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QUADRATIC TRANSFORMATIONS

PART I

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QUADRATIC TRANSFORMATIONS
PART I

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ABSTRACT

This report deals with the properties of a restricted class of homogeneous quadratic transformations, with interesting physical and biological analogues, which we have called Binary Reaction Systems. All possible transformations of this class in 3 variables have been studied numerically on a computing machine, and the limiting behavior of random initial vectors under iteration of each of these transformations is tabulated. Some examples of 4-variable Binary Reaction Systems are also studied, and a few generalizations of the notion of Binary Reaction System are investigated for particular cases. Some remarks and results concerning the behavior in the large are presented, and examples of the mode of approach to the limit are given. Several of the more interesting phenomena are illustrated graphically.

The appendix deals with a different class of homogeneous quadratic transformations (of arbitrary dimension) which arise naturally from the study of a simple evolutionary mode. For this class of transformations, the limiting behavior of arbitrary vectors under iteration can be given explicitly.

CONTENTS

		Page
Abstract		3
Introduction		7
I.	Homogeneous Quadratic Transformations	9
II.	Normalization	12
III.	Binary Reaction Systems	14
IV.	Procedure and Results	21
V.	Convergence Behavior	27
VI.	The Nature of the Interior Fixed Points	31
VII.	Periodic Limits	39
VIII.	Form Stability	43
IX.	A Specific Convergence Problem	48
X.	Further Generalizations	56
XI.	Properties in the Large	60
XII.	Connection with Ordinary Differential Equations	67
Explanation of Graphs		70
Appendix		150

TABLES

Table I.	Summary of Convergence Behavior of Three-Variable Binary Reaction Systems	81
Table II.	Three-Variable Binary Reaction Systems	83
Table III.	Examples of Binary Reaction Systems for $N > 3$	121
Table IV.	Modifications of System I.3.g	132
Table V.	Modifications of System I.3.g	139
Table VI.	Super-systems	146

INTRODUCTION

This report summarizes and discusses some recent studies of the properties of quadratic transformations in several variables under iteration. The report is of an interim nature, and consists mainly in a presentation of "experimental" (i.e., numerical) results. A general theory of quadratic transformations (in contrast to the linear case) is essentially non-existent, and from the theoretical point of view, the work summarized below does little to improve the situation. It can only be hoped that as more facts become known, some outlines of a theory--at least a classification or "descriptive theory"--will emerge.

The motivation for the considerations which follow lies in the combinatorial problems suggested by genetic or biological systems. One has to deal with large populations of individuals (or particles) present in a given generation. Those may combine in pairs and produce, in the next generation, new particles. Suppose the original particles are of N different types. Given a rule for the type $i(i=1, \ldots N)$ produced by mating of individuals of type j and k, the proportion or fraction x_i of a given type in the next generation will be a quadratic function of the two fractions x_j and x_k.

More generally, one could consider a system (gas) of physical particles with N possible characteristics which collide in pairs and produce through the collision, say, a pair of particles with new characteristics. There could be many different values of the momenta, but possibly the "type" of the particle resulting from the collision could be different from the original ones.

Our present considerations concern the averages or expected values of the fraction x_i in the next generation. The role of fluctuations or deviations from the fractions given by the quadratic formulae will be studied in a subsequent report.

As suggested in the title, this report is the first of a series (one might add, of unspecified length). By the time these remarks appear in print, much of the work will probably have been generalized and the results extended. It is hoped that at least some of the tentative conclusions presented below will stand.

I. HOMOGENEOUS QUADRATIC TRANSFORMATIONS

We begin by defining a homogeneous quadratic transformation as a set of N coupled non-linear first-order difference equations of the form:

$$x_i' = \sum_{k,\ell=1}^{N} \gamma_i^{k\ell} x_k x_\ell \qquad i = 1, \ldots N \tag{1}$$

where the γ_i^k are some real numerical coefficients with the property:

$$\gamma_i^{k\ell} = \gamma_i^{\ell k} \tag{2}$$

In the present work we restrict ourselves to the case of <u>non-negative</u> coefficients:

$$\gamma_i^{k\ell} \geq 0, \quad \text{all } k, \ell, i \tag{3}$$

Then if we choose the x_j as non-negative real numbers, the x_j' will also have this property. In the following we shall always restrict our x_j in this manner.

In order that the system 1) be of dimension N, we insist that for a given index i, not all $\gamma_i^{k\ell}$ can vanish; i.e.,

$$\sum_{k,\ell=1}^{N} \gamma_i^{k\ell} > 0 \qquad 4)$$

Systems of the form 1) (not necessarily with the restrictions of reality or non-negativeness) can be considered from two points of view:

a) As the transformation of all vectors $\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_N \end{pmatrix}$ into vectors $\begin{pmatrix} x_1' \\ \cdot \\ \cdot \\ x_N' \end{pmatrix}$ or, more generally, as a mapping of some specified region X into a region X'.

b) As a set of difference equations which determine the value of a vector $\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_N \end{pmatrix}$ at "time" n from the value of some initial vector $\begin{pmatrix} x^{(o)} \\ \cdot \\ \cdot \\ x_N^{(o)} \end{pmatrix}$.

If we take the first point of view we are, in effect, studying a single iteration, the "mapping" in explicit form. The literature does contain some sporadic work on this problem for low values of N.[1] On the other hand, the second point of view does not seem to have been considered except for the case of one dimension.[2] The general "solution" to the

[1] See, e.g., R. Sauer, Math. Ann. 106, 722 (1932)
O. Baier, Math. Ann. 112, 630 (1936)

[2] To be sure, there is an extensive literature on coupled non-linear differential equations of 1st order in 2 variables. See, e.g., Coddington and Levinson, Theory of Ordinary Differential Equations, Chapters 15 and 16, McGraw-Hill, New York, 1955. For N variables, a special class of differential equations is treated in V. Volterra, Leçons sur la Théorie Mathématique de la Lutte Pour La Vie, Gauthier-Villars, Paris, 1939

to the problem posed by b) is the explicit construction of the vector $\vec{x}^{(n)}$ in terms of the initial vector $\vec{x}^{(o)}$ and the time-variable n. Such a solution can only, in general, be presented as an iterative procedure. The most one can hope to do is to predict the limiting behavior of the system as $n \to \infty$. Luckily, for practical applications this is usually the only thing of interest. However, even this problem has hardly been touched upon.

In the limit $n \to \infty$ a variety of behaviors is possible; the vectors may, for example, converge under iteration to a limiting vector $\vec{x}$, they may oscillate between a finite set of limit vectors $\vec{x}_i$, or they may exhibit a more or less chaotic behavior, i.e., do neither, but have an ergodic behavior, i.e., the limit in time of the average $\frac{1}{N}\sum_{i=1}^{N} x_i$ will exist. (The last is familiar from Kronecker-Weyl's theorem[3] of irrational rotations; see also Fn. 4.) The type of behavior, as well as the numerical value of the limit (if it exists) may or may not depend on the initial vector $\vec{x}^{(o)}$. All the examples (with one exception--see Section XI) studied in this report turn out to be of the first two types; i.e., they either converge to a limit vector or converge to an oscillation between a finite set of vectors in a definite order, i.e., with a definite period.

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[3] See, e.g., H. Weyl, Math. Ann. 77, 313 (1916)

[4] S. Ulam and J. von Neumann, Bull. Am. Math. Soc. 53, 1120 (1947)

II. NORMALIZATION

Formally, the system 1) contains $\frac{N^2(N+1)}{2}$ parameters, the $\gamma_i^{k\ell}$. It proves convenient to reduce this number (somewhat arbitrarily) by postulating the condition:

$$\sum_{i=1}^{N} \gamma_i^{k\ell} = 1, \quad \text{all } k, \ell \tag{5}$$

This has the great practical advantage that if we now normalize the initial vector by the conditions:

$$\sum_{i=1}^{N} x_i = S \tag{6}$$

then if we divide the right-hand side of each equation of 1) by the constant S, we have:

$$\sum_{i=1}^{N} x_i' = S \tag{7}$$

In biological terminology (see below) this means that we restrict ourselves to the case of a "constant population." Of course, there is no loss of generality in taking $S = 1$. In the sequel we therefore always

impose condition 5), and furthermore, take:

$$0 \leq x^{(o)} \leq 1, \quad \text{for all } i \text{ with}$$

$$\sum_{i=1}^{N} x_i^{(o)} = 1 \tag{8}$$

This, of course, reduces the number of variables to N-1. (For an example of a different procedure when 5) is not postulated, see the last section of the Appendix.) The x_i are now restricted to lie on the positive portion of the hyperplane:

$$\sum_{i=1}^{N} x_i = 1$$

Even with all these restrictions, it has so far proved impossible to give any general theory of the limiting behavior of the systems 1). However, one sub-class, defined by certain reasonable restrictions on the coefficient $\gamma_i^{k\ell}$, has been completely studied for all N. The results for this case are summarized in the Appendix. The rest of the report is concerned with a discussion of a different sub-class which seems to be of considerable interest, but for which no general theory as yet exists.

III. BINARY REACTION SYSTEMS

1. Equations 1) have a natural interpretation in terms of biological or genetic language. Consider a large population which consists of N different "types" of individuals.[5] Let x_j represent the fraction of male individuals of type j. (We assume that there are equal numbers of males and females of every given type, hence we may represent them both by the same letter x.) x_ℓ is the fraction of the population of type ℓ. Then the system 1), defined by the coefficients $\gamma_i^{j\ell}$, determines the composition of the next generation, which results from a random pairing (once for each individual) of the population at the present generation. If we assume that the members of the old generation do not survive into the next, then for very large populations the expected value of the fractions of the individuals of type j will be given by the equations 1). By virtue of our restrictions:

$$\sum_{i=1}^{N} \gamma_i^{k\ell} = 1, \quad \text{all } k, \ell \qquad 5)$$

[5] These "types" are not, of course, meant to correspond to the genotypes or phenotypes of Mendelian genetics!

$$\sum_{i=1}^{N} x_i = 1$$

$$0 \leq x_i \leq 1 \qquad 8)$$

the size of the population is constant. The system 1) can therefore be looked upon simply as defining a "mating rule," i.e., determining the characteristics of the offspring from the characteristics of parents. The problem, of course, is to determine the composition of the limiting population.

2. As mentioned in II, the set of equations 1) is, in general, very difficult to study. Consequently, we decided to restrict ourselves initially to a sub-class of systems defined by the following additional restrictions on the coefficients $\gamma_i^{k\ell}$:

> For each pair k, ℓ, there is exactly one value of i for which $\gamma_i^{k\ell} = 1$. For all other values of i the coefficient is zero; i.e.,

$$\gamma_i^{k\ell} = \delta_{ii_1} \qquad 9)$$

> where each pair (k,ℓ) determines one value of i_1.

This means that every term in the product $(x_1 + x_2 \ . \ . \ . + x_N)^2$ will appear in exactly one row of the set (cross-terms appearing with the factor 2). For example,

$$x_1' = x_1^2 + x_2^2 + x_4^2 + 2x_1x_4 + 2x_2x_4 + 2x_3x_4$$

$$x_2' = 2x_1x_3 + 2x_2x_3$$

$$x_3' = 2x_1x_2$$

$$x_4' = x_3^2$$

Since there are $\frac{N(N+1)}{2}$ terms in the square of the sum, the number of different possible systems of this sort is clearly equal the number of ways of placing $\frac{N(N+1)}{2}$ different objects in N boxes, no box being empty. Setting $P \equiv N(N+1)/2$, this number is easily shown to be :

$$T_N = N^P - \binom{N}{1}(N-1)^P + \binom{N}{2}(N-2)^P - \ldots \qquad 10)$$

For example,

$$T_3 = 540$$

$$T_4 = 818{,}520$$

Naturally, many of these are equivalent to each other under permutation of the indices 1,2, . . . N. A <u>lower limit</u> to the number S_N of different systems, inequivalent by permutation, is:

$$T_N^* = \frac{T_N}{N!} \qquad 11)$$

In fact, the number S_N of inequivalent systems (in this sense) will be somewhat higher than this because of the fact that some of the systems are formally invariant under certain of the N! permutations. Thus, for

$N=3$, $T_N^* = 90$, but by actual enumeration it is found that there are 97 inequivalent systems, i.e., $S_N = 97$. For $N=4$, $T_N^* = 34{,}105$; at the present writing the actual number S_N of inequivalent systems has not been determined by us.

We have called systems restricted by condition 9) "Binary Reaction Systems." The reason for this name is that such a system associates with each pair (x_k, x_ℓ) a unique result, say x_j'. Symbolically:

$$k \oplus \ell \rightarrow j \qquad 12)$$

This seems to us to be a natural and simple definition for binary reactions in which "particles" of types k and ℓ produce by "collision" particles of type j. (The "genetic" case is inherently more complicated; see, for instance, the Appendix.)

More general and natural, though less simple, would be a rule of the form:

$$k \oplus \ell \rightarrow (j,m) \qquad 13)$$

i.e., a pair of particles produces a pair of not necessarily similar particles. We have studied a few generalizations of the simple scheme 12), though not yet in comparable detail (see below, Section VII.2).

The reaction rule may be presented in tabular form. Consider, e.g., the system:

$$\left.\begin{aligned} x_1' &= x_1^2 + x_2^2 + 2x_2x_3 \\ x_2' &= 2x_1x_3 + x_3^2 \\ x_3' &= 2x_1x_2 \end{aligned}\right\} \qquad 14)$$

The table for this would be:

	x_1	x_2	x_3
x_1	x_1	x_3	x_2
x_2	x_3	x_1	x_1
x_3	x_2	x_1	x_2

15)

Considered as an algebraic system with a law of composition (multiplication) given by the table, this scheme is seen to be commutative ($x_ix_j = x_jx_i$) but <u>non-associative</u>, e.g.,

$$(x_1x_2)x_3 = x_3x_3 = x_2$$

$$x_1(x_2x_3) = x_1x_1 = x_1$$

Binary reaction systems, as defined above, are always commutative (since each product occurs in only one of the set of equations) but are not in general associative. Indeed, for N=3 there are just five associative schemes of this sort:

	x_1	x_2	x_3
x_1	x_1	x_2	x_1
x_2	x_2	x_1	x_2
x_3	x_1	x_2	x_3

	x_1	x_2	x_3
x_1	x_1	x_1	x_1
x_2	x_1	x_2	x_2
x_3	x_1	x_2	x_3

	x_1	x_2	x_3
x_1	x_1	x_1	x_1
x_2	x_1	x_2	x_3
x_3	x_1	x_3	x_1

	x_1	x_2	x_3
x_1	x_1	x_1	x_1
x_2	x_1	x_2	x_1
x_3	x_1	x_1	x_3

	x_1	x_2	x_3
x_1	x_2	x_3	x_1
x_2	x_3	x_1	x_2
x_3	x_1	x_2	x_3

This last table corresponds to the finite group in 3 variables. Although this classification is suggestive, it does not appear that these associative systems are distinguished from the non-associative ones as regards their convergence properties. This is at least the case for N=3, and the presumption is that no particular significance will attach

to the associative property for higher N either. However, if the "reaction table" possesses the properties of a group table, certain special properties are easy to establish; e.g.:

The fixed point of the transformation has coordinates $x_1 = x_2 = \ldots x_N = \frac{1}{N}$ and is attractive, i.e., the iterates of any vector in its neighborhood converge to it.

IV. PROCEDURE AND RESULTS

At this point it is necessary to describe our experimental procedure and results in some detail, since they will be referred to frequently in what follows.

As stated above, for the case of 3 variables, x_1, x_2, x_3, there are 97 binary reaction systems inequivalent to each other under permutation of the labels 1, 2, 3. Each one of these has been studied numerically (on an IBM 704) by having the machine select randomly three initial vectors with coordinates $x_1^{(o)}$, $x_2^{(o)}$, $x_3^{(o)}$ (satisfying $\sum x_i^{(o)} = 1$, $0 \leq x_i^{(o)} \leq 1$) and letting the computer iterate the transformation in question "as long as necessary," i.e., until some definite limiting behavior was observed.[6] In all but two cases such limiting behavior became evident without further analysis (one "ambiguous" case is discussed in Section IX.1; the other is mentioned in Section XI). All other transformations eventually either:

a) Reached a stable distribution or a "fixed point" of the transformation, or

---

[6] Although the sum $x_1 + x_2 + x_3 = 1$ is formally conserved, it is necessary to normalize at each step to avoid loss of accuracy by round-off errors in the last digit.

b) Oscillated between two or three fixed sets of values.

Only in (relatively) few cases did the behavior depend on the choice of the initial vector. However, the rate of convergence (used in the generalized sense to refer to both fixed points and "fixed" oscillations) often varied considerably with this choice.

The main results are contained in Table II. Here each system is written down in symbolic form, the results of iteration being given below. Each system has attached to it a conventional symbol, e.g., I.5.p, II.1.d, etc. The Roman numerals I, II, III refer to a distribution of quadratic terms on the right-hand side corresponding respectively to the three partitions of 6 into exactly 3 parts, viz.: (3,2,1), (4,1,1), (2,2,2). The other symbols refer to distributions within these main divisions, and are purely conventional. (They correspond to a particular order in which we have examined these cases on computing machines.)

A few examples will serve to illustrate the notation.

a) Consider the system:

$$x_1' = 2x_1x_2 + 2x_1x_3 + 2x_2x_3$$
$$x_2' = x_1^2 + x_3^2 \qquad \text{Conventional name: I.6.b}$$
$$x_3' = x_2^2$$

In Table II this appears symbolically as:

$$\left.\begin{array}{l} 2(12) + 2(13) + 2(23) \\ (11) + (33) \\ (22) \end{array}\right\} \begin{array}{l} \text{b.d.p.} \\ \text{not degenerate} \end{array}$$

$$\begin{array}{ll} \text{i.f.p. given by } (x = x_2): & \\ \quad 2x^4 + 2x^3 - x^2 - 3x + 1 = 0 & \\ m \rightarrow \text{i.f.p.} & \end{array} \quad \left.\begin{array}{l} x_1 = .56311573 \\ x_2 = .32878482 \\ x_3 = .10809945 \end{array}\right\} \text{i.f.p.}$$

The notation b.d.p. means that if any of the initial x's = 1, the limiting configuration will be an oscillation between the two states: ($x_1 = 0$, $x_2 = 1$, $x_3 = 0$) and ($x_1 = 0$, $x_2 = 0$, $x_3 = 1$). This we call a "boundary double point" (b.d.p.). Since it is evident from the structure of the system which variables will assume these values, it is not necessary to specify the b.d.p. more completely. In some systems the b.d.p. will only be reached if either of some two rather than any of all 3 variables is initially equal to 1. These cases are always immediately obvious from the structure of the system.

The words "not degenerate" mean that if initially we have some $x_i^{(o)} = 0$, it will not automatically remain so for all time, i.e., that x_i is not a factor of the r.h.s. of the ith equation. The notation i.f.p. ($x = x_2$) followed by an equation means that there exists an <u>interior fixed point</u> (i.f.p.), that is, a fixed point with no $x_i = 0$, and that the value of one of the variables (in this case x_2) is given by the relevant root of the equation. This equation is simply gotten by suppressing the primes on the l.h.s. and eliminating two of the

variables. By relevant root we mean a real root between 0 and 1 which satisfies the set (sometimes extraneous roots are introduced in the elimination process; these do not satisfy the original set). To the right of this is given the resulting fixed point obtained from this equation. The notation $m \rightarrow$ i.f.p. means that the transformations as carried out on the machine actually converged to this value (to 8 decimal places) for three random initial vectors.

b) As a second example, consider:

$$x_1' = x_1^2 + 2x_1x_2 + 2x_1x_3$$
$$x_2' = x_2^2 + x_3^2 \qquad \text{Conventional name: I.4.a}$$
$$x_3' = 2x_2x_3$$

In Table II this appears as:

$$\left.\begin{array}{l} (11) + 2(12) + 2(13) \\ (22) + (33) \\ 2(23) \end{array}\right\} \begin{array}{l} \text{2 n.f.p.'s} \\ \text{b.f.p.} \\ \text{doubly degenerate} \end{array}$$

no i.f.p.

