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On simulation and analysis of variable-rate pumping tests

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Abstract

Analytical solutions for constant-rate pumping tests are widely used to infer aquifer properties. The aquifer parameters are estimated by fitting pressure responses observed during pumping test to appropriate analytical solutions for radial flow towards the pumping well. For mathematical simplicity, analytical solutions are commonly derived for constant-rate pumping conditions. However, the pumping rate is often varied either intentionally or due to technical difficulties during the test. Using the principle of superposition, the constant-rate analytical solutions are frequently applied to analyze pumping tests conducted with variable pumping rates by representing pumping rate variation as a series of steps of constant-pumping rate changes. In this study, we propose a methodology that approximates the time-varying pumping record as a series of segments with linearly varying pumping rates.The proposed approach is demonstrated using an analytical solution due to Hantush (1964) for confined aquifers. However, the proposed approach is also applicable to unconfined and/or leaky aquifers. We validate our approach by comparing it with sinusoidally varying pumping tests having direct analytical solution. We also apply our methodology to analyze the synthetic pumping test data by inversely estimating the apparent aquifer parameters and compare it with commonly used method where pumping rate variations are represented by series of constant rate step changes. The proposed approach is implemented for confined, unconfined and leaky aquifers in a computer program WELLS and is available upon request at http://wells.lanl.gov.

Keywords: Pumping test, Varying pumping rate, Piecewise linear pumping rate, Laplace transformation

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1. Introduction

Hydraulic properties of an aquifer are commonly inferred by fitting drawdown and/or recovery data recorded from pumping tests to analytical solutions for radial flow towards a pumping well. For mathematical simplicity, such analytical solutions are commonly derived for constant-rate pumping conditions. However, the pumping rate often varies either intentionally or due to technical difficulties during the test.

The most common approach to analyze pumping tests that are conducted with variable pumping rates is based on superposition of piecewise constant rates. Considering the confined aquifer as a linear system with time-invariant boundary conditions, Cooper and Jacob (1946) applied superposition principle to account for stepwise changes in pumping rates. Abu-Zeid and Scott (1963), Abu-Zeid et al. (1964) and Hantush (1964) proposed analytical solutions for variable-rate pumping tests assuming exponentially decreasing pumping rates. Lai et al. (1973) and Lai and Su (1974) extended the solution of Papadopulos and Cooper (1967) to include leakage from the semi-confining layers when the pumping rates are exponentially and linearly varying. Black and Kipp (1981) provided a solution to an aquifer borehole test for sinusoidal perturbation in a confined non-leaky aquifer. Rasmussen et al. (2003) extended the Hantush (1964) solution to include sinusoidal variation of pumping rates.

In some field applications, the pumping rates are varied intentionally. Butler and McElwee (1990) suggested that variable pumping rates can be used to increase the sensitivity of parameters to observed drawdown, and hence improve parameter identifiably; each time the pumping rate is increased, a new cone of depression (superimposed upon the original one) propagates out from the pumping well, producing an increase in sensitivity and a new interval of time during which the aquifer zone influences changes in drawdown.

Adequate representation of variable pumping rates in the case of real world analyses of pumping tests is also important when various natural phenomena unaccounted for in the analytical solution are causing transients in the observed drawdown records (e.g. barometric effects, infiltration events, etc.). In these cases, the analyses of the observed drawdown transients is difficult, if transients caused by variable pumping rates are not accurately captured.

In many ways, the commonly used approach of constant-rate step changes to represent pumping variability may not be sufficient to capture important details in the observed drawdown transients. For the cases where the pumping variability has increasing or decreasing linear trends the method of step changes is generally not suitable unless a large number of closely spaced step changes are introduced. Therefore, there is a general need for a methodology and computational tools that can address any pattern of temporal variability of pumping rates during field tests.

In this paper, we propose an approach to approximate a time-varying pumping history as a series of linearly varying pumping rates. The approach is demonstrated using the existing analytical solution of Hantush (1964) for the confined aquifers, but is also applicable to solutions for unconfined and/or leaky aquifers. After validating our methodology, we analyze synthetic pumping test data by inversely estimating the parameters and compare our estimates with the commonly used method of constant-rate step changes in pumping rate.

