of the theory.

The first problem is to identify the local symmetry group. Before the standard model was proposed over twenty years ago, the electromagnetic and charge-changing weak interactions were known. The smallest continuous group that can describe these is SU(2), which has a doublet representation. If the weak interactions can change electrons to electron neutrinos, which are electrically neutral, it is not possible to incorporate electrodynamics in SU(2) alone unless a heavy positively charged electron is added to the electron and its neutrino to make a triplet, because the sum of charges in an SU(2) multiplet is zero. Various schemes of this sort have been tried but do not agree with experiment. The only way to leave the electron and electron neutrino in a doublet and include electrodynamics is to add an extra U(1) interaction to the theory. The hypothesis of the extra U(1) factor was challenged many times until the discovery of the weak neutral current. That discovery established that the local symmetry of the electroweak theory had to be at least as large as SU(2) × U(1).

Let us now interpret the physical meaning of the four generators of SU(2) × U(1). The three generators of the SU(2) group are I^+, I_3, and I^-, and the generator of the U(1) group is called Y, the weak hypercharge. (The weak SU(2) and U(1) groups are distinguished from other SU(2) and U(1) groups by the label "W.") I^+ and I^- are associated with the weak charge-changing currents (the general definition of a current is described in Note 2), and the charge-changing currents couple to the W^+ and W^- charged weak vector bosons in analogy to Eq. 40b. Both I_3 and Y are related to the electromagnetic current and the weak neutral current. In order to assign the electron and its neutrino to an SU(2) doublet, the electric charge Q^{em} is defined by

$$Q^{\text{em}} = I_3 + Y/2\,, \tag{55}$$

so the sum of electric charges in an n-dimensional multiplet is $nY/2$. The charge of the weak neutral current is a different combination of I_3 and Y, as will be described below.

The Lagrangian includes many pieces. The kinetic energies of the vector bosons are described by $\mathcal{L}_{\text{Y-M}}$, in analogy to the first term in Eq. 38. The three weak bosons have masses acquired through spontaneous symmetry breaking, so we need to add a scalar piece $\mathcal{L}_{\text{scalar}}$ to the Lagrangian in order to describe the observed symmetry breaking (also see Eq. 38). The fermion kinetic energy $\mathcal{L}_{\text{fermion}}$ includes the fermion-boson interactions, analogous to the electromagnetic interactions derived in Eqs. 39 and 40. Finally, we can add terms that couple the scalars with the fermions in a term $\mathcal{L}_{\text{Yukawa}}$. One physical significance of the Yukawa terms is that they provide for masses of the quarks and charged leptons.

The standard model is then a theory with a very long Lagrangian with many fields. The electroweak Lagrangian has the terms

$$\mathcal{L}_{\text{electroweak}} = \mathcal{L}_{\text{Y-M}} + \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{Yukawa}}\,. \tag{56}$$

(The reader may find this construction to be ad hoc and ugly. If so, the motivation will be clear for searching for a more unified theory from which this Lagrangian can be derived. However, it is important to remember that, at present, the standard model is the pinnacle of success in theoretical physics and describes a broader range of natural phenomena than any theory ever has.)

The Yang-Mills kinetic energy term has the form given by Eq. 52 for the SU(2) bosons, plus a term for the U(1) field tensor similar to electrodynamics (Eqs. 36 and 38).

$$\mathcal{L}_{\text{Y-M}} = -\frac{1}{4}F_a^{\mu\nu}F^a_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \tag{57}$$

where the U(1) field tensor is

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{58}$$

and the SU(2) Yang-Mills field tensor is

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\varepsilon_{abc}W^b_\mu W^c_\nu\,, \tag{59}$$

where the ε_{abc} are the structure constants for SU(2) defined in Eq. 24 and the W^a_μ are the Yang-Mills fields.

8 continued

$SU(2) \times U(1)$ has two factors, and there is an independent coupling constant for each factor. The coupling for the SU(2) factor is called g, and it has become conventional to call the U(1) coupling $g'/2$. The two couplings can be written in several ways. The U(1) of electrodynamics is generated by a linear combination of I_3 and Y, and the coupling is, as usual, denoted by e. The other coupling can then be parameterized by an angle θ_W. The relations among g, g', e, and θ_W are

$$e \equiv gg'/\sqrt{g^2+g'^2} \text{ and } \tan\theta_W \equiv g'/g\,. \tag{60}$$

These definitions will be motivated shortly. In the electroweak theory both couplings must be evaluated experimentally and cannot be calculated in the standard model.