$m \rightarrow$ n.f.p. $(x_1 = 1)$

n.f.p. means "nodal fixed point" and refers to the fact that $x_i = 1$, for at least one value of i, is a fixed point. b.f.p. or "boundary fixed point" means that there exists a fixed point for which one $x = 0$. In view of the explanation of "not degenerate," the term "doubly

degenerate" is self-explanatory. "No i.f.p." means that there is no interior fixed point, and $m \rightarrow$ n.f.p. ($x_1 = 1$) means that the system converged to $x_1 = 1$ for 3 randomly chosen initial vectors.

c) Consider finally:

$$x_1' = x_1^2 + x_3^2$$
$$x_2' = 2x_1x_3 + 2x_2x_3 \qquad \text{Conventional name: III.1.f}$$
$$x_3' = 2x_1x_2 + x_2^2$$

In Table II (leaving out some explanatory material with regard to the b.d.p.):

$$\left.\begin{array}{l} (11) + (33) \\ 2(13) + 2(23) \\ 2(12) + (22) \end{array}\right\} \begin{array}{l} \text{n.f.p.} \\ \text{b.d.p.} \\ \text{not degenerate} \end{array}$$

i.f.p. given by ($x = x_3$):

$$1 - 4(1-x)^3 = 0$$

$m \rightarrow$ i.d.p.

$$\left.\begin{array}{l} x_1 = .16374000 \\ x_2 = .46622052 \\ x_3 = .37003948 \end{array}\right\} \text{i.f.p.}$$

$$\left.\begin{array}{ll} x_1 = .17899745 & x_1 = .40604579 \\ x_2 = .20944248 & x_2 = .47510870 \\ x_3 = .61156007 & x_3 = .11884551 \end{array}\right\} \text{i.d.p.}$$

In this case the transformation did <u>not</u> converge to the i.f.p., but rather ended up oscillating between two sets of values, that is, it achieved an "interior double point" (i.d.p.).

We hope that with these examples in mind, it will be possible to interpret Table II. In a few instances, some remarks are appended, but these are self-explanatory.

V. CONVERGENCE BEHAVIOR

To date we know of no criterion which enables one to predict combinatorially, i.e., from an inspection of the reaction table, whether or not the limiting behavior of a given system will be true convergence (fixed point attained), oscillation between a finite set of limit vectors (periodic point), or neither. For general N, such a criterion will certainly not be simple; it would, for instance, have to take into account the various boundary fixed points and boundary periodic points which are, of necessity, present in any binary reaction system. Of course, every binary reaction system has either boundary fixed points or boundary periodic points (or both). (From Brouwer's fixed point theorem it follows that at least one fixed point must exist.) In other words, the behavior of boundary points under iteration will always have to be treated specially.

An example of this type of complication is provided by system I.1.c. Transcribed from Table II, this reads:

$$x_1' = x_1^2 + x_2^2 + x_3^2$$

$$x_2' = 2x_1x_3 + 2x_2x_3$$

$$x_3' = 2x_1x_2$$

First of all, it is clear that $x_i^{(o)} = 1$ will lead to the n.f.p. $x_1 = 1$. Furthermore, it is clear that a b.d.p. exists, namely:

$x_1 = 1/2$	$x_1 = 1/2$
$x_2 = 0$	$x_2 = 1/2$
$x_3 = 1/2$	$x_3 = 0$

Experimentally, for three different randomly chosen <u>interior</u> vectors, the systems converged to the i.f.p. given in the table.

Other special cases have a behavior more difficult to discover; e.g., the system III.1.f, in addition to the i.f.p., has the periodic solution:

$x_1 = .31944846$	$x_1 = .56519772$
$x_2 = 0$	$x_2 = .43480228$
$x_3 = .68055154$	$x_3 = 0$

which is attained if $x_2^{(o)}$ or $x_3^{(o)} = 0$.

For our randomly selected initial vectors, however, the system attains the i.d.p. given in the table.

One might think that things get simpler if we consider only interior initial points. Several examples, however, show that even here nothing universally true can be asserted. For example, the system I.2.m attained its i.f.p. with two different initial vectors, but went to the b.d.p. from a third initial vector.

The situation clearly gets more complicated in higher dimensions, where the classification of special boundary solutions in general depends on a complete knowledge of the behavior of lower-dimensional systems.

One may consider, in order to determine whether the fixed point of the transformation is "attractive" or "repellent," the value of the Jacobian of the transformation or the fixed point. If, for example, the absolute value of the Jacobian is > 1, then, in general, iterates of points in every neighborhood of the fixed point will diverge from it.

A summary of the convergence behavior in all 97 three-variable systems for random initial vectors is given in Table I. In 23 systems there is convergence to $x_i = 1$, for some i. Twelve converged to a b.f.p., 15 to a b.d.p., 4 to an i.d.p., 4 to a b.t.p. (boundary triple point), one to an interior triple point, while 6 showed varying behavior depending on the initial conditions. (This class may turn out to be larger with more initial points sampled.) One does not converge at all. All the rest converged to an i.f.p.

The two systems II.1.d and II.1.f showed a continuum of i.f.p.'s and i.d.p.'s, respectively, but this behavior is easy to understand, and not specially significant (see remarks in Table II).

Except in 2 cases, systems I.2.j and III.2.3.a, convergence was numerically evident (though occasionally extremely slow). I.2.j is particularly interesting, and is discussed in detail in Section IX.1. III.2.3.a is briefly discussed in Section XI.

VI. THE NATURE OF THE INTERIOR FIXED POINTS

1. Leaving aside for the moment the question of convergence, it is of interest to inquire into the nature of the various i.f.p. Since for a given N, there are a finite number of different systems, there are only a finite number of i.f.p. These are, of course, defined by the set of algebraic equations obtained on suppressing the primes on the left-hand side of the systems in question. As mentioned above, from Brouwer's theorem it follows that there exists at least one fixed point, but it need not lie in the interior. Frequently these systems have no solutions such that $0 < x_i < 1$, all x_i, which means that no i.f.p. exists. Consider, for example, system I.1.a. The set of equations defining the fixed point is:

$$x_1 = x_1^2 + x_2^2 + x_3^2$$

$$x_2 = 2x_1x_2 + 2x_1x_3$$

$$x_3 = 2x_2x_3$$

Since, by definition of an i.f.p., $x_3 \neq 0$, we must have $x_2 = 1/2$; the second equation then implies:

$$1/2 = 2x_1(1-x_1), \quad \text{i.e.,} \quad x_1 = 1/2, \quad \text{so that}$$

$$x_1 + x_2 = 1, \quad \text{implying} \quad x_3 = 0. \quad \text{Thus no i.f.p. exists.}$$

In general, to find the i.f.p. we must eliminate two of the variables. The resulting equation is then of 4th order in the remaining variable, say x_i, although it may have factors corresponding to $x_i = 1$, $x_i = 0$, or perhaps to extraneous roots like $x_i = -1$. (In Table II the equation listed is always in "reduced" form, with these factors removed.) Occasionally the equation may have two real roots in the interval 0 to 1. For N=3, in all such cases one of the roots proved to be spurious, i.e., it did not satisfy the original system. In fact, excepting the case II.1.d, mentioned above, which had a continuum of i.f.p., no system had more than one i.f.p. Although it is doubtless possible to give a complete theory of these equations for N=3, a similar treatment for general N seems beyond reach. Here the elimination process can yield an (unreduced) equation of order 2^{N-1}.

2. Bounds for the i.f.p.

Consider an i.f.p. satisfying:

$$1 > x_1 \geq x_2 \geq x_3 \geq \cdot \cdot \cdot \geq x_N > 0 \qquad 16)$$

Clearly, we lose no generality by specifying this ordering, since we can always carry out a permutation on the system so that 16) holds. For a given N, the "largest" i.f.p. will be defined as that i.f.p. for

which $x_1 < 1$ has the largest numerical value as we range over all possible systems. (Since the number of systems is finite for finite N, there will always exist a largest i.f.p.) The question then arises: Given N, which system has the largest i.f.p., and what is the corresponding value of x_1? In view of the astronomical number of inequivalent systems (for even moderate values of N) it is of some interest that a partial answer can be given to this question.

For N=3, our complete study reveals that the system possessing the largest i.f.p.--hereafter called the "maximal system"--is II.3.d, for which the defining equations are (we interchange x_2 and x_3 for convenience):

$$\begin{aligned} x_1 &= x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \\ x_2 &= x_1^2 \\ x_3 &= x_2^2 \end{aligned} \qquad 17)$$

The (unreduced) equation is clearly ($x \equiv x_1$):

$$x + x^2 + x^4 = 1 \qquad 18)$$

which yields:

$$x = .56984029 \qquad 19)$$

The natural generalization of this system to N dimensions is:

$$
\begin{aligned}
x_1 &= x_N^{\,2} + \ldots = \left(\sum_{i=1}^{N} x_i\right)^2 - \sum_{i=1}^{N-1} x_i^{\,2} \\
x_2 &= x_1^{\,2} \\
x_3 &= x_2^{\,2} \\
&\;\;\vdots \\
x_N &= x_{N-1}^{2}
\end{aligned}
\qquad 20)
$$

The root $x = x_1$ is then given as a root of the equation:

$$
f_N(x) \equiv \sum_{N=0}^{p=N-1} x^{2^p} = 1 \qquad 21)
$$

This root converges very rapidly as $N \rightarrow \infty$; for example,

$$
\begin{aligned}
N = 4&: \quad x = .566160865 \ldots \\
N = 5&: \quad x = .566123797 \ldots \\
N = \infty&: \quad x = .566123792 \ldots
\end{aligned}
\qquad 22)
$$

It is tempting to consider this last number as an N-independent bound for all binary reaction systems. Unfortunately, this is false, as will be shown below.

Consider a system with $x_1 > 1/2$ and satisfying the ordering 16). Let us assume the "skeleton":

$$x_1 = x_1^2 + 2x_1x_2 + \dots$$

$$x_2 = 2x_1x_3 + \dots$$

$$x_3 = 2x_1x_4 + \dots$$

$$\vdots \qquad 23)$$

$$x_{N-1} = 2x_1x_N + \dots$$

$$x_N = \dots$$

Clearly:

$$1 - x_1 = x_2 + x_3 + x_4 + \dots + x_N \leq x_2\left[1 + \frac{1}{2x_1} + \frac{1}{4x_1^2} + \frac{1}{8x_1^3} \dots \frac{1}{(2x_1)^{N-2}}\right] \qquad 24)$$

But $\quad x_2 \leq \frac{1-x_1}{2} \qquad 25)$

or $\quad \frac{1-x_1}{2} \geq x_2 \geq \frac{1-x_1}{\sum_{p=0}^{N-2} (2x_1)^{-p}} \qquad 26)$

If we equate these bounds and set $y = 2x_1$, we obtain the equation:

$$y^{N-1} - 2y^{N-2} + 1 = 0 \qquad 27)$$

Calling the root of this equation y_N, it is evident from 26) that:

$$x_1 \leq \frac{y_N}{2} \qquad 28)$$

Clearly, $y_N \to 2$ as $N \to \infty$, i.e., the bound is N-dependent. One might suspect at first that this bound is a very weak one, and that the actual maximal system has a much lower i.f.p. However, that the bound is the best possible is proven by exhibiting a system for which $\frac{y_N}{2} = x_1$ is actually attained. In fact, such a system is:

$$
\begin{aligned}
x_1 &= x_1^2 + 2x_1x_2 \\
x_2 &= 2x_1x_3 \\
x_3 &= 2x_1x_4 \\
&\cdot \\
&\cdot \\
&\cdot \\
x_{N-1} &= 2x_1x_N \\
x_N &= \sum_{i=2}^{N} x_i^2 + 2\sum_{i<j=2}^{N} x_ix_j = (x_2 + \cdot\cdot\cdot x_N)^2 = (1-x_1)^2
\end{aligned}
\qquad 29)
$$

It is easily verified that this system has 27) as its i.f.p. equation.

$$
\begin{aligned}
&\text{For} \quad N = 4 \qquad x_1 = \frac{1+\sqrt{5}}{4} = .809016995 \\
&\qquad N = 5 \qquad x_1 = .919643378
\end{aligned}
\qquad 30)
$$

Experimentally (N=4,5,6), this i.f.p. is <u>not</u> attained on iteration starting from a general point, but these converge to the n.f.p. $x_N = 1$. This is to be contrasted with the behavior of the system 20), which actually attained its i.f.p. (N=3,4,5). Indeed, for system 29) it turns

out that the absolute value of the Jacobian at the i.f.p. is ($y = 2x_1$):

$$|J| = y^{N-2}(2-y)(\frac{3y}{2} - 1) > 1$$

which makes it reasonable that this i.f.p. is not attractive.

On the other hand, it is clear that for a "skeleton" of the form:

$$\begin{aligned} x_1 &= \cdot\ \cdot\ \cdot \\ x_2 &= x_1^{\ 2} + \cdot\ \cdot\ \cdot \\ x_3 &= x_2^{\ 2} + \cdot\ \cdot\ \cdot \\ &\cdot \\ &\cdot \\ &\cdot \\ x_N &= x_{N-1}^2 + \cdot\ \cdot\ \cdot \end{aligned} \qquad 31)$$

we have:

$$\begin{aligned} x_2 &\geq x_1^{\ 2} \\ x_3 &\geq x_2^{\ 2} \geq x_1^{\ 4} \\ &\cdot \\ &\cdot \\ &\cdot \\ x_N &\geq x_{N-1}^2 \geq x_1^{\ 2^{N-1}} \end{aligned}$$

whence:

$$1 = x_1 + x_2 \cdot\ \cdot\ \cdot + x_N \geq \sum_{p=0}^{N-1} x_1^{\ 2^p} = f_N(x_1) \qquad 32)$$

Therefore, for such a skeleton, the root of $f_N(x) = 1$ does indeed provide an upper bound (attained for the system 20)).

At this writing it has not yet been shown that the system 29) is actually maximal. However, a weak upper bound can be obtained for all systems such that x_1 does <u>not</u> contain the term x_1^2 (skeletons of the form 31) are a sub-class of these).

Namely, in this case:

$$x_1 \leq 2x_1(1-x_1) + (1-x_1)^2 = 1 - x_1^2 \qquad 33)$$

Therefore, clearly:

$$x_1^2 + x_1 \leq 1$$

or

$$x_1 \leq \frac{\sqrt{5} - 1}{2} = .61803399 \qquad 34)$$

However, we can do much better for this case. In fact, we can show that for $x_1 > 1/2$ we must have, under the ordering 16):

$$x_k > x_{k-1}^2 \qquad 35)$$

which then establishes the bound 21) by the previous argument.

VII. PERIODIC LIMITS

For a large number of 3-dimensional systems, randomly chosen initial vectors iterated to a periodic limit, i.e., the limiting behavior was an oscillation of period 2 or 3 between fixed points. Twenty-four systems exhibited this behavior for three initial vectors, while six others achieved a similar limiting configuration for at least <u>one</u> choice of initial vector (with no coordinates lying on the boundary). In most of these cases the final state was of the form:

$$x_i^{(n)} = 1,\ x_j^{(n)} = x_k^{(n)} = 0; \qquad x_i^{(n+1)} = 0,\ x_j^{(n+1)} = 1,\ x_k^{(n+1)} = 0;$$

$$x_i^{(n+2)} = 1,\ x_j^{(n+2)} = x_k^{(n+2)} = 0;$$

i.e., a boundary double point. A few cases of a boundary triple point were also observed (cf. Table I). Such final states we call "trivial," for the reason that the algebraic structure of the transformation alone indicates that such a final state is at least possible. We may contrast this "trivial" type of oscillatory final state with those for which the oscillation takes place between two or three <u>interior</u> points. The latter we call an "interior double point" (i.d.p.) or "interior triple point" (i.t.p.).

For N=3 we found just four examples of an i.d.p. and one of an i.t.p. There was also one case of a "non-trivial" b.d.p. (system I.2.e) for which the final state was oscillatory with period 2, but between two "non-trivial" boundary points, viz.:

$$x_1 = .56519772 \qquad x_1 = .31944846$$
$$x_2 = 0 \qquad x_2 = .68055154$$
$$x_3 = .4348022 \qquad x_3 = 0$$

The existence of an interior double (or triple) point means that the second (or third) power of the transformation possesses these limit values as fixed points. The algebraic difficulty of finding such points is in general prohibitive. For example, in the unique case of the interior triple point (system I.3.g), if we let:

$$x_1^{(n+3)} = x_1^{(n)} = x, \quad x_1^{(n+1)} = y, \quad x_1^{(n+2)} = z$$

then one coordinate of the triple point is determined by the set of equations:

$$\begin{aligned} x &= 1 - z^2 - 2y^2(1-y^2) \\ y &= 1 - x^2 - 2z^2(1-z^2) \\ z &= 1 - y^2 - 2x^2(1-x^2) \end{aligned} \qquad 36)$$

It may be verified that the successive values of x_1 given in Table II indeed satisfy this set of equations.

Although there are no oscillatory limiting configurations with periods greater than 3 for N=3, one can, of course, find such by going to higher N. Indeed, we discovered, by chance, a particularly interesting case, viz.:

$$
\begin{aligned}
x_1' &= x_2^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_4 + 2x_2x_4 + 2x_3x_4 \\
x_2' &= x_4^2 + 2x_1x_3 \\
x_3' &= x_1^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{I.5.j - ext - 1}) \\
x_4' &= x_3^2
\end{aligned}
$$

This can be considered as one particular generalization of the 3-variable system I.5.j. This generalization--which we applied to several of our original systems[7]-- consists in setting $x_4' = x_3^2$, replacing x_3^2 in the original system by x_4^2, and putting the new cross-terms $2x_4(x_1 + x_2 + x_3)$ in the top line. When we generalized the triply periodic case, I.3.g, in this manner, the resulting limiting behavior (for 3 randomly chosen initial vectors) was still periodic with period 3, but the configuration was of the "trivial" sort, i.e., the b.t.p. (1,0,0,0), (0,0,1,0), (0,0,0,1). However, in the case of I.5.j, whose resulting generalization is given above, the limiting configuration was oscillatory with period 12. (The values of the coordinates are given in Table III.) A further generalization to 5 variables, following the same prescription, yields the system:

[7] See Table III for results.

$$x_1' = x_2^2 + 2x_1x_2 + 2x_2x_3 + 2x_4(x_1 + x_2 + x_3) + 2x_5(x_1 + x_2 + x_3 + x_4)$$

$$x_2' = x_5^2 + 2x_1x_3$$

$$x_3' = x_1^2 \qquad \text{(I.5.j - ext - 2)}$$

$$x_4' = x_3^2$$

$$x_5' = x_4^2$$

In this case, 3 random initial vectors achieved an oscillatory limiting configuration of period 6. (See Table III for the numerical values.)

In our opinion, it is not likely that the behavior observed in these two cases could be predicted by means of any simple criteria.

VIII. FORM STABILITY

1. In a few cases we investigated the effect of making slight changes in the <u>form</u> of the equations themselves. One way of doing this which makes the change of form depend on a single parameter is as follows: Multiply each term by the factor $1 - \epsilon$, and add $\epsilon/2$ times the term in each of the two other rows.

For example, the system I.5.o is:

$$\begin{aligned} x_1' &= x_3^2 + 2x_1x_2 + 2x_2x_3 \\ x_2' &= x_2^2 + 2x_1x_3 \\ x_3' &= x_1^2 \end{aligned} \qquad \text{(I.5.o)}$$

This was now modified to:

$$\begin{aligned} x_1' &= \epsilon/2\, x_1^2 + \epsilon/2\, x_2^2 + (1-\epsilon)x_3^2 + 2(1-\epsilon)x_1x_2 + \epsilon\, x_1x_3 + 2(1-\epsilon)x_2x_3 \\ x_2' &= \epsilon/2\, x_1^2 + (1-\epsilon)x_2^2 + \epsilon/2\, x_3^2 + \epsilon\, x_1x_2 + 2(1-\epsilon)x_1x_3 + \epsilon\, x_2x_3 \\ x_3' &= (1-\epsilon)x_1^2 + \epsilon/2\, x_2^2 + \epsilon/2\, x_3^2 + \epsilon\, x_1x_2 + \epsilon\, x_1x_3 + \epsilon\, x_2x_3 \end{aligned}$$

In this case it is easy to carry out the elimination of x_2, x_3 to obtain the equation for the i.f.p. as a function of ϵ. There results:

$$b_1 x_1^4 + b_2 x_1^3 + b_3 x_1^2 + b_4 x_1 + b_5 = 0 \qquad 38)$$

where the b_i are given in terms of the parameter:

$$\alpha \equiv \frac{2-3\epsilon}{2} \qquad 39)$$

as follows:

$$\begin{aligned} b_1 &= \alpha^4 \\ b_2 &= 4\alpha^3 \\ b_3 &= 2\alpha^2 - 2/3\ \alpha^3(2+\alpha) \\ b_4 &= \alpha(1+2\alpha) - \frac{4\alpha^2}{3}(2+\alpha) \\ b_5 &= -\frac{\alpha}{3}(2\alpha+1) + \frac{\alpha^2}{9}(2+\alpha)^2 \end{aligned} \qquad 40)$$

For $\epsilon = 0$ (α=1) we get (dividing out x_1) the original i.f.p. equation:

$$x_1^3 + 4x_1^2 - 1 = 0 \qquad 41)$$

At $\epsilon = 2/3$, $\alpha = 0$ and all the coefficients b_i vanish. This corresponds to the set:

$$x_1' = x_2' = x_3' = 1/3(x_1^2 + x_2^2 + x_3^2) + 2/3(x_1x_2 + x_1x_3 + x_2x_3) \qquad 42)$$

which reaches the i.f.p. (1/3, 1/3, 1/3) on a single iteration starting

from any initial vector. In fact, as $\epsilon \to 2/3$, all systems to which this generalization is applied tend toward this simple case (since at $\epsilon = 2/3$, $1-\epsilon = \epsilon/2$).