The proposed approach is implemented in the computer program WELLS available at http://wells.lanl.gov. WELLS is a computer program designed to analyze the multi-well variable rate pumping tests in confined, unconfined and leaky aquifers in a finite or infinite domain through a variety of analytical solutions, including the solution of Mishra and Neuman (2011) for unconfined aquifers. The constant pressure or no-flow boundaries are implemented in the code through the method of images.

2. Analytical solution for variable pumping rate

Consider a partially penetrating well of small radius (i.e. $r_w \to 0$) that is in hydraulic contact with a surrounding confined aquifer at depths d through l below the top impermeable boundary. The aquifer is horizontal and of infinite lateral extent with uniform thickness b, uniform hydraulic properties and anisotropy ratio $K_D = K_z/K_r$ between vertical and horizontal hydraulic conductivities, K_z and K_r, respectively. Initially, drawdown $s(r,z,t)$ throughout the aquifer is zero where r is radial distance from the axis of the well, z is depth below the top impermeable boundary of the aquifer and t is time. Starting at time $t = 0$ water is withdrawn from the pumping well at a variable volumetric rate $Q(t)$.

For the case when the pumping well is discharging at constant rate Q, the Laplace transformed drawdown $s(r, z, t)$ can be expressed by Hantush (1964) solution as:

$$\overline{s}\left(r_D, z_D, p_D\right) = \frac{Q}{p}\dot{f}(r_D, z_D, p_D) = \frac{Q/p}{4\pi T}\left\{2K_0\left(\phi_0\right) + \frac{4}{\pi}\times \right. \\ \left. \sum_{n=1}^{\infty}\frac{K_0\left(\phi_0\right)\left[sin\left(n\pi l_D\right) - sin\left(n\pi d_D\right)\right]cos\left(n\pi z_D\right)}{n\left(l_D - d_D\right)}\right\} \tag{1}$$

where p is the Laplace transformation parameter, $r_D = r/b$, $T = Krb$, $z_D = z/b$, $p_D = pt$, $d_D = d/b$, $l_D = l/b$, $\phi_n = \sqrt{p_D/t_s + \beta^2 n^2 \pi^2}$, $t_s = \alpha_s t/r^2$, $\alpha_s = K_r/S_s$, $\beta = r_D K_D^{1/2}$ and K_0 and K_1 are modified Bessel functions of the second kind and order zero and one, respectively.

For constant pumping rates the Hantush (1964) solution has the form $\overline{s} = \frac{Q}{p}f(r_D, z_D, p_D)$, where Q/p is the Laplace transform of the constant pumping rate Q; many other constant-rate analytical solutions for Laplace transformed drawdown have similar form (e.g. Mishra and Neuman (2011)). For variable pumping rate $Q(t)$ with Laplace transform $\overline{Q}(p)$, the existing solutions can be directly used by replacing the Q/p with $\overline{Q}(p)$ giving Laplace space drawdown as

$$\overline{s} = \overline{Q}(p)f(r_D, z_D, p_D) \tag{2}$$

where $f(r_D, z_D, p_D)$ is part of the constant-rate solutions defined in Equation (1).

3. Simple representation of the piecewise-linear pumping rates

Consider pumping rate history recorded as Q_0, Q_1, Q_2 ... Q_n at discrete time intervals t_0, t_1, t_2, ... t_n. Expressing the pumping rate variation as a piecewise linear function allows writing $Q(t)$ as

$$Q(t) = \sum_{i=1}^{n}\{Q_{i-1} + \beta_i(t - t_{i-1})\}\left(\delta_{t_{i-1}} - \delta_{t_i}\right) \tag{3}$$

where $\beta_i = (Q_i - Q_{i-1})/(t_i - t_{i-1})$ is the slope of i^{th} linear pumping element and δ_{t_i} is unit step function which equals one when $t \geq t_i$ and remains zero elsewhere. Using Laplace transform relations $L\{\delta_{t_i}\} = \frac{1}{p}e^{-t_i p}$ and $L\{tf(t)\} = -\frac{d}{dp}F(p)$, where $F(p)$ is Laplace transform of $f(t)$, the Laplace transform of Equation 3 is given as

$$\overline{Q}(p) = \frac{1}{p}\sum_{i=1}^{n}\left(Q_{i-1} + \frac{\beta_i}{p}\right)\left(e^{-t_{i-1}p} - e^{-t_i p}\right) - \frac{1}{p}\sum_{i=1}^{n}\beta_i(t_i - t_{i-1})e^{-t_i p} \quad (4)$$

Substituting the Laplace transformed piecewise-linear pumping rate $\overline{Q}(p)$ in equation (2) gives the Laplace transformed drawdown at any location. The solution corresponding to equation (2) in the time domain, $s\left(r_D, z_D, t\right)$, is obtained through numerical inversion of the Laplace transform by means of an algorithm due to Crump (1976) as modified by de Hoog et al. (1982).