The scalar Lagrangian requires a choice of representation for the scalar fields. The choice requires that the field with a nonzero vacuum value is electrically neutral, so the photon remains massless, but it must carry nonzero values of I_3 and Y so that the weak neutral boson (the Z^0_μ) acquires a mass from spontaneous symmetry breaking. The simplest assignment is

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} \text{ and } \varphi^\dagger = (-\varphi^-, (\varphi^0)^\dagger)\,, \tag{61}$$

where φ^+ has $I_3 = 1/2$ and $Y = 1$, and φ^0 has $I_3 = -1/2$ and $Y = 1$. Since φ does not have $Y = -1$ fields, it is necessary to make φ a complex doublet, so $(\varphi^+)^\dagger = -\varphi^-$ has $I_3 = -1/2$ and $Y = -1$, and $(\varphi^0)^\dagger$ has $I_3 = 1/2$ and $Y = -1$. Then we can write down the Lagrangian of the scalar fields as

$$\mathscr{L}_{\text{scalar}} = (D^\mu\varphi)^\dagger(D_\mu\varphi) - m^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2\,, \tag{62}$$

where

$$D_\mu\varphi = \partial_\mu\varphi - i\frac{g'}{2}B_\mu\varphi - i\frac{g}{2}\tau_a W^a_\mu\varphi \tag{63}$$

is the covariant derivative. The 2-by-2 matrices τ_a are the Pauli matrices. The factor of 1/2 is required because the doublet representation of the SU(2) generators is $\tau_a/2$. The factor of 1/2 in the B_μ term is due to the convention that the U(1) coupling is $g'/2$ and the assignment that the φ doublet has $Y = 1$. After the spontaneous symmetry breaking, three of the four scalar degrees of freedom are "eaten" by the weak bosons. Thus just one scalar escapes the feast and should be observable as an independent neutral particle, called the Higgs particle. It has not (?) yet been observed experimentally, and it is perhaps the most important particle in the standard model that does not yet have a firm phenomenological basis. (The minimum number of scalar fields in the standard model is four. Experimental data could eventually require more.)

We now carry out the calculation for the spontaneous symmetry breaking of $SU(2) \times U(1)$ down to the U(1) of electrodynamics. Just as in the example worked out in Note 6, spontaneous symmetry breaking occurs when $m^2 < 0$ in Eq. 62. In contrast to the simpler case, it is rather important to set up the problem in a clever way to avoid an inordinate amount of computation. As in Eq. 41, we write the four degrees of freedom in the complex scalar doublet so that it looks like a local symmetry transformation times a simple form of the field:

$$\varphi(x) = \exp[i\pi^a(x)\tau_a/2\varphi_0]\begin{pmatrix} 0 \\ [\rho(x) + \varphi_0]/\sqrt{2} \end{pmatrix}. \tag{64}$$

We can then write the scalar fields in a new gauge where the phases of $\varphi(x)$ are removed:

$$\varphi'(x) = \exp[-i\pi^a(x)\tau_a/2\varphi_0]\varphi(x) = \begin{pmatrix} 0 \\ [\rho(x) + \varphi_0]/\sqrt{2} \end{pmatrix}, \tag{65}$$

where we have used the freedom of making local symmetry transformations to write $\varphi'(x)$ in a very simple form. This choice, called the unitary gauge, will make it easy to write out Eq. 63 in explicit matrix form. Let us drop all primes on the fields in the unitary gauge and redefine W^a_μ by the equation

$$\tau_a W^a_\mu = \begin{pmatrix} W^3_\mu & W^1_\mu - iW^2_\mu \\ W^1_\mu + iW^2_\mu & -W^3_\mu \end{pmatrix} = \begin{pmatrix} W^3_\mu & \sqrt{2}\,W^+_\mu \\ \sqrt{2}\,W^-_\mu & -W^3_\mu \end{pmatrix}, \tag{66}$$

where the definition of the Pauli matrices is used in the first step, and the $W^\pm$ fields are defined in the second step with a numerical factor that guarantees the correct normalization of the kinetic energy of the charged weak vector bosons.