For the system I.5.o, generalized in this manner, we investigated the convergence for several values of ϵ, viz.: $\epsilon = .001, .01, .05, .1$. In each case, randomly chosen initial vectors iterated to the i.f.p. predicted by equation 38). In other words, there is a sort of continuity in the convergence behavior as the form of the equation is changed in this simple manner.

Slightly more interesting is the result of applying this one-parameter generalization to the system I.3.g. As mentioned above, random initial vectors iterated according to this transformation reached an oscillatory final state with period 3. Vectors under the generalized transformation behaved in the same fashion, but as ϵ was increased the final-state oscillations decreased in amplitude, until at $\epsilon \cong .045$ general initial vectors appeared to converge to the i.f.p. predicted by the corresponding i.f.p. equation. Presumably, the final state is still oscillatory (with period 3), but the oscillations are too small to observe with 8-decimal-place accuracy.[8]

---

[8] For $\epsilon > 0$, some initial points converged to the i.f.p. In other words, the i.f.p. (which is a function of ϵ) appears to be attractive for $\epsilon > 0$. More detailed analysis would probably show that what is happening is that the area of the region of the triangle for which convergence (of a point in this region) is to the i.f.p., is increasing with increasing ϵ. Correspondingly, one is less likely to pick an initial point outside this region, i.e., a point which will iterate to the i.t.p. For further discussion of behavior in the large, see Section XI.

The conclusion to be drawn from these experiments is that binary reaction systems are "stable" under small perturbations of formal structure.

2. When more radical changes of form are made, we do get correspondingly greater changes in behavior. For instance, we took the above-mentioned system I.3.g and kept only the skeleton:

$$\begin{aligned} x_1' &= 1/2\, x_2^2 + 1/2\, x_3^2 + 2x_1x_2 + \ldots \\ x_2' &= 2x_1x_3 + 2x_2x_3 + \ldots \\ x_3' &= 1/2\, x_1^2 + \ldots \end{aligned} \qquad 43)$$

To this we added the missing terms $1/2\, x_1^2$, $1/2\, x_2^2$, $1/2\, x_3^2$ in all possible ways (i.e., 27 ways, of which one corresponds to the original system). For 25 of the resulting 26 new transformations, random initial vectors iterated to an i.f.p. (The results are summarized in Table IV.) One system, however, gave an oscillatory final state with period 3, a behavior analogous to that of the original system, viz.:

$$\begin{aligned} x_1' &= x_2^2 + 1/2\, x_3^2 + 2x_1x_2 \\ x_2' &= 1/2\, x_3^2 + 2x_1x_3 + 2x_2x_3 \\ x_3' &= x_1^2 \end{aligned} \qquad 44)$$

The configuration of the final state (starting from 3 randomly chosen initial vectors) was:

$$x_1^{(n+3)} = x_1^{(n)} = .87777286, \quad x_1^{(n+1)} = .00766150, \quad x_1^{(n+2)} = .34944206$$
$$x_2^{(n+3)} = x_2^{(n)} = .00011739, \quad x_2^{(n+1)} = .22185332, \quad x_2^{(n+2)} = .65049924$$
$$x_3^{(n+3)} = x_3^{(n)} = .12210975, \quad x_3^{(n+1)} = .77048518, \quad x_3^{(n+2)} = .00005870$$

It is clear that a "rule" which allows $x_3 \oplus x_3 \rightarrow x_1$ or x_2 (with equal probability) is a rather unnatural one. A somewhat more logical modification is to assume a skeleton in which the cross-terms appear with coefficient unity, and to add the missing terms x_1x_2, x_1x_3, x_2x_3, in all possible ways. When this was done for I.3.g, all the resulting transformations had an i.f.p., and in every case random initial vectors iterated to the i.f.p. (see Table V).

This change of rule has a natural interpretation; effectively it allows non-commutativity, since, e.g., $x_i \oplus x_j$ may give two results, which we can interpret as the respective results of x_ix_j and x_jx_i. Formally, this is an interesting generalization of the concept of a binary reaction system (as originally defined), but the convergence behavior does not appear to be startlingly different.

This same "non-commutative" generalization was tried on the system III.2.1.b; the results offered no surprises.

IX. A SPECIFIC CONVERGENCE PROBLEM

1. The Exceptional Case I.2.j

As mentioned above, there are only two systems among all the 97 which exhibit an ambiguous convergence behavior. The most interesting of these has the form:

$$\begin{aligned} x_1' &= x_1^2 + x_3^2 + 2x_1x_3 \\ x_2' &= x_2^2 + 2x_1x_2 \\ x_3' &= 2x_2x_3 \end{aligned} \qquad \text{(I.2.j)}$$

The system is degenerate, and has two n.f.p.'s; in the following we shall ignore the behavior of boundary points (since these present no problem) and only discuss the behavior under iteration of interior points.

By inspection, it is evident that the system possesses the i.f.p.:

$$\begin{aligned} x_1 &= x_3 = 1/4 \\ x_2 &= 1/2 \end{aligned} \qquad 45)$$

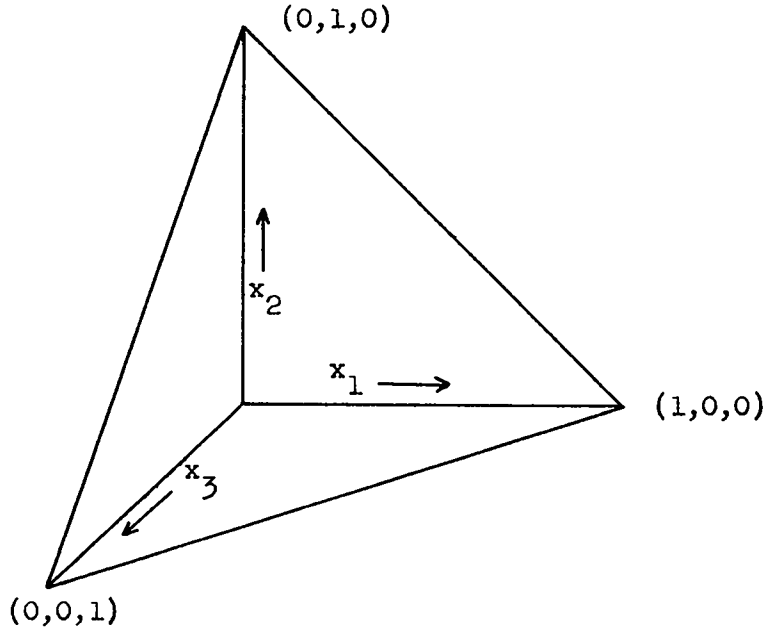
For three randomly chosen (interior) initial points, little convergence was evident even after some 85,000 iterations. In order to see better what was happening, we introduced new coordinates:

$$S = \frac{1+x_1 - x_3}{2}$$

$$\alpha = 1/2\ x_2 \qquad 46)$$

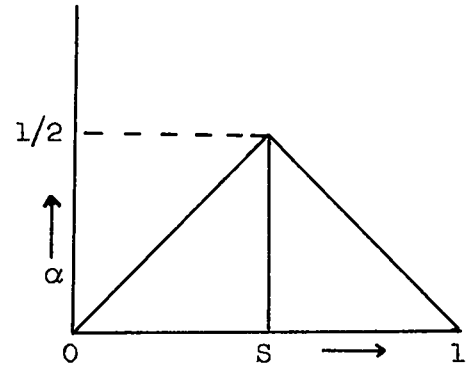
These are effectively Cartesian coordinates in the plane of the triangle formed by the three vertices (1,0,0) (0,1,0) (0,0,1).

More specifically, in the original form of the transformation, the x_i's are constrained to move on the positive portion of plane $x_1 + x_2 + x_3 = 1$:



For algebraic convenience, we distort the triangle into a 45° triangle with base unity. The coordinates of a point in this triangle are then (S,α), as shown in the sketch below. Here

(0,0,1)	(0,0)
(0,1,0)	(1/2,1/2)
(1,0,0)	(1,0)



In terms of these new coordinates, the transformation takes the form:

$$S' = 1 - 4\alpha + 4\alpha^2 + 2\alpha S$$
$$\alpha' = 2\alpha S \qquad 47)$$

and the i.f.p. is:

$$S = 1/2$$
$$\alpha = 1/4 \qquad 48)$$

Note that the Jacobian of 47) is exactly 1 at the f.p. 48). For future reference, we write down the inverse of 47):

$$S = \frac{\alpha'}{1 - \sqrt{S' - \alpha'}}$$

$$\alpha = \frac{1 - \sqrt{S' - \alpha'}}{2} \qquad 49)$$

If we now make the further transformation:

$$x = S - 1/2$$

$$y = \alpha - 1/4 \qquad 50)$$

we get:

$$x' = -y + x/2 + 4y^2 + 2xy$$

$$y' = y + x/2 + 2xy \qquad 51)$$

with i.f.p. $x = y = 0$.

Note that if we consider only <u>linear</u> terms:

$$x' = -y + x/2$$

$$y' = y + x/2 \qquad 52)$$

then an invariant ellipse exists:

$$x'^2 + x'y' + 2y'^2 = x^2 + xy + 2y^2 \qquad 53)$$

Figure 1 shows a plot of the observed iterates in the x,y plane. The three curves show the behavior of successive iterates, the initial point of the sequence being taken respectively at $n = 1550$, $n = 10{,}101$, $n = 75{,}001$. Each curve is roughly an ellipse of the form:

$$x^2 + xy + 2y^2 = C \qquad 54)$$

Numerically, at least, C appears to $\rightarrow 0$ as $n \rightarrow \infty$, or, to put it another way, the axes of the ellipse of reference are shrinking.

In order to convince ourselves that this apparent convergence was not simply the result of systematic round-off, we used the inverse transformation 48), numerically retracing our steps from $n = 10{,}101$ to $n = 1{,}550$. All coordinates agreed to 6 decimal figures, which pretty well precludes the possibility that the observed convergence is a numerical accident.

It should be noted, by the way, that no matter how close one is to the fixed point, the quadratic terms in 51) cannot be ignored, for according to 53), the linear terms by themselves will generate a sequence of iterates which will all lie on the ellipse of reference.

For the purpose of discussion, it is convenient to transform equation 51) so that the reference curve is a circle. The linear part of the transformation can be written:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 1/2 & -1 \\ 1/2 & 1 \end{pmatrix} \qquad 55)$$

If we transform this with the matrix:

$$S = \begin{pmatrix} \sqrt{7} & 1 \\ 0 & -2 \end{pmatrix} \qquad 56)$$

we find:

$$R = S^{-1} AS = \begin{pmatrix} 3/4 & \frac{\sqrt{7}}{4} \\ \frac{-\sqrt{7}}{4} & 3/4 \end{pmatrix} \qquad 57)$$

which corresponds to a rotation through an angle $\theta = \cos^{-1} \frac{3}{4}$. The transform through S of the complete mapping 50) has, then, the form:

$$\begin{aligned} x' &= 3/4x + \frac{\sqrt{7}}{4} y + \frac{10}{\sqrt{7}} y^2 - 6xy \\ y' &= \frac{-\sqrt{7}}{4} x + 3/4y + 2y^2 + 2\sqrt{7}\, xy \end{aligned} \qquad 58)$$

2. The Asymptotic Behavior of the Angle of the Radium Vector under Iteration

The transformations with which we are concerned do not preserve the ordinary measure (Lebesgue measure) of the space which is mapped into itself. Moreover, they are not one-one. In some cases they shrink a neighborhood of the fixed point into a proper part of itself and the limiting image of such a region may consist of this point alone. Obviously, an invariant measure, if it is to be constructed, would have to be of Lebesgue-Stieltjes type and assign positive values, in some cases, to sets consisting of single points.

One may be interested in the behavior of the angles (with a fixed direction issuing from a fixed point) of the vectors $T^i(\bar{x})$, where $\bar{x} = (x_1, x_2\ x_3)$; $i = 1, 2, \ldots$ more generally in the behavior of the points:

$$S_i = \frac{T^i(\bar{x})}{|T^i(\bar{x})|} \qquad 59)$$

on the unit sphere. Something can be said about it even in cases where T does not transform any bounded region into itself. Thus, for example, if T is an arbitrary linear transformation of the n-dimensional Euclidean space into itself, $T(0) = 0$. Then the ergodic limit of the average of S_i exists. In other words, if C is an arbitrary "cone" of directions in space, i.e., a sub-set of the unit sphere $f_C(s)$, its characteristic function being:

$$f_C(s) = 0 \quad \text{if} \quad s \notin C; \quad f_C(s) = 1 \quad \text{if} \quad s \in C$$

then for almost all s, the limit:

$$\lim_{N=\infty} \frac{1}{N} \sum_{i=1}^{N} f_C(T^i(s)) \qquad 60)$$

exists. This follows for a general linear transformation, T, from the well-known theorems giving, in fact, more precise information--in the two extreme cases: If T is an orthogonal transformation we have the Kronecker-Weyl theorem on equipartition; if T, considered as a matrix, has all coefficients positive, the Perron-Frobenius theorem asserts convergence to a unique direction. In the general case one obtains, by considering a decomposition of the space into sub-regions where one or the other behavior dominates, at least the <u>existence</u> of the ergodic limit.

Presumably, the theorem still holds true if T is a general homogeneous quadratic transformation of the n-dimensional space.

In our very special quadratic transformations of the plane, more can be said: The transformation of the previous section possesses the Kronecker-Weyl property: The angle described by the iterates of almost every point covers the circumference of the unit circle densely and uniformly.

We hope to show that if the linear part of a quadratic transformation Q, which has the origin as its fixed point, consists of a rotation through an irrational angle, then the iterates of Q converge to the origin provided one starts with points in a circle of sufficiently small radius.

A detailed discussion of these matters will be given in a subsequent report.

X. FURTHER GENERALIZATIONS

In view of the impracticability of studying all possible Binary Reaction Systems for any $N > 3$, we thought it worthwhile to generalize a few of our 3-variable systems to higher dimension by arbitrary but fixed rules. One such generalization is mentioned in Section VII (see I.5.j - ext - 1 and subsequent discussion). Another essentially different way to generate "interesting" systems is to construct for any given 3-variable transformation the corresponding "super-system." This is constructively defined as follows:

We introduce nine variables, $y_1, y_2, \cdots y_9$, according to the prescription:

$$\begin{aligned} x_1 &= y_1 + y_2 + y_3 \\ x_2 &= y_4 + y_5 + y_6 \\ x_3 &= y_7 + y_8 + y_9 \end{aligned} \qquad 61)$$

and substitute these in the original transformation. We then have three transformations for the three triads of variables $y_1 + y_2 + y_3$, $y_4 + y_5 + y_6$, $y_7 + y_8 + y_9$. Consider, e.g., the system I.2.e. The last line of the transformation now reads:

$$y_7' + y_8' + y_9' = 2(y_1 + y_2 + y_3)(y_4 + y_5 + y_6)$$

In order to convert this into 3 separate expressions for y_7', y_8', y_9', we could, for example, <u>formally</u> identify the variables modulo 3, i.e.,

$$\begin{aligned} &y_1 \sim y_4 \sim y_7 \\ &y_2 \sim y_5 \sim y_8 \\ &y_3 \sim y_6 \sim y_9 \end{aligned} \qquad 62)$$

Correspondingly, on the right-hand side, we could make the identification:

$$\begin{aligned} &2y_1y_4 \sim y_1^2 \\ &2y_2y_5 \sim y_2^2 \\ &2y_2y_6 + 2y_3y_5 \sim 2y_2y_3 \\ &2y_3y_6 \sim y_3^2 \\ &2y_1y_6 + 2y_3y_4 \sim 2y_1y_3 \\ &2y_1y_5 + 2y_2y_4 \sim 2y_1y_2 \end{aligned} \qquad 63)$$

We can now write expressions for y_7', y_8', y_9' so that, with these formal identifications, the resulting sub-system will have the same form as the original 3-variable system, i.e.,

$$\begin{aligned} y_7' \sim y_1' &= y_1^2 + y_2^2 + 2y_2y_3 \sim 2y_1y_4 + 2y_2y_5 + 2y_2y_6 + 2y_3y_5 \\ y_8' \sim y_2' &= y_3^2 + 2y_1y_3 \sim 2y_3y_6 + 2y_1y_6 + 2y_3y_4 \\ y_9' \sim y_3' &= 2y_1y_2 \sim 2y_1y_5 + 2y_2y_4 \end{aligned} \qquad 64)$$

In this way we (arbitrarily) obtain equations for y_7', y_8', y_9' in terms of the y_i's. When this is done for each triad, a 9-dimensional B.R.S. results.

In the present case, using the symbolic notation of Table II:

(11) + (44) + 2(47) + (22) + (55) + 2(58) + 2(23) + 2(56) + 2(59) + 2(68)

(33) + (66) + 2(69) + 2(13) + 2(46) + 2(49) + 2(67)

2(12) + 2(45) + 2(48) + 2(57)

(77) + 2(17) + (88) + 2(28) + 2(29) + 2(38) + 2(89)

(99) + 2(39) + 2(79) + 2(19) + 2(37)

2(78) + 2(18) + 2(27)

2(14) + 2(25) + 2(26) + 2(35) (I.2.e - Super)

2(36) + 2(16) + 2(34)

2(15) + 2(24)

As a second example, we quote the result of treating system I.3.g in the same manner:

(55) + 2(25) + (88) + (66) + (99) + 2(36) + 2(45) + 2(78) + 2(15) + 2(24)

2(46) + 2(79) + 2(16) + 2(34) + 2(56) + 2(89) + 2(26) + 2(35)

(44) + (77) + 2(14)

2(28) + 2(58) + 2(39) + 2(69) + 2(18) + 2(48) + 2(27) + 2(57)

2(19) + 2(37) + 2(49) + 2(67) + 2(29) + 2(38) + 2(59) + 2(68)

2(17) + 2(47)

(22) + (33) + 2(12)

2(13) + 2(23)

(11) (I.3.g - Super)

When these transformation were iterated for randomly chosen initial vectors, the sums:

$$x_1 = y_1 + y_2 + y_3, \quad x_2 = y_4 + y_5 + y_6, \quad x_3 = y_7 + y_8 + y_9$$

reached, of course, as they should, the same limiting configuration as was observed in the original 3-variable system. However, the actual values of the individual y_i varied with the initial configuration. The results are given in Table VI.

XI. PROPERTIES IN THE LARGE

1. Although it is, in general, possible to discuss the behavior of points under iteration in the neighborhood of a fixed point, the iteration behavior of such points over the whole domain (positive portion of the hyperplane) can, at present, be treated only experimentally. In what follows, we shall take the variables to be S, α, and the domain to be the corresponding 45° triangle with unit base (see Section IX.1 for a definition of this coordinate system).

As stated above, we have not found any general criteria for determining which of several possible limiting behaviors will be realized for a given 3-variable system, starting with a general point in the triangle. On the basis of a rather small sample (~ 3 random initial points for each system), it appears that, excluding boundary points, the limiting behavior is independent of the initial point for the large majority of systems. As shown in Table I, however, there are (at least) 6 systems in which this limiting behavior depends on the initial point (we exclude from consideration the "pathological" systems II.1.d and II.1.f; see the discussions under these entries in Table II). Two of these, I.2.m and I.5.h, have been examined in greater detail. What was

done was to look for boundaries which separate regions of different limiting behavior. This was accomplished by programming the computing machine to "search" the whole triangle in a systematic manner. On the first pass a crude net was used ($\Delta S = \Delta\alpha = .05$). Then, when the boundaries had been approximately located, a more refined interval was employed in the appropriate neighborhoods. For each trial point in the triangle, sufficient iterations had to be performed to identify the limiting behavior. Despite the apparent magnitude of the task (several hundred trial points had to be followed for some 70 iterations each), a complete search (first crude, then appropriately refined) takes only about 15 minutes of computing time per system.[9] The results for the two systems studied are shown in figures 2 and 3. In these, each calculated boundary point is determined to within an absolute error $< .0025$, i.e., to within 1/4% of the length of the base of the triangle. In these figures, initial points lying in the region marked "OSC." will iterate to the appropriate boundary oscillation, while all points lying outside these regions will converge under iteration to the i.f.p. The boundaries appear to be complicated. We have not attempted to study what happens to points actually lying on the boundary curves; for this an analytical treatment is necessary.

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[9] On the average, the machine will perform 50 iterations/sec for a 3-variable Binary Reaction System. The systematic search of the triangle is somewhat slower for various reasons connected with input-output requirements.