To demonstrate the validity of the proposed approach, consider a sinusoidal pumping rate $Q(t) = 2.0 + sin\left(30t/\pi\right)\ m^3/day$ which has Laplace transform of $\overline{Q}(p) = 2.0/p + \frac{30/\pi}{(30/\pi)^2+p^2}$. Figure 1 presents the pumping rate variation and drawdown at 1.0 m from a fully penetrating pumping well of zero radius in an isotropic uniform aquifer with transmissivity $10.0 m^2/d$ and storativity of 1.0×10^{-5}. Figure 1 compares drawdown computed directly using the Laplace transform of sinusoidal pumping rate variation (red curve) and the drawdown computed by fitting a piecewise-linear function (blue curve) between pumping rates at every 2 hour (black line). The close correspondence between analytically computed drawdowns with and without piecewise-linear approximations validates the proposed methodology. It is intuitive to achieve the similar accuracy in estimated drawdown using constant-rate step changes would require an excessively large number of constant-step changes.

However, the simple piecewise-linear representation of the variable pumping rates presented in equation (4) does not always produce satisfactory results. Consider hypothetical pumping test where pumping rate varies rapidly (shown in green lines in Figure 2). Figure 2 also shows the computed drawdown at a point located 1.0 m from the fully penetrating pumping will of zero radius in an isotropic uniform aquifer with transmissivity 10 m^2/d and storativity of 1.0×10^{-5}. It is noted that drawdown computed using piecewise-linear approximation (red lines) shows oscillatory instability near the time where the abrupt change in pumping rate (slope of linear element $\beta_i \to \infty$) occurs; for example, this can occur if the pumping is discontinued abruptly.

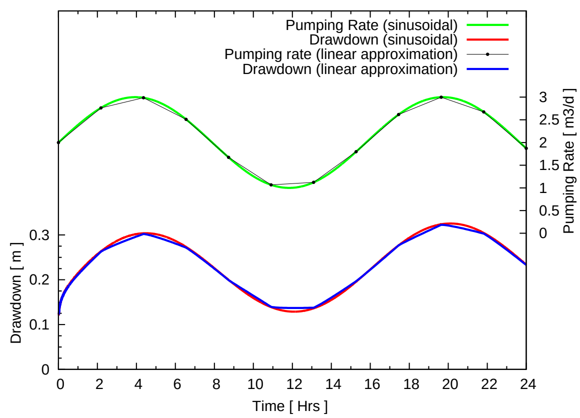

Figure 1: Comparison of analytically evaluated drawdown due to sinusoidal pumping rate variation (green) with (blue) and without (red) piecewise linear approximations

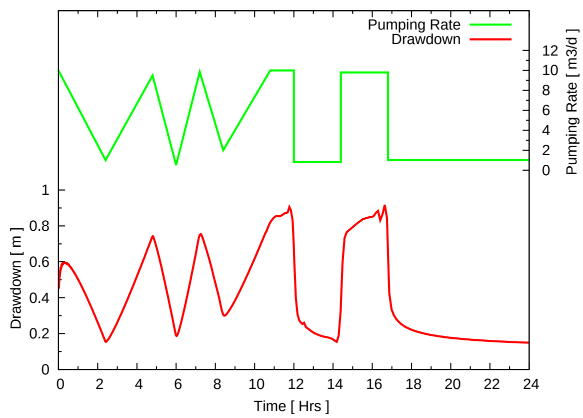

Figure 2: Drawdown evaluated using simple representation of piecewise-linear variation of pumping rate (red curve).