Next, we write out the $D_\mu\varphi$ in explicit matrix form, using Eqs. 63, 65, and 66:

$$D_\mu\varphi = \frac{1}{\sqrt{2}}\begin{pmatrix} -i\sqrt{2}gW^+_\mu(\rho + \varphi_0)/2 \\ \partial_\mu\rho - i(g'B_\mu - gW^3_\mu)(\rho + \varphi_0)/2 \end{pmatrix}. \tag{67}$$

Finally, we substitute Eqs. 65 and 67 into Eq. 63 and obtain

$$\mathscr{L}_{\text{scalar}} = \frac{g^2}{4} W^{\mu}_{-} W^{+}_{\mu} (\rho + \varphi_0)^2 + \frac{1}{2}\, \partial^{\mu}\rho\partial_{\mu}\rho$$

$$+ \frac{1}{8}\, (g' B^{\mu} - g W^{\mu}_{3})(g' B_{\mu} - g W^{3}_{\mu})(\rho + \varphi_0)^2$$

$$+ \frac{m^2}{2}\, (\rho + \varphi_0)^2 + \frac{\lambda}{4}\, (\rho + \varphi_0)^4\,, \tag{68}$$

where ρ is the, as yet (?), unobserved Higgs field.

It is clear from Eq. 68 that the W fields will acquire a mass equal to $g\varphi_0/2$ from the term quadratic in the W fields, $(g^2/4)\varphi_0^2 W^{\mu}_{-} W^{+}_{\mu}$. The combination $g'B_\mu - gW^3_\mu$ will also have a mass. Thus, we "rotate" the B_μ and W^3_μ fields to the fields Z^0_μ for the weak neutral boson and A_μ for the photon so that the photon is massless.

$$\begin{pmatrix} Z^0_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \sin\theta_W & -\cos\theta_W \\ \cos\theta_W & \sin\theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix}, \tag{69}$$

where

$$\cos\theta_W = g/\sqrt{g^2 + g'^2} \quad \text{and} \quad \sin\theta_W = g'/\sqrt{g^2 + g'^2}\,. \tag{70}$$

Upon substituting Eqs. 69 and 70 into Eq. 68, we find that the Z^0_μ mass is $\tfrac{1}{2}\,\varphi_0\,\sqrt{g^2 + g'^2}$, so the ratio of the W and Z masses is

$$M_W/M_Z = \cos\theta_W\,. \tag{71}$$

Values for M_W and M_Z have recently been measured at the CERN proton-antiproton collider: $M_W = (80.8 \pm 2.7)$ GeV/c^2 and $M_Z = (92.9 \pm 1.6)$ GeV/c^2. The ratio M_W/M_Z calculated with these values agrees well with that given by Eq. 71. (The angle θ_W is usually expressed as $\sin^2\theta_W$ and is measured in neutrino-scattering experiments to be $\sin^2\theta_W = 0.224 \pm 0.015$.) The photon field A_μ does not appear in $\mathscr{L}_{\text{scalar}}$, so it does not become massive from spontaneous symmetry breaking. Note, also, that the $\pi^a(x)$ fields appear nowhere in the Lagrangian; they have been eaten by three weak vector bosons, which have become massive from the feast.

The next term in Eq. 56 is $\mathscr{L}_{\text{fermion}}$. Its form is analogous to Eqs. 39 and 40 for electrodynamics:

$$\mathscr{L}_{\text{fermion}} = i\bar{\psi}\gamma^{\mu} D_{\mu}\psi\,. \tag{72}$$

The physical problem is to assign the left- and right-handed fermions to multiplets of SU(2); the assignments rely heavily on experimental data and are listed in "Particle Physics and the Standard Model." Our purpose here will be to write out Eq. 72 explicitly for the assignments.