In a few simple cases it is possible to give an analytical treatment of such boundary regions. As an example, we cite the system III.2.2.a. In the S,α coordinates (we use these to conform with our treatment of the other systems; the argument can be carried out in the original coordinates with equal ease) the transformation takes the form:

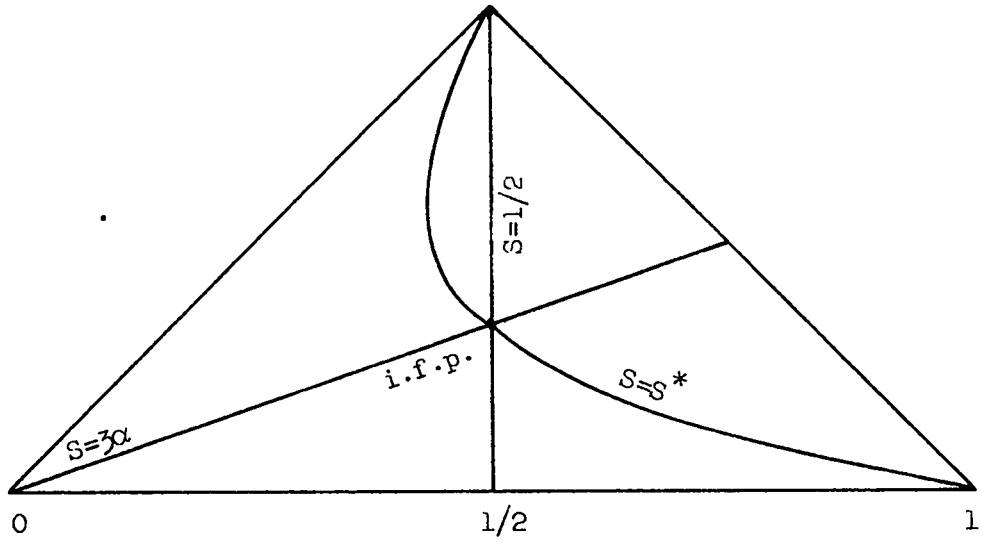
$$\begin{aligned} S' &= 2S - S^2 + 3\alpha^2 - 4\alpha S \\ \alpha' &= 2\alpha(1-S) \end{aligned} \qquad 65)$$

There are 3 n.f.p.'s, namely, $(S=0,\ \alpha=0)$, $(S=1,\ \alpha=0)$, and $(S=1/2,\ \alpha=1/2)$, while the i.f.p. is $(S=1/2,\ \alpha=1/6)$. The boundaries $\alpha=0$ and $S=\alpha$ are clearly transformed into themselves. In addition, there exists an invariant line:

$$S = 3\alpha$$

All points lying on this line can easily be shown to iterate to the i.f.p.--with the exception of $S=\alpha=0$, which is a non-attractive f.p. This line is shown on the diagram below. The curve $S=S^*$ is the locus of all points such that $S'=S$; its equation is:

$$S^* = \frac{1 - 4\alpha + \sqrt{(1-4\alpha)^2 + 12\alpha^2}}{2} \qquad 66)$$



It is easily shown that a point lying in the region below the line $S=3\alpha$, i.e., such that $S > 3\alpha$, remains in this region under iteration; similarly for points lying above the line. Further, all points lying to the left of the curve $S=S^*$ remain to the left under iteration. It can be further shown that points lying to the right of $S=S^*$ and not situated at the corners of the triangle or on the line $S=3\alpha$ will eventually cross the curve. Furthermore, in the neighborhood of the i.f.p., the linear approximation to the transformation is ($x \equiv S-1/2$, $y \equiv \alpha-1/6$):

$$\begin{aligned} x' &= 1/3x - y \\ y' &= -1/3x + y \end{aligned} \qquad 67)$$

from which it can be deduced that the i.f.p. is attractive only along the line $x = 3y$. Finally, using the fact that for $S < 1/2$, we always

have $\alpha' - \alpha > 0$ and that correspondingly for $S > 1/2$, $\alpha' - \alpha < 0$, it can be shown that all points lying above the line $S=3\alpha$ will iterate to $S=\alpha=1/2$, while all points below the line will iterate to $S=1$, $\alpha=0$.

The reason why this system can be treated so simply is, of course, that the boundary curves are explicitly known. When this is not the case, it may be helpful to have before one a picture of the mapping (single iteration) of the entire triangle, as well as the curves along which α and S are stationary. A few interesting examples are given in figures 4, 5, and 6.

2. There is one case which does not appear to converge either to a fixed point or to a finite oscillation. This is the system III.2.3.a:

$$\begin{aligned} x_1' &= x_1^2 + 2x_1x_2 \\ x_2' &= x_2^2 + 2x_2x_3 \\ x_3' &= x_3^2 + 2x_1x_3 \end{aligned} \qquad 68)$$

or, in terms of the coordinates S,α:

$$\begin{aligned} S' &= 2\alpha + S^2 - 3\alpha^2 \\ \alpha' &= 2\alpha(1-S) \end{aligned} \qquad 69)$$

The situation is illustrated in figure 7. The i.f.p. is non-attractive and the 3 corners of the triangle are attractive only along the boundaries in a clockwise direction. Under iteration, points will spiral out, approaching arbitrarily close to the boundaries but, e.g.,

as the transformation $\alpha' = 2\alpha(1-S)$ shows (for the bottom boundary), no point inside the triangle can ever reach the boundary. Thus a general point will continue to spiral indefinitely.

Numerically, a spurious convergence was observed owing to the fact that the several random initial points chosen rapidly iterated to within a distance less than 10^{-8} from one or another boundary line.

If one transforms the triangle into the unit circle in an appropriate manner, the situation can be viewed as follows: The center is a non-attractive fixed point, and all points lying in the circle spiral outwards towards the circumference. On the circumference itself, there are 3 fixed points located, say, at $\Theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$, which in turn define 3 arcs. Any point lying on one of these arcs will move under iteration in a clockwise direction, ultimately converging to the fixed point which constitutes the right-hand boundary of the arc in question. Interior points, however, can never reach the boundary. In general, the sequence of iterates of any interior point (excluding the center) does not converge.

3. We have not studied in detail the <u>rate</u> of approach to the limiting configuration. In the neighborhood of a fixed point, this rate is usually easy to obtain, but for oscillating configurations the algebra is more difficult. For most of the cases studied, convergence (to 8 decimal places) was either attained within 100 iterations (some were much faster) or else not for many thousands of iterations. We shall not

discuss this further except to remark that the path of approach to a fixed point may depend very critically on the initial point. As an example, we refer the reader to figure 8. For this transformation, two of the corners of the triangle are non-attractive fixed points; initial points in their neighborhoods iterate smoothly to the i.f.p. In contrast, points in the neighborhood of the origin iterate to the i.f.p. in an oscillatory manner; the existence of a limiting line through the i.f.p. is clearly evident.

XII. CONNECTION WITH ORDINARY DIFFERENTIAL EQUATIONS

No doubt it will have occurred to the reader that Binary Reaction Systems and their extensions have an obvious connection with systems of ordinary differential equations. Consider, for instance, the system of differential equations:

$$\begin{aligned}
\frac{dx_1}{dt} &= -x_1 + f_1(x_1 \ldots x_N) \\
\frac{dx_2}{dt} &= -x_2 + f_2(x_1 \ldots x_N) \\
&\cdot \\
&\cdot \\
&\cdot \\
\frac{dx_N}{dt} &= -x_N + f_N(x_1 \ldots x_N)
\end{aligned} \qquad 70)$$

where the $f_i(x_1 \ldots x_N)$ are homogeneous quadratic functions of the variables. A straightforward finite difference approximation to the system is:

$$x_1^{(n+1)} = (1 - \Delta t)x_1^{(n)} + \Delta t\, f_1^{(n)}$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 71)$$

$$x_N^{(n+1)} = (1 - \Delta t)x_N^{(n)} + \Delta t\, f_N^{(n)}$$

If we now restrict the f_i to be disjoint partial sums of the terms in $(x_1 + \ldots + x_N)^2$, such that $\sum_{i=1}^{N} f_i = (x_1 + \ldots + x_N)^2$, i.e., just the terms that occur in our binary reaction transformation, then the above set of difference equations goes over into a Binary Reaction System for $\Delta t = 1$. With this restriction on the f_i, the system of difference equations has the property $\sum x_i^{(n)} = 1$ for all n if $\sum x_i^{(o)} = 1$, independent of the time-step Δt. It should be observed that for $\Delta t > 0$, the system has the same fixed points as the corresponding Binary Reaction System.

As a very simple example, consider the system:

$$\begin{aligned} x_1' &= x_1^2 + x_2^2 \\ x_2' &= 2x_1x_2 \end{aligned} \qquad \begin{aligned} &\text{with i.f.p. } x_1 = x_2 = 1/2, \\ &\text{and b.f.p. } x_1 = 1,\ x_2 = 0. \end{aligned} \qquad 72)$$

Any interior point $(x_1 \neq 1, 0)$ will iterate to the i.f.p. The corresponding differential equation is (eliminating x_2):

$$\frac{dx_1}{dt} = 2x_1^2 - 3x_1 + 1 \qquad 73)$$

of which the solution is:

$$\left| \frac{1-x_1}{1/2-x_1} \right| = \left| \frac{1-x_1^{(o)}}{1/2-x_1^{(o)}} \right| e^t \qquad 74)$$

Clearly, as $t \to \infty$, $x_1 \to 1/2$, $x_1^{(o)} \neq 1, 1/2$. Thus, in this case, the asymptotic behavior of the differential equation is the same as that of the corresponding Binary Reaction System. This, however, is not generally so. For example, the transformation I.5.d (see Table II) possesses a boundary double point which is attained for a certain set of initial points. The corresponding differential equation system, when integrated according to the finite difference scheme 71) with $\Delta t = 2^{-6}$, converged to the n.f.p. $x_1 = 1$ regardless of the initial point.

In such cases the Binary Reaction System, viewed as a finite difference approximation to the corresponding system of differential equations, is clearly an "unstable" scheme. We hope to discuss the point further in a subsequent report.

EXPLANATION OF GRAPHS

1. Figure 1 is a plot of successive iterates of a point under the transformation equation 53) of the text. The outer curve shows the iterates from cycle n = 1550 (point i_1) to cycle n = 1570 (point f_1); the next curve goes from n = 10,101 (point i_2) to n = 10,114 (point f_2), while the innermost curve goes from n = 75,001 (point i_3) to n = 75,011 (point f_3).

2. Figures 2 and 3 show the (experimentally determined) boundaries between two types of limiting behaviors for two different Binary Reaction Systems (see discussion in Section XI). In terms of the S,α coordinates, these transformations are:

I.2.m (Figure 2):

$$S' = 1 + \frac{3S^2 - \alpha^2}{2} + 3\alpha S - 2S$$

$$\alpha' = \frac{S^2 - 3\alpha^2}{2} - 3\alpha S + 2\alpha$$

I.5.h (Figure 3):

$$S' = \frac{1 + \alpha^2 - 3S^2}{2} - \alpha + S + 3\alpha S$$

$$\alpha' = \frac{1 + S^2 - 3\alpha^2}{2} + \alpha - S - \alpha S$$

In each case, points in the inner region (marked "CONV.") will iterate to the fixed point, while points outside this region will reach an oscillatory final state (in I.5.h, all three outer regions are oscillatory, though only one is so marked).

3. Figures 4, 5, and 6. These figures give the S,α mapping explicitly for three different Binary Reaction Systems:

I.2.j (Figure 4):

$$S' = 1 - 4\alpha + 4\alpha^2 + 2\alpha S$$
$$\alpha' = 2\alpha S$$

(cf. equation 47) of the text)

I.5.h (Figure 5):

See 2. above.

III.2.1.a (Figure 6):

$$S' = S + 3\alpha - 6\alpha S$$
$$\alpha' = S - \alpha + 3\alpha^2 - S^2$$

In Figures 4 and 5, the numerically labelled lines are the transforms of lines of constant α; e.g., in I.5.h the curve labelled .25 is the transform of the horizontal line $\alpha = .25$. The curves labelled $\alpha' = \alpha$ and $S' = S$ are lines of constant α and constant S, respectively.

Figure 6 is plotted in a 60°, 30° reference triangle, i.e., in terms of the variables S and $t = \sqrt{3}\,\alpha$. The labelled curves are the transforms of lines of constant t, and the arrows indicate the order of the transformed points as the relevant lines of constant t are traversed from left to right (direction of increasing S). Note the change of

direction for $t > \frac{\sqrt{3}}{6}$. (The vertical line with two arrows labelled 1/6 is the transform of $\alpha = 1/6$ or $t = \frac{\sqrt{3}}{6}$. This line is doubly covered, first upwards, then downwards, as S increases.)

4. Figure 7 illustrates the "non-convergent" case. See Section XI, equation 69), and the accompanying text.

5. Figure 8 illustrates different modes of convergence to a fixed point. See part 3 of Section XI for discussion.

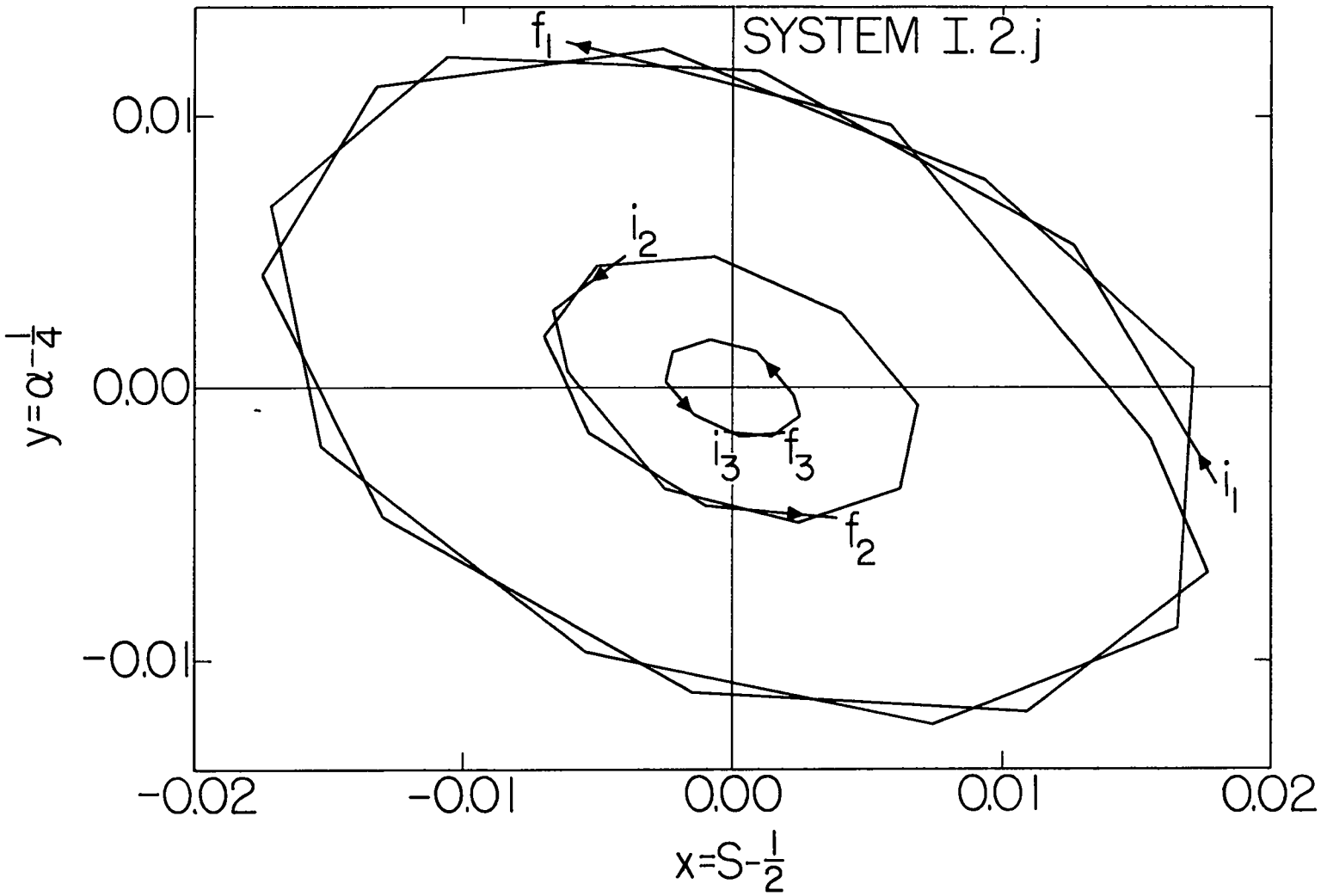

SYSTEM I. 2.j
f_1
i_2
i_3
f_3
f_2
i_1
0.01
0.00
-0.01
-0.02
-0.01
0.00
0.01
0.02
$y=\alpha-\frac{1}{4}$
$x=S-\frac{1}{2}$