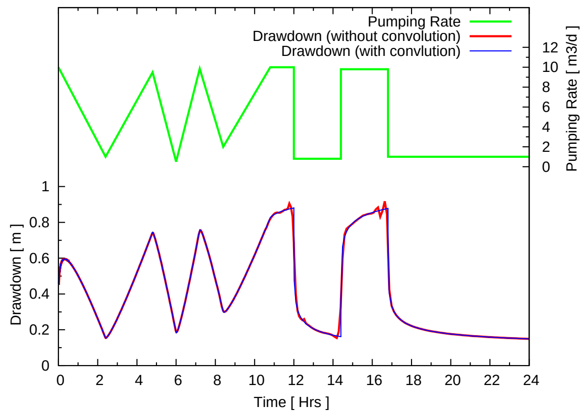

Figure 3: Comparison of drawdown evaluated using simple representation of piecewise-linear variation of pumping rate (red curve) with the drawdown evaluated using convoluted representation of piecewise-linear pumping rate variation (blue curve)

4. Convoluted representation of the piecewise-linear pumping rates

To avoid numerical instabilities presented in Figure 2, we propose a convolution method based on a linear combination of pumping and injection events. It is apparent that each period of linear pumping rate change can be decomposed into a combination of linear pumping and injection events. Consider linear pumping variation from Q_{i-1} at t_{i-1} to Q_i at time t_i , i.e. $q(t) = Q_{i-1} + \beta_i(t - t_{i-1})(\delta_{t_{i-1}} - \delta_{t_i})$. These linear elements can be decomposed into set of pumping $q_a(t) = Q_{i-1} + \beta_i(t - t_{i-1})\delta_{t_{i-1}}$ and injection $q_b(t) = -Q_i - \beta_i(t - t_i)\delta_{t_i}$ events such that $q(t) = q_a(t) + q_b(t)$. Superposing these set of pumping and injection events will result in Laplace transformed pumping rate variation as

$$\overline{Q}(p) = \frac{1}{p}\sum_{i=1}^{n}\left(Q_{i-1} + \frac{\beta_i}{p}\right)e^{-t_{i-1}p} - \frac{1}{p}\sum_{i=1}^{n}\left(Q_i + \frac{\beta_i}{p}\right)e^{-t_i p} \tag{5}$$

The drawdown computed using superposition of an equivalent set of pumping and injection wells were found to have similar numerical instabilities as found in the method with direct implementation of piecewis-linear variation. The same can also be inferred by comparing (5) with (4) as both of them have similar mathematical form.

Instead of superposing the linear elements in Laplace space as done in equation 5, superposition can be done in real time space using a discrete convolution integral resulting in an expression for drawdown as,

$$s(t) = \sum_{i=1}^{n} \{ s_{a'}(t - t_{i-1}) + s_{b'}(t - t_i) \} \tag{6}$$

where $s_{a'}(t)$ and $s_{b'}(t)$ are Laplace transformed drawdown due to pumping $Q_{a'}(t) = Q_{i-1} + \beta_i t$ and injection $Q_{b'}(t) = -Q_i - \beta_i t$ with corresponding Laplace transforms $\overline{Q}_{a'}(p) = \frac{1}{p}\left(Q_{i-1} + \frac{\beta_i}{p}\right)$ and $\overline{Q}_{a'}(p) = \frac{-1}{p}\left(Q_i + \frac{\beta_i}{p}\right)$ respectively.

As shown in Figure 3, numerical instabilities observed when a simple piecewise-linear approach of representing pumping rate variation is applied (red line; Equation 4) can be entirely avoided by applying the method based on convolution of a linear set of pumping and injection events (blue line).

It is important to note that the method based on convoluting the drawdowns due to a combination of pumping and injection events (equation 5) is computationally more robust but is computationally more expensive than the simple approach utilizing equation 4. The code WELLS has implementation of both simple and convoluted schemes.