Consider the electron and its neutrino. (The quark and remaining lepton contributions can be worked out in a similar fashion.) The left-handed components are assigned to a doublet and the right-handed components are singlets. (Since a neutral singlet has no weak charge, the right-handed component of the neutrino is invisible to weak, electromagnetic, or strong interactions. Thus, we can neglect it here, whether or not it actually exists.) We adopt the notation

$$\psi_L = \begin{pmatrix} \nu_L \\ e^-_L \end{pmatrix} \quad \text{and} \quad \psi_R = (e^-_R)\,, \tag{73}$$

where L and R denote left- and right-handed. Then the explicit statement of Eq. 72 requires constructing D_μ for the left- and right-handed leptons.

$$\mathscr{L}_{\text{lepton}} = i\bar{\psi}_R\gamma^{\mu}(\partial_\mu + ig'B_\mu)\psi_R$$

$$+ i\bar{\psi}_L\gamma^{\mu}[\partial_\mu + \frac{i}{2}\,(g'B_\mu - g\tau_a W^a_\mu)]\psi_L\,. \tag{74}$$

The weak hypercharge of the right-handed electron is -2 so the coefficient of B_μ in the first term of Eq. 74 is $(-g'/2)\times(-2) = g'$. We leave it to the reader to check the rest of Eq. 74. The absence of a mass term is not an error. Mass terms are of the form $\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$. Since ψ_L is a doublet and $\bar{\psi}_R$ is a singlet, an electron mass term must violate the SU(2)$\times$U(1) symmetry. We will see later that the electron mass will reappear as a result of modification of $\mathscr{L}_{\text{Yukawa}}$ due to spontaneous symmetry breaking.

The next task is exciting, because it will reveal how the vector bosons interact with the leptons. The calculation begins with Eq. 74 and requires the substitution of explicit matrices for $\tau_a W^a_\mu$, ψ_R, and ψ_L. We use the definitions in Eqs. 66, 69, and 73. The expressions become quite long, but the calculation is very straightforward. After simplifying some expressions, we find that $\mathscr{L}_{\text{lepton}}$ for the electron lepton and its neutrino is

$$\mathscr{L}_{\text{lepton}} = i\bar{e}\gamma^{\mu}\partial_\mu e + i\bar{\nu}_L\gamma^{\mu}\partial_\mu\nu_L - e\,\bar{e}\gamma^{\mu}eA_\mu$$

$$+ \frac{g}{\sqrt{2}}\,(\bar{\nu}_L\gamma^{\mu}e_L W^{+}_{\mu} + \bar{e}_L\gamma^{\mu}\nu_L W^{-}_{\mu})$$

$$- \frac{g^2}{2\sqrt{g^2 + g'^2}}\,[\tan^2\theta_W(2\bar{e}_R\gamma^{\mu}e_R + \bar{e}_L\gamma^{\mu}e_L) - \bar{e}_L\gamma^{\mu}e_L]Z_\mu$$

$$- \frac{1}{2}\,\sqrt{g^2 + g'^2}\;\bar{\nu}_L\gamma^{\mu}\nu_L Z_\mu\,. \tag{75}$$

8 continued

The first two terms are the kinetic energies of the electron and the neutrino. (Note that $e = e_L + e_R$.) The third term is the electromagnetic interaction (cf. Eq. 40) with electrons of charge $-e$, where e is defined in Eq. 60. The coupling of A_μ to the electron current does not distinguish left from right, so electrodynamics does not violate parity. The fourth term is the interaction of the $W^\pm$ bosons with the weak charged current of the neutrinos and electrons. Note that these bosons are blind to right-handed electrons. This is the reason for maximal parity violation in beta decay. The final terms predict how the weak neutral current of the electron and that of the neutrino couple to the neutral weak vector boson Z^0.

If the left- and right-handed electron spinors are written out explicitly, with $e_L = \frac{1}{2}(1 - \gamma_5)e$, the interaction of the weak neutral current of the electron with the Z^0 is proportional to $\bar{e}\gamma^\mu[(1 - 4\sin^2\theta_W) - \gamma_5]eZ_\mu$. This prediction provided a crucial test of the standard model. Recall from Eq. 71 that $\sin^2\theta_W$ is very nearly $\frac{1}{4}$, so that the weak neutral current of the electron is very nearly a purely axial current, that is, a current of the form $\bar{e}\gamma^\mu\gamma_5 e$. This crucial prediction was tested in deep inelastic scattering of polarized electrons and in atomic parity-violation experiments. The results of these experiments went a long way toward establishing the standard model. The tests also ruled out models quite similar to the standard model. We could discuss many more tests and predictions of the model based on the form of the weak currents, but this would greatly lengthen our discussion. The electroweak currents of the quarks will be described in the next section.