Fig. 1

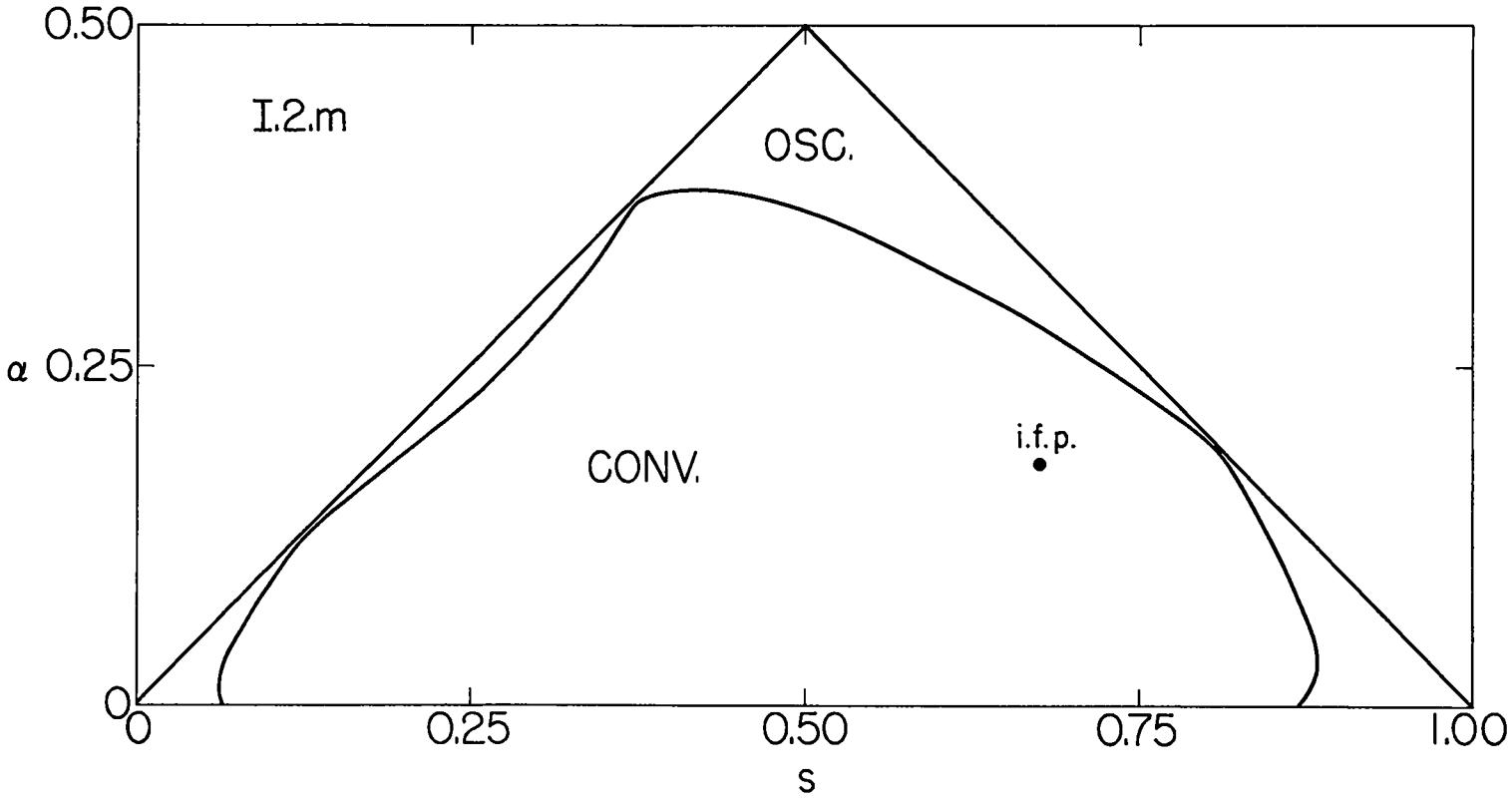


Fig. 2

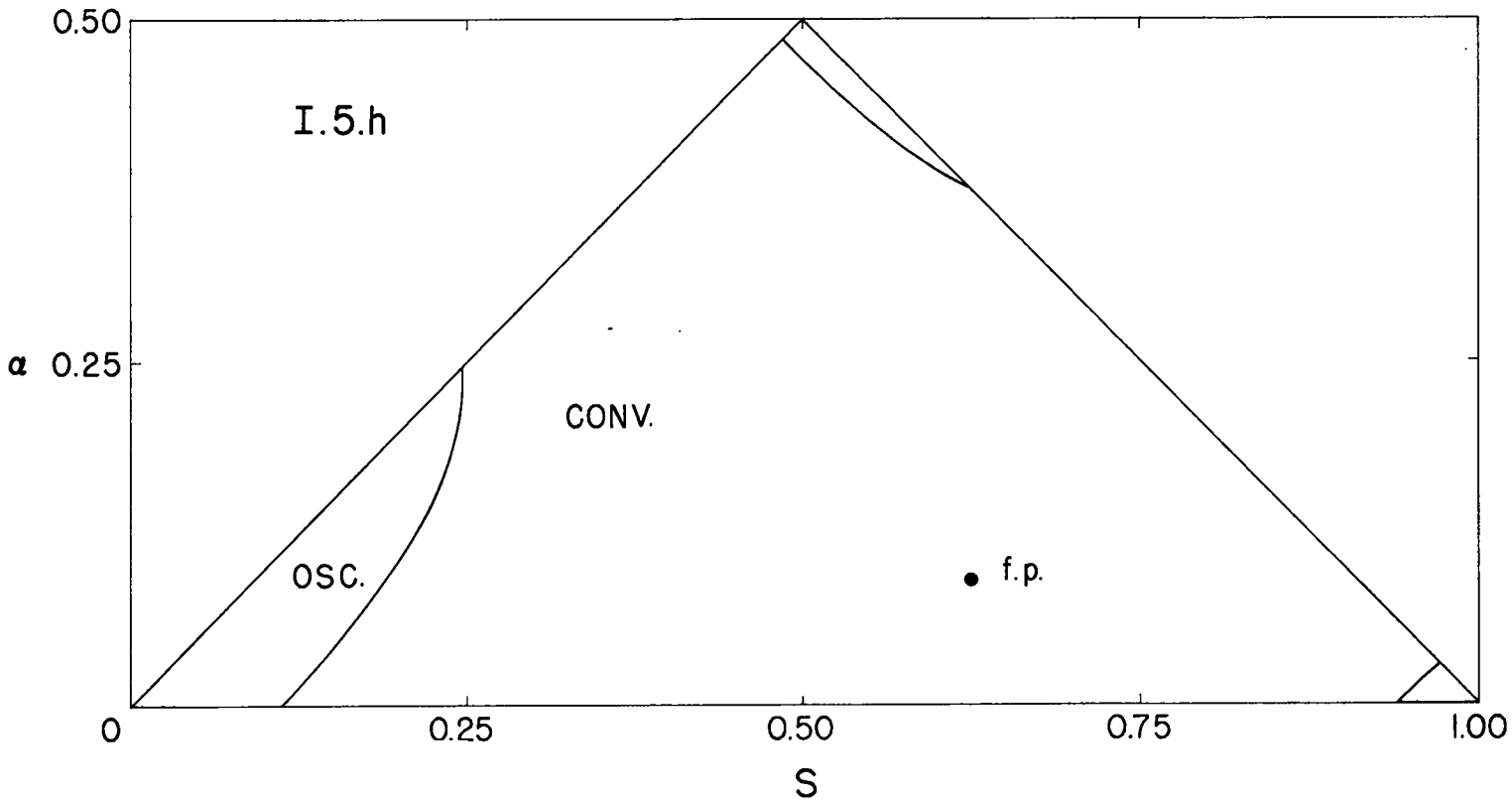


Fig. 3

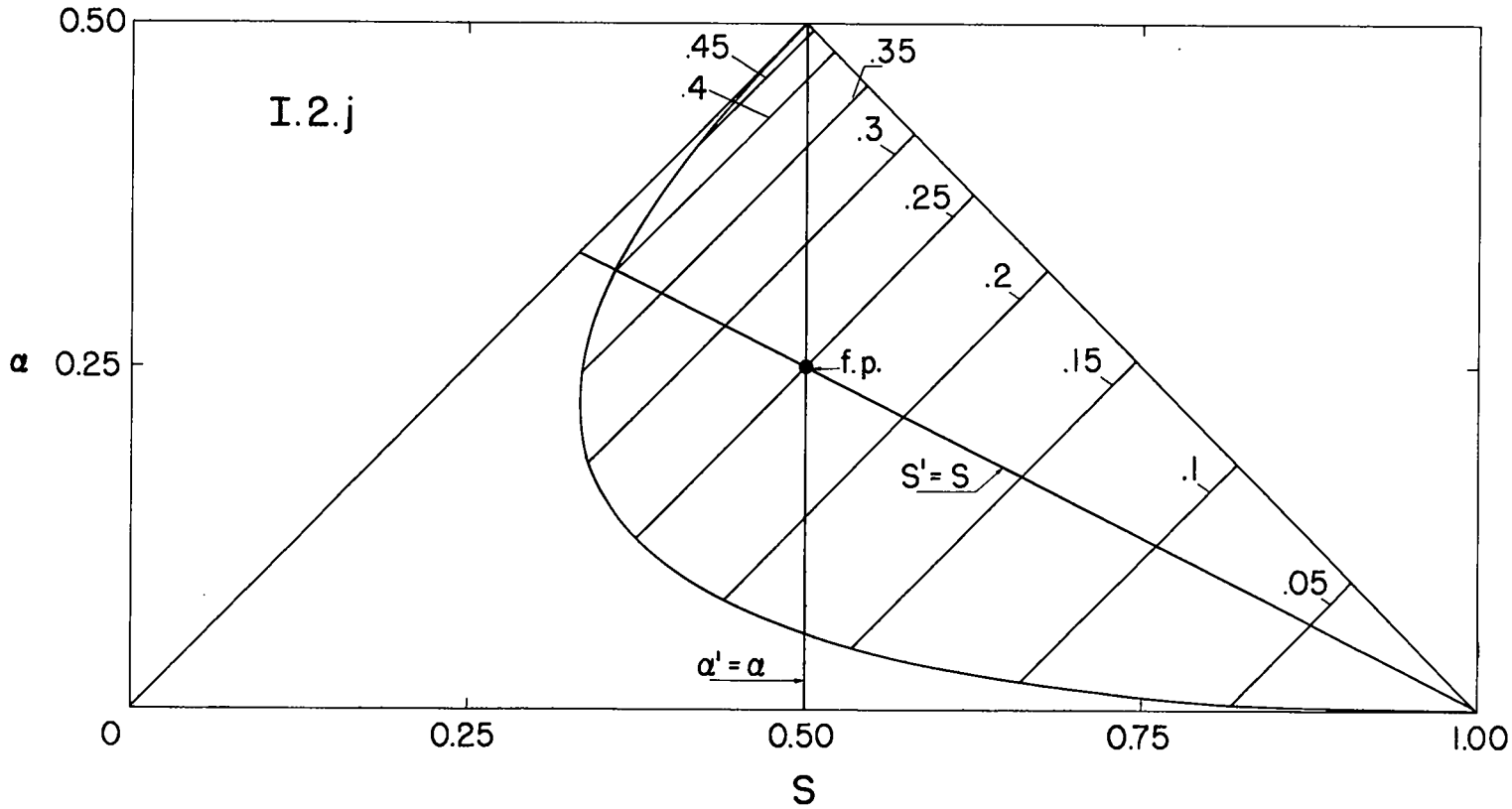


Fig. 4

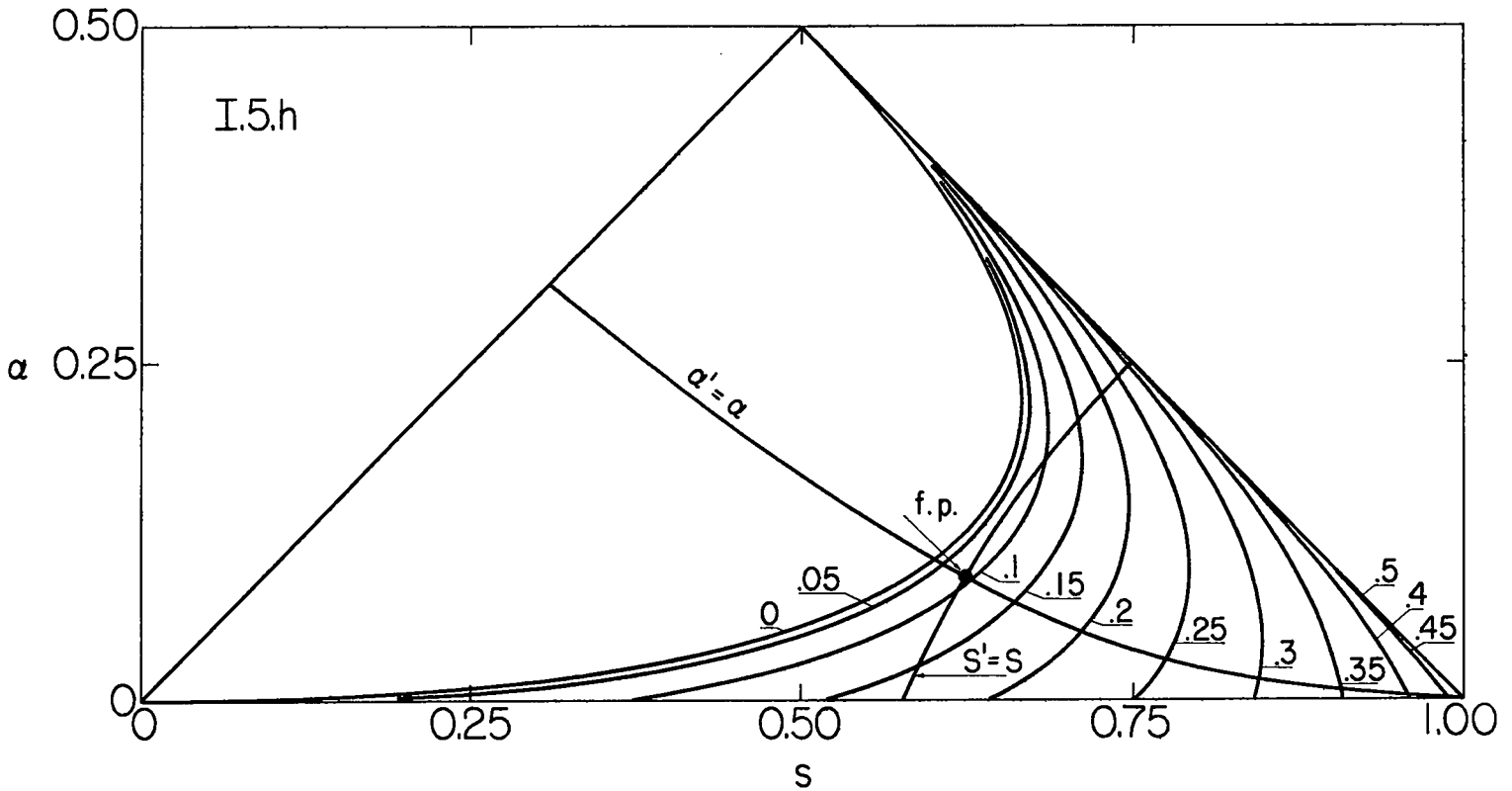


Fig. 5

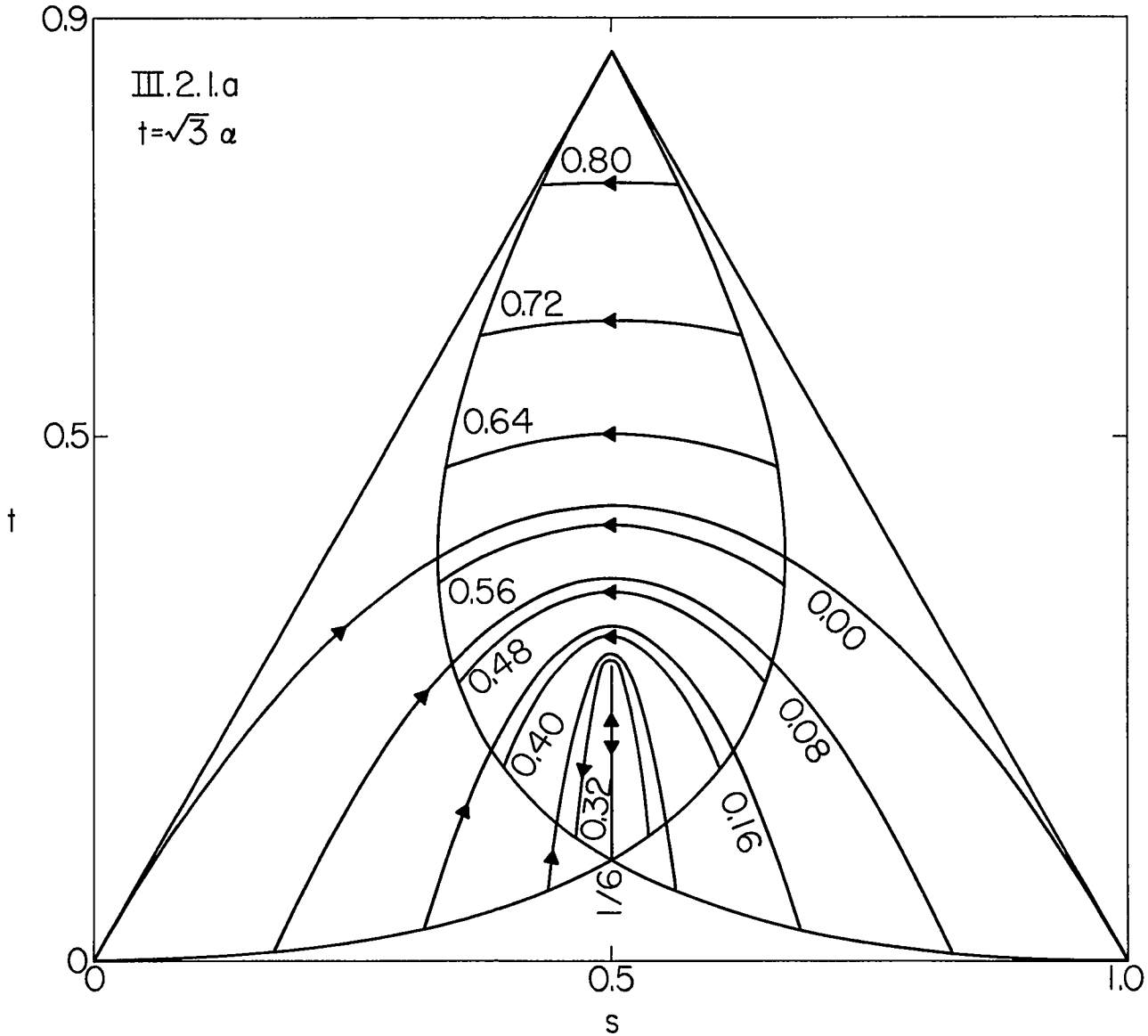

III.2.1.a
t=√3 α
0.9
0.5
0
t
0
0.5
1.0
s
0.80
0.72
0.64
0.56
0.48
0.40
0.32
1/6
0.16
0.08
0.00

Fig. 6

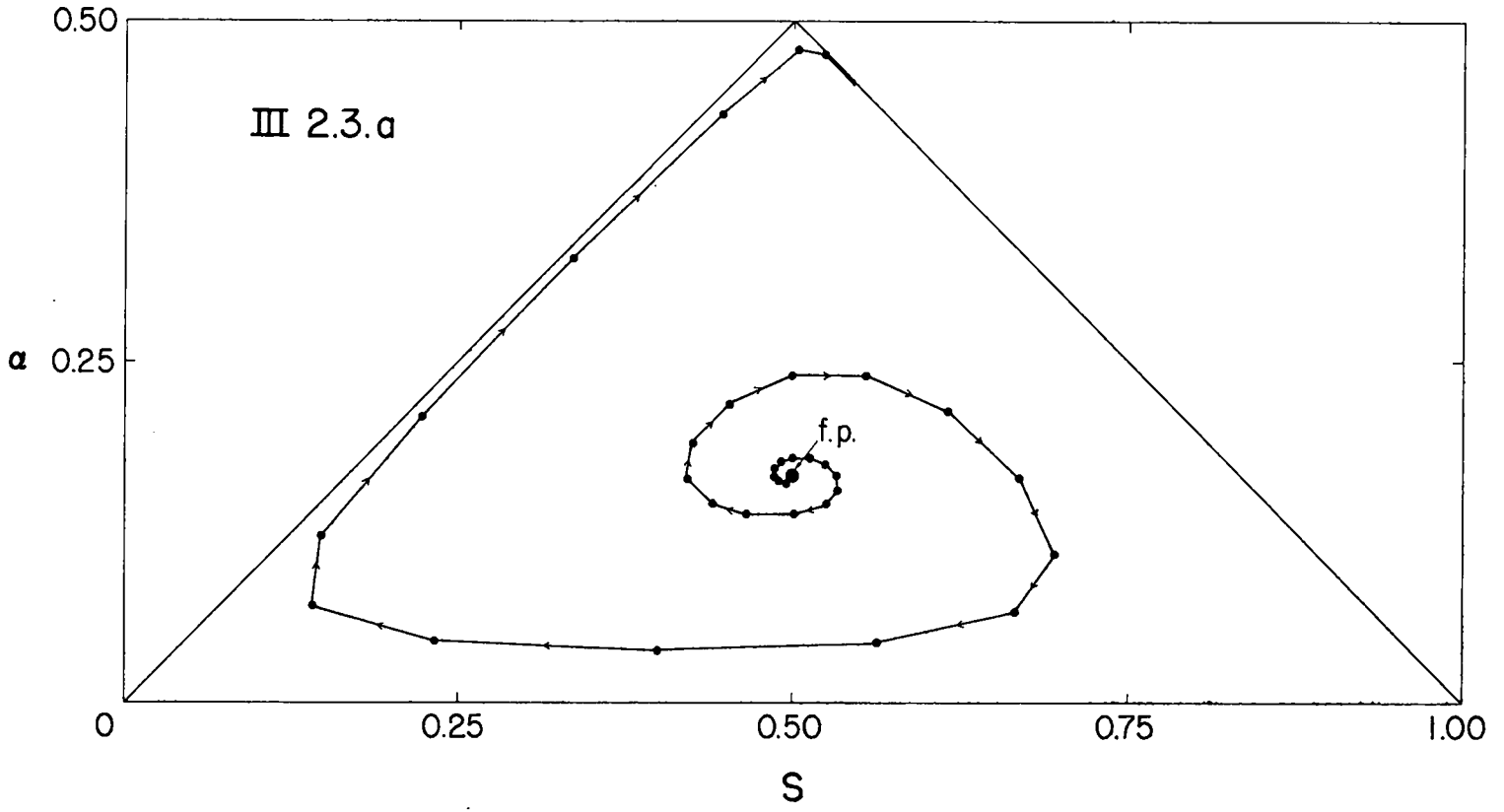


Fig. 7

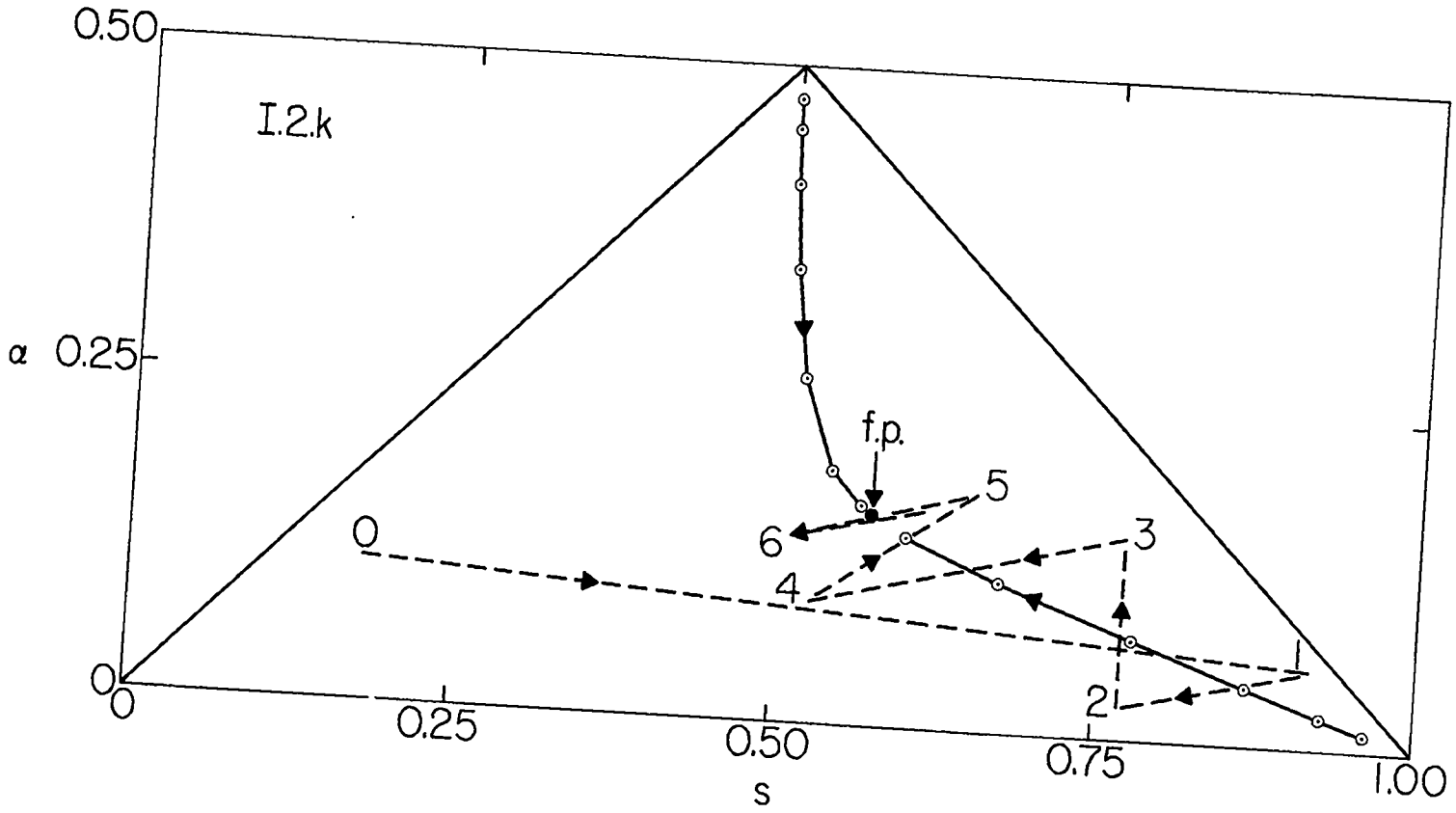


Fig. 8

TABLE I

SUMMARY OF CONVERGENCE BEHAVIOR OF THREE-VARIABLE BINARY REACTION SYSTEMS

1 n.f.p.	2 b.f.p.	3 i.f.p.	4 b.d.p.	5 b.t.p.	6 i.d.p.	Mixed Behavior
I.2.b	I.1.a	I.1.c	I.2.e	I.5.j	I.5.l	I.2.m(3,4)
I.2.c	I.1.b	I.2.a	I.2.n	I.5.r	I.5.p	I.2.p(3,4)
I.2.h	I.2.d	I.2.j	I.2.q	III.2.3.e	II.1.f	
I.2.i	I.2.f	I.2.k	I.2.r	III.2.3.f	III.1.f	I.f.h(3,5)
I.2.l	I.2.g	I.2.o	I.3.i			I.5.q(1,4)
I.3.a	I.3.b	I.3.e	I.4.d			III.2.2.a
I.3.c	I.4.g	I.3.f	I.4.e			(2 different values of l)
I.3.d	I.4.h	I.3.h				
I.4.a	I.6.a	I.4.b	I.5.g			
I.4.c	II.1.a	I.4.f	I.5.i			III.2.3.b(4,1)
I.5.a	II.2.e	I.4.i	I.5.k			
I.5.b		I.5.f	I.5.m			
I.5.c	III.1.a	I.5.n	II.2.g			
I.5.d		I.5.o	II.2.h			
I.5.e		I.6.b	II.3.c			
II.1.b		I.6.c	III.2.2.b			
II.2.a		II.1.d				
II.2.c		II.2.b				
II.2.d		II.2.i				
II.2.f		II.3.d				
II.3.a		III.1.c				
II.3.b		III.1.d				
III.1.b		III.1.e				
		III.1.g				
		III.1.h				
		III.1.i				
		III.2.1.a				
		III.2.1.b				
		III.2.1.e				
		III.2.2.c				
		III.2.2.e				