5. Parameter Evaluation using Synthetic Aquifer Test

Consider a 7 m thick isotropic confined aquifer ($K_D = 1.0$) with horizontal hydraulic conductivity $K_r = 5.01$ m/d and specific storage $S_s = 5.01 \times 10^{-6}$ m^{-1}. The pumping well with infinitesimal diameter, penetrates the upper 3.5 m of the confined aquifer and discharges at variable rate. The pumping rate is assumed to vary linearly and the changes occur at every hour as shown in Figure 4 (blue stars). Drawdowns were recorded at over 1000 temporal values uniformly spaced in log space spanning from 10^{-4} to 1.0 day. To pose the problem similar to a real pumping-test analysis, a random noise of $\pm 5\%$ magnitude was added to the recorded drawdowns. The goal of the synthetic test analysis is to estimate the aquifer parameters based on the pumping test data applying two different approaches to characterize pumping rate variability: (1) the piecewise-linear approach proposed here, and (2) piecewise-constant step approach. The two approximations of the pumping rate variability are presented in Figures 4 and 5 (red lines). The approximate pumping rates are adjusted so that the total amount of water pumped during the pumping test in both cases is the same. Note that in Figure 4, the pumping rate is represented by 8 piecewise linear regions. In Figure 5, the pumping rate is represented by 5 stepwise regions with constant pumping rate. Also, it is important to emphasize the true solution is computed assuming linear changes of the pumping rates every hour. Therefore it is expected that piecewise linear approach will produce better representation of the true drawdowns as well as closer matches to the true model parameters. The major question of this synthetic analysis is whether the piecewise-constant step approach is good enough to represent true drawdowns and estimate the true model parameters.

The parameters were then inversely estimated by minimizing the sum of squared difference between model predicted drawdowns and synthetic drawdowns using PEST code (Doherty, 1994) for the case with piecewise-linear and piecewise-constant approximation of variable pumping rate. Figures 4 and 5 compare the best fit model predicted drawdown with synthetic drawdown, and Table 1 lists the estimated parameters. Table 1 also lists the %-error in the estimated parameters (values in closed brackets) and sum of squared error (SSE) in estimated drawdown. The piecewise-linear approximation improves the hydraulic conductivity estimates by a factor of about 3 and specific storage by a factor of about 2, it also results in an order of magnitude lower SSE and better representation of the actual drawdowns observed during the pumping test (based on a comparison of simulated drawdowns presented in Figures 4 and 5). This demonstrates that for the cases where pumping rate variations are better represented by piecewise linear changes will result in better posed problem for parameter estimation. As pumping rate variations in many pumping tests are not adequately represented by constant-rate step changes, the piecewise-constant approach for representing pumping transients is not always sufficient to accurately represent the observed drawdowns and reliably estimate aquifer parameters.

6. Summary and conclusions

Our work leads to the following major conclusions:

1. A new approach was developed to include piecewise-linear variation of pumping rates. This approach does not require to fit observed pumping records to mathematical form

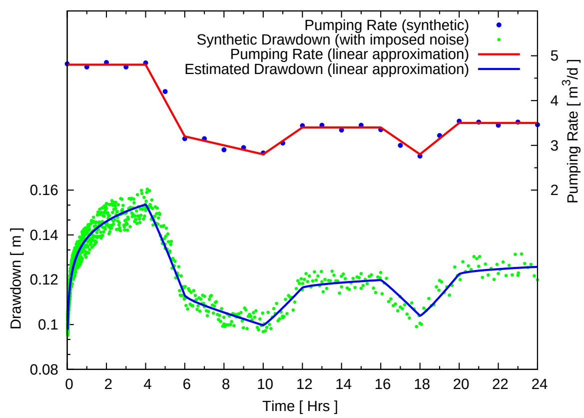

Figure 4: Comparison of inversely estimated drawdown (blue line) with synthetic drawdown (green dots) when pumping rate variation (blue dots) are approximated by equivalent linear changes (red line)

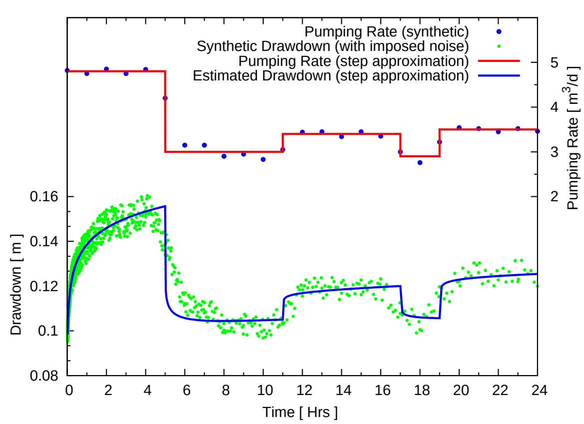

Figure 5: Comparison of inversely estimated drawdown (blue line) with synthetic drawdown (green dots) when pumping rate variation (blue dots) are approximated by equivalent step changes (red line)