We now discuss the last term in Eq. 56, $\mathscr{L}_{\text{Yukawa}}$. In a locally symmetric theory with scalars, spinors, and vectors, the interactions between vectors and scalars, vector and spinors, and vectors and vectors are determined from the local invariance by replacing ∂_μ by D_μ. *In contrast,* $\mathscr{L}_{\text{Yukawa}}$, which is the interaction between the scalars and spinors, has the same form for both local and global symmetries:

$$\begin{aligned}\mathscr{L}_{\text{Yukawa}} &= G_Y\bar{\psi}\varphi\psi \\ &= G_Y(\bar{\psi}_L\varphi\psi_R + \bar{\psi}_R\varphi^\dagger\psi_L)\,. \end{aligned}\tag{76}$$

This form for $\mathscr{L}_{\text{Yukawa}}$ is rather schematic; to make it explicit we must specify the multiplets and then arrange the component fields so that the form of $\mathscr{L}_{\text{Yukawa}}$ does not change under a local symmetry transformation.

Let us write Eq. 76 explicitly for the part of the standard model we have examined so far: φ is a complex doublet of scalar fields that has the form in the unitary gauge given by Eq. 65. The fermions include the electron and its neutrino. If the neutrino has no right-handed component, then it is not possible to insert it into Eq. 76. Since the neutrino has no mass term in $\mathscr{L}_{\text{lepton}}$, the neutrino remains massless in this theory. (If ν_R is included, then the neutrino mass is a free parameter.) The Yukawa terms for the electron are

$$\begin{aligned}\mathscr{L}_{\text{Yukawa}} &= G_Y\left[(\bar{\nu}_L, \bar{e}_L)\begin{pmatrix}0 \\ (\rho + \varphi_0)/\sqrt{2}\end{pmatrix}(e_R)\right. \\ &\qquad \left. + (\bar{e}_R)\,(0, (\rho + \varphi_0)/\sqrt{2})\begin{pmatrix}\nu_L \\ e_L\end{pmatrix}\right] \\ &= \frac{1}{\sqrt{2}}\,G_Y\bar{e}e(\rho + \varphi_0)\,, \end{aligned}\tag{77}$$

where we have used the fact that $\bar{e}_Le_L = \bar{e}_Re_R = 0$, and $e = e_L + e_R$ is the electron Dirac spinor. Note that Eq. 77 includes an electron mass term,

$$m_e = \frac{1}{\sqrt{2}}\,G_Y\varphi_0\,, \tag{78}$$

so the electron mass is proportional to the vacuum value of the scalar field. The Yukawa coupling is a free parameter, but we can use the measured electron mass to evaluate it. Recall that

$$M_W = \frac{g\varphi_0}{2} = \frac{e\varphi_0}{2\sin\theta_W} \simeq 81\text{ GeV}\,,$$

where $e^2/4\pi \simeq 1/137$. This implies that $\varphi_0 = 251$ GeV. Since $m_e = 0.000511$ GeV, $G_Y = 2.8 \times 10^{-6}$ for the electron. There are more than five Yukawa couplings, including those for the μ and τ leptons and the three quark doublets as well as terms that mix different quarks of the same electric charge. The standard model in no way determines the values of these Yukawa coupling constants. Thus, the study of fermion masses may turn out to have important hints on how to extend the standard model.

9 Quarks

Discovery of the fundamental fields of the strong interactions was not straightforward. It took some years to realize that the hadrons, such as the nucleons and mesons, are made up of subnuclear constituents, primarily quarks. Quarks originated from an effort to provide a simple physical picture of the "Eightfold Way," which is the SU(3) symmetry proposed by M. Gell-Mann and Y. Ne'eman to generalize strong isotopic spin. The hadrons could not be classified by the fundamental three-dimensional representations of this SU(3) but instead are assigned to eight- and ten-dimensional representations. These larger representations can be interpreted as products of the three-dimensional representations, which suggested to Gell-Mann and G. Zweig that hadrons are composed of constituents that are assigned to the three-dimensional representations: the u (up), d (down), and s (strange) quarks. At the time of their conception, it was not clear whether quarks were a physical reality or a mathematical trick for simplifying the analysis of the Eightfold-Way SU(3). The major breakthrough in the development of the present theory of strong interactions came with the realization that, in addition to electroweak and Eightfold-Way quantum numbers, quarks carry a new quantum number, referred to as color. This quantum number has yet to be observed experimentally.