Note: In addition, we have the following:

I.3.g i.t.p.

III.2.3.a non-convergent

INTRODUCTION TO TABLE II

This table summarizes the properties of all 97 Binary Reaction Systems in 3 variables. The notation is explained in Section IV of the text. For a few systems, the behavior of arbitrary vectors under iteration can be predicted theoretically owing to the fact that the system reduces to a difference equation in a single variable (e.g., I.1.b, I.2.q, etc.).[10] We have not thought it worthwhile to note these instances explicitly; anyone using the table will immediately discover them for himself.

For the 3-variable case, the coordinates of all fixed points could be written explicitly in terms of radicals, since the f.p. equation is at most of 4th degree. There are many interesting relationships between the roots of the various f.p. equations. We have not investigated these relationships systematically, although it could easily be done using standard tools (cf. L. E. Dickson[11]).

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[10] In some other cases, one may observe that one of the variables will obviously iterate to zero, in which case the limiting behavior is also evident.

[11] L. E. Dickson, Modern Algebraic Theories, Chapter II, Sanborn, Chicago, 1930.

TABLE II

THREE-VARIABLE BINARY REACTION SYSTEMS

I.1.a

(11) + (22) + (33)	n.f.p.
2(12) + 2(13)	b.f.p.
2(23)	degenerate

no i.f.p.

m → b.f.p.

I.1.b

(11) + (22) + (33)	n.f.p.
2(12) + 2(23)	2 b.f.p.'s
2(13)	doubly degenerate

no i.f.p.

m → b.f.p. ($x_3 = 0$)

I.1.c

(11) + (22) + (33)	n.f.p.
2(13) + 2(23)	b.d.p.
2(12)	not degenerate

i.f.p. given by ($x = x_3$)

$8x^3 - 20x^2 + 16x - 3 = 0$

or ($y = 2x_1$)

$y^3 + y - 1 = 0$

$$\left.\begin{aligned} x_1 &= .34116390 \\ x_2 &= .39162172 \\ x_3 &= .26721438 \end{aligned}\right\} \text{ i.f.p.}$$

m → i.f.p.

TABLE II (cont.)

I.2.a

$$\left.\begin{array}{l}(11) + (22) + 2(12)\\ (33) + 2(23)\\ 2(13)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\ \text{degenerate}\end{array}$$

i.f.p. given by $(x = x_2)$

$$x^2 + x - 1/4 = 0$$

$m \rightarrow$ i.f.p.

$$\left.\begin{array}{l}x_1 = 1/2\\ x_2 = \dfrac{\sqrt{2} - 1}{2} = .20710678\\ x_3 = 1 - \dfrac{\sqrt{2}}{2} = .29289322\end{array}\right\}\begin{array}{l}\text{i.f.p.}\\ \text{(cf. I.4.e)}\\ \text{(cf. III.1.g)}\end{array}$$

I.2.b

$$\left.\begin{array}{l}(11) + (22) + 2(12)\\ (33) + 2(13)\\ 2(23)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\ \text{degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ n.f.p.

I.2.c

$$\left.\begin{array}{l}(11) + (22) + 2(13)\\ (33) + 2(23)\\ 2(12)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\ \text{not degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ n.f.p.

TABLE II (cont.)

I.2.d

(11) + (22) + 2(13)	n.f.p.
(33) + 2(12)	b.f.p.
2(23)	degenerate

no i.f.p.

m → b.f.p.

I.2.e

(11) + (22) + 2(23)	n.f.p.
(33) + 2(13)	b.d.p. (see below)
2(12)	not degenerate

no i.f.p.

one coordinate of b.d.p ($x = x_1$) given by

$4x^3 - 4x^2 + 4x - 1 = 0$

$x = .31944847$

m → b.d.p.

$$\left.\begin{array}{ll} x_1 = .56519772, & x_1 = .31944846 \\ x_2 = 0, & x_2 = .68055154 \\ x_3 = .43480228, & x_3 = 0 \end{array}\right\} \text{b.d.p.}$$

I.2.f

(11) + (22) + 2(23)	n.f.p.
(33) + 2(12)	b.f.p.
2(13)	degenerate

no i.f.p.

m → b.f.p.

TABLE II (cont.)

I.2.g

$$\left.\begin{array}{l}(11) + (33) + 2(12)\\(22) + 2(23)\\2(13)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\\text{b.f.p.}\\\text{doubly degenerate}\end{array}$$

no i.f.p.

m → b.f.p.

I.2.h

$$\left.\begin{array}{l}(11) + (33) + 2(12)\\(22) + 2(13)\\2(23)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\\text{degenerate}\end{array}$$

no i.f.p.

m → n.f.p. ($x_1 = 1$)

I.2.i

$$\left.\begin{array}{l}(11) + (33) + 2(13)\\(22) + 2(23)\\2(12)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\\text{degenerate}\end{array}\qquad\left.\begin{array}{l}x_1 = 1/4\\x_2 = 1/2\\x_3 = 1/4\end{array}\right\}\text{i.f.p.}$$

m → n.f.p. ($x_1 = 1$)

I.2.j

$$\left.\begin{array}{l}(11) + (33) + 2(13)\\(22) + 2(12)\\2(23)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\\text{degenerate}\end{array}\qquad\left.\begin{array}{l}x_1 = 1/4\\x_2 = 1/2\\x_3 = 1/4\end{array}\right\}\text{i.f.p.}$$

m → i.f.p. (See Section IX.1.of text for discussion)

TABLE II (cont.)

I.2.k

$$\left.\begin{array}{l}(11) + (33) + 2(23)\\ (22) + 2(13)\\ 2(12)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\ \text{not degenerate}\end{array}$$

i.f.p. given by $(x = x_1)$

$8x^2 + 4x - 3 = 0$

$m \rightarrow$ i.f.p.

$$\left.\begin{array}{l} x_1 = \frac{\sqrt{7} - 1}{4} = .41143783\\ x_2 = .32287566\\ x_3 = .26568651\end{array}\right\}\text{i.f.p.}$$

I.2.l

$$\left.\begin{array}{l}(11) + (33) + 2(23)\\ (22) + 2(12)\\ 2(13)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\ \text{b.f.p.}\\ \text{doubly degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ n.f.p. $(x_2 = 1)$

I.2.m

$$\left.\begin{array}{l}(22) + (33) + 2(12)\\ (11) + 2(23)\\ 2(13)\end{array}\right\}\begin{array}{l}\text{b.d.p.}\\ \text{degenerate}\end{array}\qquad \left.\begin{array}{l} x_1 = 1/2\\ x_2 = \frac{1}{\sqrt{8}} = .35355339\\ x_3 = \frac{2-\sqrt{2}}{4} = .14644661\end{array}\right\}\begin{array}{l}\text{i.f.p.}\\ \text{(cf. I.4.b)}\\ \text{(cf. III.1.d)}\end{array}$$

$m \rightarrow$ i.f.p. with initial vectors:

$$\begin{array}{lcl} x_1^{(o)} = 1/3 & & x_1^{(o)} = .10066270\\ x_2^{(o)} = 1/3 & \text{and} & x_2^{(o)} = .17982822\\ x_3^{(o)} = 1/3 & & x_3^{(o)} = .71950908\end{array}$$

TABLE II (cont.)

I.2.m (cont.)

m → b.d.p. with initial vector: (See Section XI for discussion)

$x_1^{(o)} = .35613727$

$x_2^{(o)} = .62067802$

$x_3^{(o)} = .02318471$

I.2.n

$$\left.\begin{array}{l}(22) + (33) + 2(12)\\ (11) + 2(13)\\ 2(23)\end{array}\right\} \begin{array}{l}\text{b.d.p.}\\ \text{degenerate}\end{array}$$

The only f.p. is the b.f.p., which is the i.f.p. of the 2-variable system:

$$\text{System S:} \quad \left\{\begin{array}{l}(22) + 2(12)\\ (11)\end{array}\right., \text{ i.e., } \left.\begin{array}{l} x_1 = \frac{\sqrt{5} - 1}{2} = .61803399\\ x_2 = \frac{3 - \sqrt{5}}{2} = .38196601\\ (x_3 = 0)\end{array}\right\}$$

This f.p. is easily shown not to be attractive. And, experimentally: m → b.d.p.

TABLE II (cont.)

I.2.o

$$\left.\begin{array}{l}(22) + (33) + 2(13)\\ (11) + 2(23)\\ 2(12)\end{array}\right\}\ \begin{array}{l}\text{b.d.p.}\\ \text{not degenerate}\end{array} \qquad \left.\begin{array}{l}x_1 = .39624265\\ x_2 = .33682695\\ x_3 = .26693040\end{array}\right\}\ \text{i.f.p.}$$

i.f.p. given by ($x = x_1$):

$$4x^4 + 8x^3 - 5x^2 + 3x - 1 = 0$$

m → i.f.p.

I.2.p

$$\left.\begin{array}{l}(22) + (33) + 2(13)\\ (11) + 2(12)\\ 2(23)\end{array}\right\}\ \begin{array}{l}\text{b.f.p. (cf. I.2.n)}\\ \text{b.d.p.}\\ \text{degenerate}\end{array}$$

i.f.p. given by ($x = x_1$):

$$x^2 + x - 1/2 = 0$$

$$\left.\begin{array}{l}x_1 = \frac{\sqrt{3} - 1}{2} = .36602540\\ x_2 = 1/2\\ x_3 = 1 - \frac{\sqrt{3}}{2} = .13397460\end{array}\right\}\ \begin{array}{l}\text{i.f.p.}\\ \text{(cf. I.3.h)}\\ \text{(cf. II.2.b)}\end{array}$$

m → i.f.p. with initial vectors:

$$\begin{array}{lcl} x_1^{(o)} = 1/3 & & x_1^{(o)} = \frac{5}{16} & & x_1^{(o)} = .35613727\\ x_2^{(o)} = 1/3 & \text{and} & x_2^{(o)} = \frac{9}{16} & \text{and} & x_2^{(o)} = .62067802\\ x_3^{(o)} = 1/3 & & x_3^{(o)} = \frac{2}{16} & & x_3^{(o)} = .02318471 \end{array}$$

m → b.d.p. with initial vector:

$$x_1^{(o)} = .04815165$$
$$x_2^{(o)} = .21990982$$
$$x_3^{(o)} = .73193853$$

TABLE II (cont.)

I.2.q

$$\left.\begin{array}{l}(22) + (33) + 2(23)\\(11) + 2(13)\\2(12)\end{array}\right\}\begin{array}{l}\text{b.d.p.}\\\text{not degenerate}\end{array}$$

i.f.p. given by $(x = x_1)$

$x^2 - 3x + 1 = 0$

$m \rightarrow$ b.d.p.

$$\left.\begin{array}{l}x_1 = \frac{3-\sqrt{5}}{2} = .38196601\\x_2 = \frac{\sqrt{5} - 1}{8-2\sqrt{5}} = .35037291\\x_3 = .26766108\end{array}\right\}\text{i.f.p.}$$

I.2.r

$$\left.\begin{array}{l}(22) + (33) + 2(23)\\(11) + 2(12)\\2(13)\end{array}\right\}\begin{array}{l}\text{b.f.p. (cf. I.2.n)}\\\text{b.d.p.}\\\text{degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ b.d.p.

I.3.a

$$\left.\begin{array}{l}(11) + (22) + 2(12)\\2(13) + 2(23)\\(33)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\\text{degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ n.f.p. $(x_1 = 1)$

TABLE II (cont.)

I.3.b

(11) + (22) + 2(13) 2(12) + 2(23) (33)	2 n.f.p.'s b.f.p. doubly degenerate

no i.f.p.

m → b.f.p.

I.3.c

(11) + (22) + 2(23) 2(12) + 2(13) (33)	2 n.f.p.'s b.f.p. degenerate

no i.f.p.

m → n.f.p. ($x_1 = 1$)

I.3.d

(11) + (33) + 2(12) 2(13) + 2(23) (22)	n.f.p. not degenerate

no i.f.p.

m → n.f.p.

TABLE II (cont.)

I.3.e

$$\left.\begin{array}{l}(11) + (33) + 2(13)\\ 2(12) + 2(23)\\ (22)\end{array}\right\}\ \begin{array}{l}\text{n.f.p.}\\ \text{degenerate}\end{array}\qquad \left.\begin{array}{l}x_1 = 1/4\\ x_2 = 1/2\\ x_3 = 1/4\end{array}\right\}\ \text{i.f.p.}$$

i.f.p. exists

m $\rightarrow$ i.f.p.

I.3.f

$$\left.\begin{array}{l}(11) + (33) + 2(23)\\ 2(12) + 2(13)\\ (22)\end{array}\right\}\ \begin{array}{l}\text{n.f.p.}\\ \text{not degenerate}\end{array}$$

i.f.p. given by ($x = x_2$):

$$2x^3 + 4x^2 - 1 = 0$$

m $\rightarrow$ i.f.p.

$$\left.\begin{array}{l}x_1 = .34444609\\ x_2 = .45160596\\ x_3 = .20394795\end{array}\right\}\ \text{i.f.p.}$$

I.3.g

$$\left.\begin{array}{l}(22) + (33) + 2(12)\\ 2(13) + 2(23)\\ (11)\end{array}\right\}\ \begin{array}{l}\text{b.d.p.}\\ \text{not degenerate}\end{array}\qquad \left.\begin{array}{l}x_1 = .45823825\\ x_2 = .33177946\\ x_3 = .20998229\end{array}\right\}\ \text{i.f.p.}$$

i.f.p. given by ($x = x_1$):

$$2x^4 - 3x^2 - x + 1 = 0$$

TABLE II (cont.)

I.3.g (cont.)

m → interior triple point:

$$\left.\begin{array}{lll} x_1^{(n+3)} = x_1^{(n)} = .73924369, & x_1^{(n+1)} = .08071790, & x_1^{(n+2)} = .49780563 \\ x_2^{(n+3)} = x_2^{(n)} = .01294586, & x_2^{(n+1)} = .37280086, & x_2^{(n+2)} = .49567899 \\ x_3^{(n+3)} = x_3^{(n)} = .24781045, & x_3^{(n+1)} = .54648124, & x_3^{(n+2)} = .00651538 \end{array}\right\} \text{i.t.p.}$$

I.3.h

$$\left.\begin{array}{l} (22) + (33) + 2(13) \\ 2(12) + 2(23) \\ (11) \end{array}\right\} \begin{array}{l} \text{b.f.p. (cf. I.2.n)} \\ \text{b.d.p.} \\ \text{degenerate} \end{array}$$

i.f.p. given by ($x = x_1$):

$x^2 + x - 1/2 = 0$

m → i.f.p.

$$\left.\begin{array}{l} x_1 = \frac{\sqrt{3} - 1}{2} = .36602540 \\ x_2 = 1/2 \\ x_3 = 1 - \frac{\sqrt{3}}{2} = .13397460 \end{array}\right\} \begin{array}{l} \text{i.f.p.} \\ \text{(cf. I.2.p)} \end{array}$$

I.3.i

$$\left.\begin{array}{l} (22) + (33) + 2(23) \\ 2(12) + 2(13) \\ (11) \end{array}\right\} \begin{array}{l} \text{b.d.p.} \\ \text{not degenerate} \end{array}$$

i.f.p. given by ($x = x_1$):

$x^2 - 3x + 1 = 0$

m → b.d.p.

$$\left.\begin{array}{l} x_1 = \frac{3-\sqrt{5}}{2} = .38196601 \\ x_2 = 2\sqrt{5} - 4 = .47213596 \\ x_3 = \frac{7-3\sqrt{5}}{2} = .14589803 \end{array}\right\} \begin{array}{l} \text{i.f.p.} \\ \text{(cf. I.5.f)} \end{array}$$

TABLE II (cont.)

I.4.a

$$\left.\begin{array}{l}(11) + 2(12) + 2(13)\\ (22) + (33)\\ 2(23)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\ \text{b.f.p.}\\ \text{doubly degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ n.f.p. ($x_1 = 1$)

I.4.b

$$\left.\begin{array}{l}(11) + 2(12) + 2(23)\\ (22) + (33)\\ 2(13)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\ \text{degenerate}\end{array}$$

i.f.p. given by ($x = x_2$):

$2x^2 - 2x + 1/4 = 0$

$m \rightarrow$ i.f.p.

$$\left.\begin{array}{l}x_1 = 1/2\\ x_2 = \frac{2-\sqrt{2}}{4} = .14644661\\ x_3 = \frac{\sqrt{2}}{4} = .35355339\end{array}\right\}\begin{array}{l}\text{i.f.p.}\\ \text{(cf. I.2.m)}\end{array}$$

I.4.c

$$\left.\begin{array}{l}(11) + 2(13) + 2(23)\\ (22) + (33)\\ 2(12)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\ \text{not degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ n.f.p. ($x_1 = 1$)

TABLE II (cont.)

I.4.d

$$\left.\begin{array}{l}(22) + 2(12) + 2(13)\\ (11) + (33)\\ 2(23)\end{array}\right\}\begin{array}{l}\text{b.f.p. (cf. I.2.n)}\\ \text{b.d.p.}\\ \text{degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ b.d.p.

I.4.e

$$\left.\begin{array}{l}(22) + 2(12) + 2(23)\\ (11) + (33)\\ 2(13)\end{array}\right\}\begin{array}{l}\text{b.f.p. (cf. I.2.n)}\\ \text{b.d.p.}\\ \text{degenerate}\end{array}$$

i.f.p. given by $(x = x_2)$:

$x^2 - 2x + 1/2 = 0$

$m \rightarrow$ b.d.p.

$$\left.\begin{array}{l}x_1 = 1/2\\ x_2 = 1 - \frac{\sqrt{2}}{2} = .29289322\\ x_3 = \frac{\sqrt{2} - 1}{2} = .20710678\end{array}\right\}\begin{array}{l}\text{i.f.p.}\\ \text{(cf. I.2.a)}\end{array}$$

I.4.f

$$\left.\begin{array}{l}(22) + 2(13) + 2(23)\\ (11) + (33)\\ 2(12)\end{array}\right\}\begin{array}{l}\text{b.d.p.}\\ \text{not degenerate}\end{array}$$

i.f.p. given by $(x = x_2)$

$4x^4 - 12x^3 + x^2 - 3x + 1 = 0$

$m \rightarrow$ i.f.p.

$$\left.\begin{array}{l}x_1 = .46164837\\ x_2 = .27991085\\ x_3 = .25844078\end{array}\right\}\text{i.f.p.}$$

TABLE II (cont.)