Table 1: Comparison of estimated parameters and sum of squared errors (SSE) in estimated drawdowns with the synthetic true case (column 2) when time varying pumping rate is approximated as piecewise linear (column 3) and step function (column 4)

Quantity	True	Linear changes	Step changes
K_r [m/d]	5.01	5.04 (0.60%)	5.10 (1.79%)
S_s $\times 10^{-6}$ $[m^{-1}]$	5.01	4.53 (9.58%)	3.99 (20.36%)
SSE (m^2)	–	1.65 $\times 10^{-2}$	1.84 $\times$ 10^{-1}

such as equivalent peicewise-constant steps, sinusoidal or exponential form; instead, the measured pumping transients can be directly used to estimate the transient drawdown in an aquifer. The approach is demonstrated here using the confined aquifer solution due to Hantush (1964).

2. The proposed piecewise-linear approximation of time-varying pumping rate is implemented for confined, unconfined and leaky aquifers in the computer program WELLS (http://wells.lanl.gov) which is written in ANSI-C for the multi-well variable-rate analysis of pumping test data.
3. The piecewise-linear approximation can represent fairly well any time-varying pumping rates and can reproduce the drawdown for sinusoidal tests with relatively few piecewise-linear discretization.
4. The use of piecewise-linear approach is important in real world analyses when various naturally occurring factors are causing transients in observed drawdown records (e.g. barometric effects, infiltration events, etc).
5. For the case when the slope of the linear pumping event is very large (i.e. $\beta_i \to \infty$) , the convolution integral approach (Equation 5) can be applied to superimpose a combination of pumping and injection steps in order to avoid any instabilities of the

numerical Laplace inversion.

6. The piecewise linear approximation can reduce the uncertainty associated with parameter estimation by providing better representation of varying pumping rates which is otherwise ignored using standard approach of implementing varying pumping rates using piecewise-constant step changes.

References

Abu-Zeid, M., Scott, V. H., 1963. Nonsteady flow to a well with decreasing discharge. Jour, Hyd. Div. Proc. 89 (HY3), 119–132.

Abu-Zeid, M., Scott, V. H., Aron, G., 1964. Modified solutions for decreasing discharge wells. Jour, Hyd. Div. Proc. 90 (HY6), 145–160.

Black, J., Kipp, K., 1981. Determination of hydrogeological parameters using sinusoidal pressure tests: a theoretical appraisal. Water Resources Research 17 (3), 686–692.

Butler, J., McElwee, C., 1990. Variable-rate pumping tests for radially symmetric nonuniform aquifers. Water Resour. Res 26 (2), 291–306.

Cooper, H., Jacob, C., 1946. A generalized graphical method of evaluating formation constants and summarizing well-field history. US Dept. of the Interior, Geological Survey, Water Resources Division, Ground Water Branch.

Crump, K., 1976. Numerical inversion of Laplace transforms using a Fourier series approximation. Journal of the ACM (JACM) 23 (1), 89–96.

de Hoog, F., Knight, J., Stokes, A., 1982. An improved method for numerical inversion of Laplace transforms. SIAM Journal on Scientific and Statistical Computing 3, 357.

Doherty, J., 1994. Pest: Model-independent parameter estimation. Watermark Numerical Computing,Corinda, Qld., Australia.

Hantush, M., 1964. Hydraulics of wells. Advances in hydroscience 1, 281–432.

Lai, R., Karadi, G., Williams, R., 1973. Drawdown at time-dependent flowrate. Journal of the American Water Resources Association 9 (5), 892–900.

Lai, R., Su, C., 1974. Nonsteady flow to a large well in a leaky aquifer. Journal of Hydrology 22 (3-4), 333–345.

Mishra, P., Neuman, S., 2011. Saturated-unsaturated flow toward a well with storage in a compressible unconfined aquifer. Water Resources Research 86 (7), W12508.

Papadopulos, I., Cooper, H., 1967. Drawdown in a well of large diameter. Water Resources Res 3 (1), 241–244.

Rasmussen, T., Haborak, K., Young, M., 2003. Estimating aquifer hydraulic properties using sinusoidal pumping at the Savannah River site, South Carolina, USA. Hydrogeology Journal 11 (4), 466–482.