We begin this lecture with a description of the Lagrangian of a strong-interaction theory of quarks formulated in terms of their color quantum numbers. Called quantum chromodynamics, or QCD, it is a Yang-Mills theory with local color-SU(3) symmetry in which each quark belongs to a three-dimensional color multiplet. The eight color-SU(3) generators commute with the electroweak SU(2) $\times$ U(1) generators, and they also commute with the generators of the Eightfold Way, which is a different SU(3). (Like SU(2), SU(3) is a recurring symmetry in physics, so its various roles need to be distinguished. Hence we need the label "color.") We conclude with a discussion of the weak interactions of the quarks.

The QCD Lagrangian. The interactions among the quarks are mediated by eight massless vector bosons (called gluons) that are required to make the SU(3) symmetry local. As we have already seen, the assumption of local symmetry leads to a Lagrangian whose form is highly restricted. As far as we know, only the quark and gluon fields are necessary to describe the strong interactions, and so the most general Lagrangian is

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + i\sum_i \bar{\psi}_i \gamma^\mu D_\mu \psi_i + \sum_{i,j} \bar{\psi}_i M_{ij} \psi_j\,, \tag{79}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f_{abc} A^b_\mu A^c_\nu\,. \tag{80}$$

The sum on a in the first term is over the eight gluon fields A^a_μ. The second term represents the coupling of each gluon field to an SU(3) current of the quark fields, called a color current. This term is summed over the index i, which labels each quark type and is independent of color. Since each quark field ψ_i is a three-dimensional column vector in color space, D_μ is defined by

$$D_\mu \psi_i = \partial_\mu \psi_i - \frac{1}{2} i g_s A^a_\mu \lambda_a \psi_i\,, \tag{81}$$

where λ_a is a generalization of the three 2-by-2 Pauli matrices of SU(2) to the eight 3-by-3 Gell-Mann matrices of SU(3), and g_s is the QCD coupling. Thus, the color current of each quark has the form $\bar{\psi}\lambda_a\gamma^\mu\psi$. The left-handed quark fields couple to the gluons with exactly the same strength as the right-handed quark fields, so parity is conserved in the strong interactions.

The gluons are massless because the QCD Lagrangian has no spinless fields and therefore no obvious possibility of spontaneous symmetry breaking. Of course, if motivated for experimental reasons, one can add scalars to the QCD Lagrangian and spontaneously break SU(3) to a smaller group. This modification has been used, for example, to explain the reported observation of fractionally charged particles. The experimental situation, however, still remains murky, so it is not (yet) necessary to spontaneously break SU(3) to a smaller group. For the remainder of the discussion, we assume that QCD is not spontaneously broken.

The third term in Eq. 79 is a mass term. In contrast to the electroweak theory, this mass term is now allowed, even in the absence of spontaneous symmetry breaking, because the left- and right-handed quarks are assigned to the same multiplet of SU(3). The numerical coefficients M_{ij} are the elements of the quark mass matrix; they can connect quarks of equal electric charge. The $\mathcal{L}_{\mathrm{QCD}}$ of Eq. 79 permits us to redefine the QCD quark fields so that $M_{ij} = m_i\delta_{ij}$. The

9 continued

mass matrix is then diagonal and each quark has a definite mass, which is an eigenvalue of the mass matrix. We will reappraise this situation below when we describe the weak currents of the quarks.

After successfully extracting detailed predictions of the electroweak theory from its complicated-looking Lagrangian, we might be expected to perform a similar feat for the $\mathscr{L}_{QCD}$ of Eq. 79 without too much difficulty. This is not possible. Analysis of the electroweak theory was so simple because the couplings g and g' are always small, regardless of the energy scale at which they are measured, so that a classical analysis is a good first approximation to the theory. The quantum corrections to the results in Note 8 are, for most processes, only a few percent.