I.4.g

$$\left.\begin{array}{l}(33) + 2(12) + 2(13)\\(11) + (22)\\2(23)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\\text{b.f.p.}\\\text{degenerate}\end{array}$$

no i.f.p.

m → b.f.p.

I.4.h

$$\left.\begin{array}{l}(33) + 2(12) + 2(23)\\(11) + (22)\\2(13)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\\text{b.f.p.}\\\text{degenerate}\end{array}$$

no i.f.p.

m → b.f.p.

I.4.i

$$\left.\begin{array}{l}(33) + 2(13) + 2(23)\\(11) + (22)\\2(12)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\\text{not degenerate}\end{array}$$

i.f.p. given by ($x = x_3$)

$$2x^3 - 10x^2 + 14x - 3 = 0$$

m → i.f.p.

$$\left.\begin{array}{l}x_1 = .45253634\\x_2 = .28737153\\x_3 = .26009213\end{array}\right\}\text{i.f.p.}$$

TABLE II (cont.)

I.5.a

(11) + 2(12) + 2(13)	3 n.f.p.'s
(22) + 2(23)	triply degenerate
(33)	

no i.f.p.

m → n.f.p. ($x_1 = 1$)

I.5.b

(11) + 2(12) + 2(13)	n.f.p.
(33) + 2(23)	b.f.p. (cf. I.2.n)
(22)	b.d.p.
	degenerate

no i.f.p.

m → n.f.p.

I.5.c

(11) + 2(12) + 2(23)	3 n.f.p.'s
(22) + 2(13)	degenerate
(33)	

no i.f.p.

m → n.f.p. ($x_1 = 1$)

TABLE II (cont.)

I.5.d

$$\left.\begin{array}{l}(11) + 2(12) + 2(23)\\ (33) + 2(13)\\ (22)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\ \text{b.d.p. (see below)}\\ \text{not degenerate}\end{array}$$

no i.f.p. In addition to the "trivial" b.d.p. (0,1,0), (0,0,1), there is a b.d.p. which is a f.p. of the transformation:

$$x_3'' = x_3^2(2-x_3)^2, \text{ viz.:}$$

$$\left.\begin{array}{ll} x_1 = \frac{\sqrt{5}-1}{2}, & x_1 = \frac{3-\sqrt{5}}{2} \\ x_2 = 0 \quad , & x_2 = \frac{\sqrt{5}-1}{2} \\ x_3 = \frac{3-\sqrt{5}}{2}, & x_3 = 0 \end{array}\right\} \text{b.d.p.}$$

This is not an attractive f.p. If $x_2^{(o)} = 0$, $x_3^{(o)} < \frac{3-\sqrt{5}}{2}$, the initial vector $\rightarrow x_1 = 1$. If $x_2^{(o)} = 0$, $x_3^{(o)} > \frac{3-\sqrt{5}}{2}$, the initial vector $\rightarrow$ the trivial b.d.p. This behavior has been verified numerically.

I.5.e

$$\left.\begin{array}{l}(11) + 2(13) + 2(23)\\ (22) + 2(12)\\ (33)\end{array}\right\}\begin{array}{l}\text{3 n.f.p.'s}\\ \text{doubly degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ n.f.p. ($x_2 = 1$)

TABLE II (cont.)

I.5.f

$$\left.\begin{array}{l}(11) + 2(13) + 2(23)\\ (33) + 2(12)\\ (22)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\ \text{b.d.p.}\\ \text{not degenerate}\end{array}$$

i.f.p. given by $(x = x_2)$:

$x^2 - 3x + 1 = 0$

$m \rightarrow$ i.f.p.

$$\left.\begin{array}{l} x_1 = 2\sqrt{5} - 4 = .47213596\\ x_2 = \dfrac{3-\sqrt{5}}{2} = .38196601\\ x_3 = \dfrac{7-3\sqrt{5}}{2} = .14589803\end{array}\right\}\begin{array}{l}\text{i.f.p.}\\ \text{(cf. I.3.i)}\end{array}$$

I.5.g

$$\left.\begin{array}{l}(22) + 2(12) + 2(13)\\ (11) + 2(23)\\ (33)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\ \text{b.f.p. (cf. I.2.n)}\\ \text{b.d.p.}\\ \text{degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ b.d.p.

I.5.h

$$\left.\begin{array}{l}(22) + 2(12) + 2(13)\\ (33) + 2(23)\\ (11)\end{array}\right\}\begin{array}{l}\text{b.t.p.}\\ \text{not degenerate}\end{array}$$

i.f.p. given by $(x = x_1)$:

$x^3 + 3x^2 - 1 = 0$

$$\left.\begin{array}{l} x_1 = .53208889\\ x_2 = .18479253\\ x_3 = .28311858\end{array}\right\}\text{i.f.p.}$$

TABLE II (cont.)

I.5.h (cont.)

$m \rightarrow$ i.f.p. with initial vectors:

$$x_1^{(o)} = 1/3 \qquad x_1^{(o)} = .08662751$$
$$x_2^{(o)} = 1/3 \quad \text{and} \quad x_2^{(o)} = .32501609$$
$$x_3^{(o)} = 1/3 \qquad x_3^{(o)} = .58835640$$

$m \rightarrow$ b.t.p. with initial vector:

$$x_1^{(o)} = .02476836$$
$$x_2^{(o)} = .01004474$$
$$x_3^{(o)} = .96518690$$

(See Section XI for discussion.)

I.5.i

$$\left.\begin{array}{l} (22) + 2(12) + 2(23) \\ (11) + 2(13) \\ (33) \end{array}\right\} \begin{array}{l} \text{n.f.p.} \\ \text{b.f.p. (cf. I.2.n)} \\ \text{b.d.p.} \\ \text{degenerate} \end{array}$$

no i.f.p.

$m \rightarrow$ b.d.p.

I.5.j

$$\left.\begin{array}{l} (22) + 2(12) + 2(23) \\ (33) + 2(13) \\ (22) \end{array}\right\} \begin{array}{l} \text{b.t.p.} \\ \text{not degenerate} \end{array}$$

i.f.p. given by $(x = x_1)$:

$$x^4 + 2x^3 + x^2 + x - 1 = 0$$

$m \rightarrow$ b.t.p.

$$\left.\begin{array}{l} x_1 = .48402830 \\ x_2 = .28168830 \\ x_3 = .23428340 \end{array}\right\} \text{i.f.p.}$$

TABLE II (cont.)

I.5.k

(22) + 2(13) + 2(23)	n.f.p.
(11) + 2(12)	b.f.p. (cf. I.2.n)
(33)	b.d.p.
	degenerate

no i.f.p.

m → b.d.p.

I.5.l

(22) + 2(13) + 2(23)	b.t.p.	$x_1 = .44504187$	i.f.p. (cf. I.5.n for value of x_2) (cf. I.5.r)
(33) + 2(12)	not degenerate	$x_2 = .35689587$	
(11)		$x_3 = .19806226$	

i.f.p. given by ($x = x_1$):

$$x^3 - x^2 - 2x + 1 = 0$$

m → i.d.p.:

$$\left.\begin{aligned} x_1^{(n+2)} &= x_1^{(n)} = .67046846, & x_1^{(n+1)} &= .13146927 \\ x_2^{(n+2)} &= x_2^{(n)} = .31224737, & x_2^{(n+1)} &= .41900277 \\ x_3^{(n+2)} &= x_3^{(n)} = .01728417, & x_3^{(n+1)} &= .44952796 \end{aligned}\right\} \text{ i.d.p.}$$

I.5.m

(33) + 2(12) + 2(13)	n.f.p.
(22) + 2(23)	b.d.p.
(11)	b.f.p. (cf. I.2.n)
	degenerate

no i.f.p.

m → b.d.p.

TABLE II (cont.)

I.5.n

$$\left.\begin{array}{l}(33) + 2(12) + 2(13)\\ (11) + 2(23)\\ (22)\end{array}\right\}\begin{array}{l}\text{b.t.p.}\\ \text{not degenerate}\end{array}$$

i.f.p. given by $(x = x_2)$

$x^3 + 3x^2 - 4x + 1 = 0$

$m \rightarrow$ i.f.p.

$$\left.\begin{array}{l}x_1 = .51572947\\ x_2 = .35689587\\ x_3 = .12737466\end{array}\right\}\begin{array}{l}\text{i.f.p.}\\ \text{(cf. I.5.1 for}\\ \text{value of } x_2)\end{array}$$

I.5.o

$$\left.\begin{array}{l}(33) + 2(12) + 2(23)\\ (22) + 2(13)\\ (11)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\ \text{b.d.p.}\\ \text{not degenerate}\end{array}$$

i.f.p. given by $(x = x_1)$

$x^3 + 4x^2 - 1 = 0$

$m \rightarrow$ i.f.p.

$$\left.\begin{array}{l}x_1 = .47283391\\ x_2 = .30359418\\ x_3 = .22357191\end{array}\right\}\text{i.f.p.}$$

I.5.p

$$\left.\begin{array}{l}(33) + 2(12) + 2(23)\\ (11) + 2(13)\\ (22)\end{array}\right\}\begin{array}{l}\text{b.t.p.}\\ \text{not degenerate}\end{array}$$

i.f.p. given by $(x = x_2)$

$x^4 - 2x^3 + x^2 - 3x + 1 = 0$

$$\left.\begin{array}{l}x_1 = .52645706\\ x_2 = .35061327\\ x_3 = .12292967\end{array}\right\}\text{i.f.p.}$$

TABLE II (cont.)

I.5.p (cont.)

$m \rightarrow$ i.d.p.:

$$\left.\begin{array}{l} x_1^{(n+2)} = x_1^{(n)} = .41717428, \; x_1^{(n+1)} = .49907610 \\ x_2^{(n+2)} = x_2^{(n)} = .53818709, \; x_2^{(n+1)} = .21127856 \\ x_3^{(n+2)} = x_3^{(n)} = .04463863, \; x_3^{(n+1)} = .28964534 \end{array}\right\} \text{ i.d.p.}$$

I.5.q

$$\left.\begin{array}{l} (33) + 2(13) + 2(23) \\ (22) + 2(12) \\ (11) \end{array}\right\} \begin{array}{l} \text{n.f.p.} \\ \text{b.f.p. (cf. I.2.n)} \\ \text{b.d.p.} \\ \text{degenerate} \end{array}$$

no i.f.p.

$m \rightarrow$ n.f.p. with initial vectors:

$$\begin{array}{lcl} x_1^{(o)} = 1/3 & & x_1^{(o)} = .28491108 \\ x_2^{(o)} = 1/3 & \text{and} & x_2^{(o)} = .47627166 \\ x_3^{(o)} = 1/3 & & x_3^{(o)} = .23881726 \end{array}$$

$m \rightarrow$ b.d.p. with initial vector:

$$\begin{array}{l} x_1^{(o)} = .43634693 \\ x_2^{(o)} = .16692265 \\ x_3^{(o)} = .39673042 \end{array}$$

I.5.r

$$\left.\begin{array}{l} (33) + 2(13) + 2(23) \\ (11) + 2(12) \\ (22) \end{array}\right\} \begin{array}{l} \text{b.t.p.} \\ \text{not degenerate} \end{array}$$

TABLE II (cont.)

I.5.r (cont.)

i.f.p. given by $(x = x_2)$

$x^3 - x^2 - 2x + 1 = 0$

$m \rightarrow$ b.t.p.

$$\left.\begin{aligned} x_1 &= .35689587 \\ x_2 &= .44504187 \\ x_3 &= .19806226 \end{aligned}\right\} \text{i.f.p. (cf. I.5.1)}$$

I.6.a

$$\left.\begin{aligned} &2(12) + 2(13) + 2(23) \\ &(11) + (22) \\ &(33) \end{aligned}\right\} \begin{aligned} &\text{2 n.f.p.'s} \\ &\text{b.f.p.} \\ &\text{degenerate} \end{aligned}$$

no i.f.p.

$m \rightarrow$ b.f.p.

I.6.b

$$\left.\begin{aligned} &2(12) + 2(13) + 2(23) \\ &(11) + (33) \\ &(22) \end{aligned}\right\} \begin{aligned} &\text{b.d.p.} \\ &\text{not degenerate} \end{aligned}$$

i.f.p. given by $(x = x_2)$

$2x^4 + 2x^3 - x^2 - 3x + 1 = 0$

$m \rightarrow$ i.f.p.

$$\left.\begin{aligned} x_1 &= .56311573 \\ x_2 &= .32878482 \\ x_3 &= .10809945 \end{aligned}\right\} \text{i.f.p.}$$

I.6.c

$$\left.\begin{aligned} &2(12) + 2(13) + 2(23) \\ &(22) + (33) \\ &(11) \end{aligned}\right\} \begin{aligned} &\text{n.f.p.} \\ &\text{not degenerate} \end{aligned}$$

TABLE II (cont.)

I.6.c (cont.)

i.f.p. given by ($x = x_1$)

$2x^3 + 2x^2 - 1 = 0$

m → i.f.p.

$$\left.\begin{array}{l} x_1 = .56519772 \\ x_2 = .11535382 \\ x_3 = .31944846 \end{array}\right\} \text{i.f.p.}$$

II.1.a

$$\left.\begin{array}{l} (11) + (22) + (33) + 2(12) \\ 2(23) \\ 2(13) \end{array}\right\} \begin{array}{l} \text{n.f.p.} \\ \text{b.f.p.} \\ \text{doubly degenerate} \end{array}$$

no i.f.p.

m → b.f.p.

II.1.b

$$\left.\begin{array}{l} (11) + (22) + (33) + 2(13) \\ 2(23) \\ 2(12) \end{array}\right\} \begin{array}{l} \text{n.f.p.} \\ \text{degenerate} \end{array}$$

no i.f.p.

m → n.f.p.

II.1.d

$$\left.\begin{array}{l} (11) + (22) + (33) + 2(23) \\ 2(12) \\ 2(13) \end{array}\right\} \begin{array}{l} \text{n.f.p.} \\ \text{2 b.f.p.'s} \\ \text{doubly degenerate} \end{array}$$

TABLE II (cont.)

II.1.d (cont.)

There is a continuum of i.f.p.'s, since $\frac{x_2'}{x_3'} = \frac{x_2^{(o)}}{x_3^{(o)}} = \alpha$.

The i.f.p. corresponding to this α is

$$\left.\begin{aligned} x_1 &= 1/2 \\ x_2 &= \frac{\alpha}{2(1+\alpha)} \\ x_3 &= \frac{1}{2(1+\alpha)} \end{aligned}\right\} \text{ i.f.p.}$$

Note that if we write $x_2 + x_3 = y_2$, $x_1 = y_1$, then the system takes the form:

$$\begin{aligned} y_1' &= y_1^2 + y_2^2 \\ y_2' &= 2y_1y_2 \end{aligned}$$

which, for a general point, $\rightarrow$ $y_1 = y_2 = 1/2$

m $\rightarrow$ appropriate i.f.p.

II.1.f

$$\left.\begin{aligned} &(11) + (22) + (33) + 2(23) \\ &2(13) \\ &2(12) \end{aligned}\right\} \begin{aligned} &\text{n.f.p.} \\ &\text{not degenerate} \end{aligned}$$

There is a continuum of i.d.p.'s owing to the fact that $\frac{x_2''}{x_3''} = \frac{x_2}{x_3}$. Thus, if $\frac{x_2^{(o)}}{x_3^{(o)}} = \alpha$, the i.d.p. is given by

TABLE II (cont.)

II.1.f (cont.)

$$\left.\begin{array}{l} x_1^{(n+2)} = x_1^{(n)} = 1/2, \quad x_1^{(n+1)} = 1/2 \\ x_2^{(n+2)} = x_2^{(n)} = \dfrac{1}{2(1+\alpha)}, \quad x_2^{(n+1)} = \dfrac{\alpha}{2(1+\alpha)} \\ x_3^{(n+2)} = x_3^{(n)} = \dfrac{\alpha}{2(1+\alpha)}, \quad x_3^{(n+1)} = \dfrac{1}{2(1+\alpha)} \end{array}\right\} \text{i.d.p. (cf. II.1.d)}$$

This system reduces to the same 2-variable system as does II.1.c.

m → appropriate i.d.p.

II.2.a

$$\left.\begin{array}{l} (11) + (22) + 2(12) + 2(13) \\ (33) \\ 2(23) \end{array}\right\} \text{n.f.p. degenerate}$$

no i.f.p.

m → n.f.p.

II.2.b

$$\left.\begin{array}{l} (11) + (22) + 2(12) + 2(23) \\ (33) \\ 2(13) \end{array}\right\} \text{n.f.p. degenerate}$$

i.f.p. given by $(x = x_3)$

$$x^2 + x - 1/2 = 0$$

m → i.f.p.

$$\left.\begin{array}{l} x_1 = 1/2 \\ x_2 = \dfrac{\sqrt{3} - 1}{2} = .36602540 \\ x_3 = 1 - \dfrac{\sqrt{3}}{2} = .13397460 \end{array}\right\} \text{i.f.p. (cf. I.2.p)}$$

TABLE II (cont.)

II.2.c

$$\left.\begin{array}{l}(11) + (22) + 2(13) + 2(23)\\ (33)\\ 2(12)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\ \text{not degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ n.f.p.

II.2.d

$$\left.\begin{array}{l}(11) + (33) + 2(12) + 2(13)\\ (22)\\ 2(23)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\ \text{doubly degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ n.f.p. ($x_1 = 1$)

II.2.e

$$\left.\begin{array}{l}(11) + (33) + 2(12) + 2(23)\\ (22)\\ 2(13)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\ \text{b.f.p.}\\ \text{doubly degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ b.f.p.

TABLE II (cont.)

II.2.f

$$\left.\begin{array}{l}(11) + (33) + 2(13) + 2(23)\\ (22)\\ 2(12)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\ \text{degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ n.f.p. ($x_1 = 1$)

II.2.g

$$\left.\begin{array}{l}(22) + (33) + 2(12) + 2(13)\\ (11)\\ 2(23)\end{array}\right\}\begin{array}{l}\text{b.f.p. (cf. I.2.n)}\\ \text{b.d.p.}\\ \text{degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ b.d.p.

II.2.h

$$\left.\begin{array}{l}(22) + (33) + 2(12) + 2(23)\\ (11)\\ 2(13)\end{array}\right\}\begin{array}{l}\text{b.f.p. (cf. I.2.n)}\\ \text{b.d.p.}\\ \text{degenerate}\end{array}$$

i.f.p. exists:

$m \rightarrow$ b.d.p.

$$\left.\begin{array}{l}x_1 = 1/2\\ x_2 = 1/4\\ x_3 = 1/4\end{array}\right\}\text{i.f.p.}$$

TABLE II (cont.)

II.2.i

$$\left.\begin{array}{l}(22) + (33) + 2(13) + 2(23)\\ (11)\\ 2(12)\end{array}\right\}\begin{array}{l}\text{b.d.p.}\\ \text{not degenerate}\end{array}$$

i.f.p. exists:

m → i.f.p.

$$\left.\begin{array}{l}x_1 = 1/2\\ x_2 = 1/4\\ x_3 = 1/4\end{array}\right\}\begin{array}{l}\text{i.f.p.}\\ \text{(cf. II.2.h)}\end{array}$$

II.3.a

$$\left.\begin{array}{l}(11) + 2(12) + 2(13) + 2(23)\\ (22)\\ (33)\end{array}\right\}\begin{array}{l}\text{3 n.f.p.'s}\\ \text{doubly degenerate}\end{array}$$

no i.f.p.

m → n.f.p. ($x_1 = 1$)

II.3.b

$$\left.\begin{array}{l}(11) + 2(12) + 2(13) + 2(23)\\ (33)\\ (22)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\ \text{b.d.p.}\\ \text{not degenerate}\end{array}$$

no i.f.p.

m → n.f.p.

TABLE II (cont.)

II.3.c

(22) + 2(12) + 2(13) + 2(23)	n.f.p.
(11)	b.f.p. (cf. I.2.n)
(33)	b.d.p.
	degenerate

no i.f.p.

m → b.d.p.

II.3.d

(22) + 2(12) + 2(13) + 2(23)	b.t.p.
(33)	not degenerate
(11)	

i.f.p. given by ($x = x_1$)

$$x^3 - x^2 + 2x - 1 = 0$$

or (unreduced):

$$x^4 + x^2 + x - 1 = 0 \quad \text{(cf. Eq. 21) of Text)}$$

or ($y = x_3$)

$$y^3 + 3y^2 + 2y - 1 = 0$$

m → i.f.p.

$$\left.\begin{aligned} x_1 &= .56984029 \\ x_2 &= .10544175 \\ x_3 &= .32471796 \end{aligned}\right\} \text{i.f.p.}$$

III.1.a

(11) + (22)	2 n.f.p.'s
2(12) + 2(23)	b.f.p.
2(13) + (33)	doubly degenerate

no i.f.p.

m → b.f.p.