In QCD processes that probe the *short-distance* structure of hadrons, the quarks inside the hadrons interact weakly, and here the classical analysis is again a good first approximation because the coupling g_s is small. However, for Yang-Mills theories in general, the renormalization group equations of quantum field theory require that g_s increases as the momentum transfer decreases until the momentum transfer equals the masses of the vector bosons. Lacking spontaneous symmetry breaking to give the gluons mass, QCD contains no mechanism to stop the growth of g_s, and the quantum effects become more and more dominant at larger and larger distances. Thus, analysis of the *long-distance* behavior of QCD, which includes deriving the hadron spectrum, requires solving the full quantum theory implied by Eq. 79. This analysis is proving to be very difficult.

Even without the solution of $\mathscr{L}_{QCD}$, we can, however, draw some conclusions. The quark fields ψ_i in Eq. 79 must be determined by experiment. The Eightfold Way has already provided three of the quarks, and phenomenological analyses determine their masses (as they appear in the QCD Lagrangian). The mass of the u quark is nearly zero (a few MeV/c^2), the d quark is a few MeV/c^2 heavier than the u, and the mass of the s quark is around 300 MeV/c^2. If these results are substituted into Eq. 79, we can derive a beautiful result from the QCD Lagrangian. In the limit that the quark mass differences can be ignored, Eq. 79 has a global SU(3) symmetry that is identical to the Eightfold-Way SU(3) symmetry. Moreover, in the limit that the u, d, and s masses can be ignored, the left-handed u, d, and s quarks can be transformed by one SU(3) and the right-handed u, d, and s quarks by an independent SU(3). Then QCD has the "chiral" SU(3) × SU(3) symmetry that is the basis of current algebra. The sums of the corresponding SU(3) generators of chiral SU(3) × SU(3) generate the Eightfold-Way SU(3). Thus, the QCD Lagrangian incorporates in a very simple manner the symmetry results of hadronic physics of the 1960s. The more recently discovered c (charmed), b (bottom), and t (top) quarks are easily added to the QCD Lagrangian. Their masses are so large and so different from one another that the SU(3) and SU(3) × SU(3) symmetries of the Eightfold-Way and current algebra cannot be extended to larger symmetries. (The predictions of, say, SU(4) and chiral SU(4) × SU(4) do not agree well with experiment.)

It is important to note that the quark masses are undetermined parameters in the QCD Lagrangian and therefore must be derived from some more complete theory or indicated phenomenologically. The Yukawa couplings in the electroweak Lagrangian are also free parameters. Thus, we are forced to conclude that the standard model alone provides no constraints on the quark masses, so they must be obtained from experimental data.

The mass term in the QCD Lagrangian (Eq. 79) has led to new insights about the neutron-proton mass difference. Recall that the quark content of a neutron is *udd* and that of a proton is *uud*. If the u and d quarks had the same mass, then we would expect the proton to be more massive than the neutron because of the electromagnetic energy stored in the *uu* system. (Many researchers have confirmed this result.) Since the masses of the u and d quarks are arbitrary in both the QCD and the electroweak Lagrangians, they can be adjusted phenomenologically to account for the fact that the neutron mass is 1.293 MeV/c^2 greater than the proton mass. This experimental constraint is satisfied if the mass of the d quark is about 3 MeV/c^2 greater than that of the u quark. In a way, this is unfortunate, because we must conclude that the famous puzzle of the *n-p* mass difference will not be solved until the standard model is extended enough to provide a theory of the quark masses.

Weak Currents. We turn now to a discussion of the weak currents of the quarks, which are determined in the same way as the weak currents of the leptons in Note 8. Let us begin with just the u and d quarks. Their electroweak assignments are as follows: the left-handed components u_L and d_L form an SU(2) doublet with $Y = 1/3$, and the right-handed components u_R and d_R are SU(2) singlets with $Y = 4/3$

and −⅔, respectively (recall Eq. 55).

The steps followed in going from Eq. 73 to Eq. 75 will yield the electroweak Lagrangian of quarks. The contribution to the Lagrangian due to interaction of the weak neutral current $J_\mu^{(\mathrm{nc})}$ of the u and d quarks with Z^0 is

$$\mathscr{L}^{(\mathrm{nc})} = \frac{e}{\sin\theta_W \cos\theta_W} J_\mu^{(\mathrm{nc})} Z_0^\mu \,, \tag{82}$$

where

$$J_\mu^{(\mathrm{nc})} = \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_W\right)\bar{u}_L\gamma_\mu u_L - \frac{2}{3}\sin^2\theta_W \bar{u}_R\gamma_\mu u_R$$

$$+ \left(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W\right)\bar{d}_L\gamma_\mu d_L + \frac{1}{3}\sin^2\theta_W \bar{d}_R\gamma_\mu d_R \,. \tag{83}$$

The reader will enjoy deriving this result and also deriving the contribution of the weak charged current of the quarks to the electroweak Lagrangian. Equation 83 will be modified slightly when we include the other quarks.

So far we have emphasized in Notes 8 and 9 the construction of the QCD and electroweak Lagrangians for just one lepton-quark "family" consisting of the electron and its neutrino together with the u and d quarks. Two other lepton-quark families are established experimentally: the muon and its neutrino along with the c and s quarks and the τ lepton and its neutrino along with the t and b quarks. Just like $(\nu_e)_L$ and e_L, $(\nu_\mu)_L$ and μ_L and $(\nu_\tau)_L$ and τ_L form weak-SU(2) doublets; e_R, μ_R and τ_R are each SU(2) singlets with a weak hypercharge of −2. Similarly, the weak quantum numbers of c and s and of t and b echo those of u and d: c_L and s_L form a weak-SU(2) doublet as do t_R and b_L. Like u_R and d_R, the right-handed quarks c_R, s_R, t_R, and b_R are all weak-SU(2) singlets.

This triplication of families cannot be explained by the standard model, although it may eventually turn out to be a critical fact in the development of theories of the standard model. The quantum numbers of the quarks and leptons are summarized in Tables 2 and 3 in "Particle Physics and the Standard Model."

All these quark and lepton fields must be included in a Lagrangian that incorporates both the electroweak and QCD Lagrangians. It is quite obvious how to do this: the standard model Lagrangian is simply the sum of the QCD and electroweak Lagrangians, except that the terms occurring in both Lagrangians (the quark kinetic energy terms $i\bar{\psi}_i\gamma^\mu\partial_\mu\psi_i$ and the quark mass terms $\bar{\psi}_i M_{ij}\psi_j$) are included just once. Only the mass term requires comment.

The quark mass terms appear in the electroweak Lagrangian in the form $\mathscr{L}_{\mathrm{Yukawa}}$ (Eq. 77). In the electroweak theory quarks acquire masses only because SU(2) × U(1) is spontaneously broken. However, when there are three quarks of the same electric charge (such as d, s, and b), the general form of the mass terms is the same as in Eq. 79, $\bar{\psi}_i M_{ij}\psi_j$, because there can be Yukawa couplings between d and s, d and b, and s and b. The problem should already be clear: when we speak of quarks, we think of fields that have a definite mass, that is, fields for which M_{ij} is diagonal. Nevertheless, there is no reason for the fields obtained directly from the electroweak symmetry breaking to be mass eigenstates.

The final part of the analysis takes some care: the problem is to find the most general relation between the mass eigenstates and the fields occurring in the weak currents. We give the answer for the case of two families of quarks. Let us denote the quark fields in the weak currents with primes and the mass eigenstates without primes. There is freedom in the Lagrangian to set $u = u'$ and $c = c'$. If we do so, then the most general relationship among d, s, d', and s' is

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_C & -\sin\theta_C \\ \sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix} . \tag{84}$$

The parameter θ_C, the Cabibbo angle, is not determined by the electroweak theory (it is related to ratios of various Yukawa couplings) and is found experimentally to be about 13°. (When the b and t ($=t'$) quarks are included, the matrix in Eq. 84 becomes a 3-by-3 matrix involving four parameters that are evaluated experimentally.) The correct weak currents are then given by Eq. 83 if all quark families are included and primes are placed on all the quark fields. The weak currents can be written in terms of the quark mass eigenstates by substituting Eq. 84 (or its three-family generalization) into the primed version of Eq. 83. The ratio of amplitudes for $s \to u$ and $d \to u$ is tan θ_C; the small ratio of the strangeness-changing to non-strangeness-changing charged-current amplitudes is due to the smallness of the Cabibbo angle. It is worth emphasizing again that the standard model alone provides no understanding of the value of this angle.