TABLE II (cont.)

III.1.b

$$\left.\begin{array}{l}(11) + (22)\\ 2(12) + 2(13)\\ 2(23) + (33)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\ \text{b.f.p.}\\ \text{degenerate}\end{array}$$

no i.f.p.

$m \rightarrow$ n.f.p. ($x_3 = 1$)

III.1.c

$$\left.\begin{array}{l}(11) + (22)\\ 2(13) + 2(23)\\ 2(12) + (33)\end{array}\right\}\begin{array}{l}\text{2 n.f.p.'s}\\ \text{not degenerate}\end{array}$$

i.f.p. exists:

$m \rightarrow$ i.f.p.

$$\left.\begin{array}{l}x_1 = 1/4\\ x_2 = \frac{\sqrt{3}}{4} = .43301270\\ x_3 = \frac{3-\sqrt{3}}{4} = .31698730\end{array}\right\}\text{i.f.p.}$$

III.1.d

$$\left.\begin{array}{l}(11) + (33)\\ 2(12) + 2(23)\\ 2(13) + (22)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\ \text{b.f.p.}\\ \text{degenerate}\end{array}$$

i.f.p. exists:

$m \rightarrow$ i.f.p.

$$\left.\begin{array}{l}x_1 = \frac{2-\sqrt{2}}{4} = .14644661\\ x_2 = 1/2\\ x_3 = \frac{\sqrt{2}}{4} = .35355339\end{array}\right\}\begin{array}{l}\text{i.f.p.}\\ \text{(cf. I.2.m)}\end{array}$$

TABLE II (cont.)

III.1.e

$$\left.\begin{array}{l}(11) + (33)\\ 2(12) + 2(13)\\ 2(23) + (22)\end{array}\right\} \begin{array}{l}\text{n.f.p.}\\ \text{not degenerate}\end{array}$$

i.f.p. given by $(x = x_1)$

$4x^3 - 8x^2 + 6x - 1 = 0$

$m \rightarrow$ i.f.p.

$$\left.\begin{array}{l}x_1 = .22815553\\ x_2 = .35220117\\ x_3 = .41964330\end{array}\right\} \text{i.f.p.}$$

III.1.f

$$\left.\begin{array}{l}(11) + (33)\\ 2(13) + 2(23)\\ 2(12) + (22)\end{array}\right\} \begin{array}{l}\text{n.f.p.}\\ \text{b.d.p. (see below)}\\ \text{not degenerate}\end{array}$$

i.f.p. given by $(x = x_3)$

$1 - 4(1-x)^3 = 0$

$$\left.\begin{array}{l}x_1 = .16374000\\ x_2 = .46622052\\ x_3 = 1 - \sqrt[3]{\frac{1}{4}} = .37003948\end{array}\right\} \text{i.f.p.}$$

A non-trivial b.d.p. exists, which is a f.p. of the transformation:

$$x_3'' = 4x_3(1-x_3)\left[1 - x_3(1-x_3)\right]$$

This f.p. is given by the root of $(x = x_3)$

$$4x^3 - 8x^2 + 8x - 3 = 0$$

which leads to

TABLE II (cont.)

III.1.f (cont.)

$$x_1^{(n+2)} = x_1^{(n)} = .31944846, \quad x_1^{(n+1)} = .56519772$$
$$x_2^{(n+2)} = x_2^{(n)} = 0 \quad , \quad x_2^{(n+1)} = .43480228$$
$$x_3^{(n+2)} = x_3^{(n)} = .68055154, \quad x_3^{(n+1)} = 0$$

Actually

$m \rightarrow$ i.d.p.:

$$\left.\begin{array}{ll} x_1^{(n+2)} = x_1^{(n)} = .17899745, & x_1^{(n+1)} = .40604579 \\ x_2^{(n+2)} = x_2^{(n)} = .20944248, & x_2^{(n+1)} = .47510870 \\ x_3^{(n+2)} = x_3^{(n)} = .61156007, & x_3^{(n+1)} = .11884551 \end{array}\right\} \text{i.d.p.}$$

III.1.g

$$\left.\begin{array}{l} (22) + (33) \\ 2(12) + 2(23) \\ 2(13) + (11) \end{array}\right\} \begin{array}{l} \text{b.f.p. (cf. I.2.n)} \\ \text{b.d.p.} \\ \text{degenerate} \end{array}$$

i.f.p. exists:

$m \rightarrow$ i.f.p.

$$\left.\begin{array}{l} x_1 = \dfrac{2-\sqrt{2}}{2} = .29289322 \\ x_2 = 1/2 \\ x_3 = \dfrac{\sqrt{2}-1}{2} = .20710678 \end{array}\right\} \begin{array}{l} \text{i.f.p.} \\ \text{(cf.I.2.a)} \end{array}$$

III.1.h

$$\left.\begin{array}{l} (22) + (33) \\ 2(12) + 2(13) \\ 2(23) + (11) \end{array}\right\} \begin{array}{l} \text{b.d.p.} \\ \text{not degenerate} \end{array}$$

i.f.p. given by $(x = x_1)$

TABLE II (cont.)

III.1.h (cont.)

$8x^4 - 20x^3 + 17x^2 - 7x + 1 = 0$

$m \rightarrow$ i.f.p.

$$\left.\begin{aligned} x_1 &= .26882443 \\ x_2 &= .39311571 \\ x_3 &= .33805986 \end{aligned}\right\} \text{i.f.p.}$$

III.1.i

$$\left.\begin{aligned} &(22) + (33) \\ &2(13) + 2(23) \\ &2(12) + (11) \end{aligned}\right\} \begin{aligned} &\text{b.d.p.} \\ &\text{not degenerate} \end{aligned}$$

i.f.p. given by $(x = x_3)$

$4x^4 - 8x^3 + 3x^2 + 3x - 1 = 0$

$m \rightarrow$ i.f.p.

$$\left.\begin{aligned} x_1 &= .27204695 \\ x_2 &= .42351278 \\ x_3 &= .30444027 \end{aligned}\right\} \text{i.f.p.}$$

III.2.1.a

$$\left.\begin{aligned} &(11) + 2(23) \\ &(22) + 2(13) \\ &(33) + 2(12) \end{aligned}\right\} \begin{aligned} &\text{3 n.f.p.'s} \\ &\text{not degenerate} \end{aligned}$$

i.f.p.: $x_1 = x_2 = x_3 = 1/3$

$m \rightarrow$ i.f.p.

TABLE II (cont.)

III.2.1.b

$$\left.\begin{array}{l}(22) + 2(13)\\(11) + 2(23)\\(33) + 2(12)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\\text{b.d.p.}\\\text{not degenerate}\end{array}$$

i.f.p.: $x_1 = x_2 = x_3 = 1/3$

m $\rightarrow$ i.f.p.

III.2.1.e

$$\left.\begin{array}{l}(33) + 2(12)\\(11) + 2(23)\\(22) + 2(13)\end{array}\right\}\begin{array}{l}\text{b.t.p.}\\\text{not degenerate}\end{array}$$

i.f.p.: $x_1 = x_2 = x_3 = 1/3$

m $\rightarrow$ i.f.p.

III.2.2.a

$$\left.\begin{array}{l}(11) + 2(13)\\(22) + 2(23)\\(33) + 2(12)\end{array}\right\}\begin{array}{l}\text{3 n.f.p.'s}\\\text{doubly degenerate}\end{array}$$

i.f.p.: $x_1 = x_2 = x_3 = 1/3$

m $\rightarrow$ n.f.p. ($x_1 = 1$) with initial vectors

$$\begin{array}{lcl} x_1 = .37165698 & & x_1 = .56881955 \\ x_2 = .08887955 & \text{and} & x_2 = .00173087 \\ x_3 = .53946347 & & x_3 = .42944958 \end{array}$$

TABLE II (cont.)

III.2.2.a (cont.)

$m \rightarrow$ n.f.p. ($x_2 = 1$) with initial vector:

$x_1 = .02489604$

$x_2 = .18806313$ (See Section XI for discussion)

$x_3 = .78704084$

III.2.2.b

$$\left.\begin{array}{l}(22) + 2(23)\\(11) + 2(13)\\(33) + 2(12)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\\text{b.d.p.}\\\text{not degenerate}\end{array}$$

i.f.p.: $x_1 = x_2 = x_3 = 1/3$

$m \rightarrow$ b.d.p.

III.2.2.c

$$\left.\begin{array}{l}(33) + 2(12)\\(22) + 2(23)\\(11) + 2(13)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\\text{b.f.p. (cf. I.2.n)}\\\text{b.d.p.}\\\text{degenerate}\end{array}$$

i.f.p.: $x_1 = x_2 = x_3 = 1/3$

$m \rightarrow$ i.f.p.

III.2.2.e

$$\left.\begin{array}{l}(33) + 2(12)\\(11) + 2(13)\\(22) + 2(23)\end{array}\right\}\begin{array}{l}\text{b.t.p.}\\\text{not degenerate}\end{array}$$

i.f.p.: $x_1 = x_2 = x_3 = 1/3$ $m \rightarrow$ i.f.p.

TABLE II (cont.)

III.2.3.a

$$\left.\begin{array}{l}(11) + 2(12)\\(22) + 2(23)\\(33) + 2(13)\end{array}\right\}\begin{array}{l}\text{3 n.f.p.'s}\\\text{triply degenerate}\end{array}$$

i.f.p.: $x_1 = x_2 = x_3 = 1/3$

m → no convergence (see Section XI for discussion)

III.2.3.b

$$\left.\begin{array}{l}(22) + 2(23)\\(11) + 2(12)\\(33) + 2(13)\end{array}\right\}\begin{array}{l}\text{n.f.p.}\\\text{b.f.p. (cf. I.2.n)}\\\text{b.d.p.}\\\text{degenerate}\end{array}$$

i.f.p.: $x_1 = x_2 = x_3 = 1/3$

m → n.f.p. with initial vectors:

$$\begin{array}{lcl} x_1 = .29517934 & & x_1 = .05841813 \\ x_2 = .10369878 & \text{and} & x_2 = .25242368 \\ x_3 = .60112188 & & x_3 = .68915819 \end{array}$$

m → b.d.p. with initial vector:

$x_1 = .35613727$

$x_2 = .62067802$

$x_3 = .02318471$

TABLE II (cont.)

III.2.3.e

$$\left.\begin{array}{l}(33) + 2(13)\\(11) + 2(12)\\(22) + 2(23)\end{array}\right\}\begin{array}{l}\text{b.t.p.}\\\text{not degenerate}\end{array}$$

i.f.p.: $x_1 = x_2 = x_3 = 1/3$

m $\rightarrow$ b.t.p.

III.2.3.f

$$\left.\begin{array}{l}(22) + 2(23)\\(33) + 2(13)\\(11) + 2(12)\end{array}\right\}\begin{array}{l}\text{b.t.p.}\\\text{not degenerate}\end{array}$$

i.f.p.: $x_1 = x_2 = x_3 = 1/3$

m $\rightarrow$ b.t.p.

INTRODUCTION TO TABLE III

This table lists a few 4-variable generalizations (and two 5-variable ones) of selected 3-variable systems. The method of generalization is explained in the text (Section VII, I.5.j - ext - 1 and subsequent discussion). The basic notation is that of Table II, but degeneracy, b.f.p.'s, n.f.p.'s, etc., are not noted.

TABLE III

EXAMPLES OF BINARY REACTION SYSTEMS FOR N > 3

I.1.c - ext - 1

(11) + (22) + (44) + 2(14) + 2(24) + 2(34)

2(13) + 2(23)

2(12)

(33)

i.f.p. given by ($x = x_1$):

$16x^3(1+x) - 1 = 0$

$m \rightarrow$ i.f.p.

$$\left.\begin{aligned} x_1 &= .35833637 \\ x_2 &= .33933212 \\ x_3 &= .24319009 \\ x_4 &= .05914142 \end{aligned}\right\} \text{i.f.p.}$$

I.2.a - ext - 1

(11) + (22) + 2(12) + 2(14) + 2(24) + 2(34)

(44) + 2(23)

2(13)

(33)

i.f.p. given by ($x = x_3$):

$x^4 - 2x^3 - x^2 + 2x - 1/2 = 0$

$m \rightarrow$ i.f.p.

$$\left.\begin{aligned} x_1 &= 1/2 \\ x_2 &= .04258212 \\ x_3 &= .34108137 \\ x_4 &= .11633651 \end{aligned}\right\} \text{i.f.p.}$$

TABLE III (cont.)

I.2.k - ext - 1

(11) + (44) + 2(23) + 2(14) + 2(24) + 2(34)

(22) + 2(13)

2(12)

(33)

i.f.p. given by ($x = x_1$):

$8x^2(8x^3-4x-1) + 3 = 0$

m → i.f.p.

$$\left.\begin{array}{l} x_1 = .42397159 \\ x_2 = .28099237 \\ x_3 = .23826556 \\ x_4 = .05677048 \end{array}\right\} \text{i.f.p.}$$

I.2.p - ext - 1

(22) + (44) + 2(13) + 2(14) + 2(24) + 2(34)

(11) + 2(12)

2(23)

(33)

m → b.d.p. (1,0,0,0), (0,1,0,0)

$$\left.\begin{array}{l} x_1 = \frac{\sqrt{3} - 1}{2} = .36602540 \\ x_2 = 1/2 \\ x_3 = \frac{-1 + \sqrt{5-2\sqrt{3}}}{2} = .11965684 \\ x_4 = x_3^2 = .01431776 \end{array}\right\} \text{i.f.p.}$$

I.3.g - ext - 1

(22) + (44) + 2(12) + 2(14) + 2(24) + 2(34)

2(13) + 2(23)

(11)

(33)

i.f.p. given by ($x = x_1$)

$2x^5 + 2x^4 + 3x^3 + 3x^2 - 1 = 0$

$$\left.\begin{array}{l} x_1 = .45003654 \\ x_2 = .30641100 \\ x_3 = .20253289 \\ x_4 = .04101957 \end{array}\right\} \text{i.f.p.}$$

m → b.t.p. (1,0,0,0), (0,0,1,0), (0,0,0,1)

TABLE III (cont.)

I.3.i - ext - 1

(22) + (44) + 2(23) + 2(14) + 2(24) + 2(34)

2(12) + 2(13)

(11)

(33)

i.f.p. given by ($x = x_1$):

$2x^4 + x^3 + x^2 + 2x - 1 = 0$

$$\left.\begin{aligned} x_1 &= .37973157 \\ x_2 &= .45527986 \\ x_3 &= .14419607 \\ x_4 &= .02079250 \end{aligned}\right\} \text{i.f.p.}$$

m → i.f.p. with initial vectors:

$$\begin{aligned} x_1^{(o)} &= .23580752 & & x_1^{(o)} &= .35656919 \\ x_2^{(o)} &= .45570511 & \text{and} \quad & x_2^{(o)} &= .01450795 \\ x_3^{(o)} &= .16718123 & & x_3^{(o)} &= .34834313 \\ x_4^{(o)} &= .14130614 & & x_4^{(o)} &= .28057973 \end{aligned}$$

m → b.t.p. with initial vector:

$$\begin{aligned} x_1^{(o)} &= .03633558 \\ x_2^{(o)} &= .15482236 \\ x_3^{(o)} &= .04547126 \\ x_4^{(o)} &= .76337080 \end{aligned}$$

I.4.e - ext - 1

(22) + 2(12) + 2(23) + 2(14) + 2(24) + 2(34)

(11) + (44)

2(13)

(33)

TABLE III (cont.)

I.4.e - ext - 1 (cont.)

i.f.p. given by ($x = x_3$):

$x^4 + x^2 + x - 1/4 = 0$

$m \rightarrow$ b.d.p.

$$\left.\begin{aligned} x_1 &= 1/2 \\ x_2 &= .25179510 \\ x_3 &= .20583631 \\ x_4 &= .04236859 \end{aligned}\right\} \text{i.f.p.}$$

I.5.j - ext - 1

(22) + 2(12) + 2(23) + 2(14) + 2(24) + 2(34)

(44) + 2(13)

(11)

(33)

i.f.p. given by ($x = x_1$):

$x^8 + x^4 + 2x^3 + x^2 + x - 1 = 0$

$$\left.\begin{aligned} x_1 &= .48325036 \\ x_2 &= .22868204 \\ x_3 &= .23353091 \\ x_4 &= .05453669 \end{aligned}\right\} \text{i.f.p.}$$

(Continued on following page.)

TABLE III (cont.)

I.5.j - ext - 1 (cont.)

m → interior periodic point of period 12:

Cycle	x_1	x_2	x_3	x_4
n	.66002500	.33689080	.00014037	.00294383
n+1	.56417301	.00019396	.43563301	.00000002
n+2	.00038792	.49154477	.31829119	.18977612
n+3	.86242865	.03626192	.00000015	.10130928
n+4	.24595298	.01026383	.74378319	0(i.e.$< 5\times10^{-9}$)
n+5	.02042231	.36587139	.06049287	.55321343
n+6	.68740763	.30851591	.00041707	.00365939
n+7	.52688378	.00058679	.47252926	.00000017
n+8	.00117358	.49793601	.27760650	.22328391
n+9	.87242595	.05050729	.00000138	.07706538
n+10	.23293147	.00594148	.76112705	0(i.e.$< 5\times10^{-9}$)
n+11	.01184765	.35458089	.05425707	.57931439

and $x_i^{(n+12)} = x_i^{(n)}$

TABLE III (cont.)

I.5.j - ext - 2

(22) + 2(12) + 2(23) + 2(14) + 2(24) + 2(34) + 2(15) + 2(25)

+ 2(35) + 2(45)

(55) + 2(13) $\qquad x_1 = .48324807$

(11) $\qquad x_2 = .22571344$

(33) $\qquad x_3 = .23352870$ } i.f.p.

(44) $\qquad x_4 = .05453565$

i.f.p. given by $(x = x_1)$ $\qquad x_5 = .00297414$

$$x^{16} + x^8 + x^4 + 2x^3 + x^2 + x - 1 = 0$$

m → interior periodic point of period 6:

Cycle	x_1	x_2	x_3	x_4	x_5
n	.83355924	.03412438	.01389264	.02095486	.09746888
n+1	.27188601	.03266085	.69482102	.00019301	.00043911
n+2	.06547728	.37782442	.07392200	.48277626	.00000004
n+3	.74749492	.00968042	.00428727	.00546446	.23307293
n+4	.38047069	.06073242	.55874865	.00001838	.00002986
n+5	.11786703	.42517497	.14475794	.31220006	0(i.e. $< 5\times10^{-9}$)

and $x_i^{(n+6)} = x_i^{(n)}$

TABLE III (cont.)

I.5.1 - ext - 1

(22) + 2(13) + 2(23) + 2(14) + 2(24) + 2(34)

(44) + 2(12) $\qquad x_1 = .49177311$

(11) $\qquad x_2 = .20789913$

(33) $\qquad x_3 = .24184079$ $\Big\}$ i.f.p.

i.f.p. given by ($x = x_1$): $\qquad x_4 = .05848697$

$$x^8 - 2x^5 + x^4 - 2x^3 - x^2 + 3x - 1 = 0$$

m → i.f.p.

I.6.b - ext - 1

2(12) + 2(13) + 2(14) + 2(23) + 2(24) + 2(34)

(11) + (44) $\qquad x_1 = .56609505$

(22) $\qquad x_2 = .32057516$

(33) $\qquad x_3 = .10276844$ $\Big\}$ i.f.p.

i.f.p. given by ($x = x_2$): $\qquad x_4 = .01056135$

$$2x^8 + 2x^6 + 2x^5 - x^4 + 2x^3 - x^2 - 3x + 1 = 0$$

m → i.f.p.

This i.f.p. is very close to that of II.3.d-ext-1 (q.v.), as might be expected.

TABLE III (cont.)

I.6.c - ext - 1

2(12) + 2(13) + 2(23) + 2(14) + 2(24) + 2(34)

(22) + (44)

(11)

(33)

i.f.p. given by ($x = x_1$):

$x^7 + x^5 + x^4 + x^2 - 1/2 = 0$

$m \rightarrow$ i.f.p.

(cf. I.6.b - ext - 1)

$$\left.\begin{array}{l} x_1 = .56612299 \\ x_2 = .01066456 \\ x_3 = .32049525 \\ x_4 = .10271720 \end{array}\right\} \text{i.f.p.}$$

II.3.d - ext - 1

(44) + 2(12) + 2(13) + 2(14) + 2(23) + 2(24) + 2(34)

(33)

(11)

(22)

i.f.p. given by ($x = x_1$):

$x + x^2 + x^4 + x^8 - 1 = 0$

(cf. Eq. 21) of the text)

$$\left.\begin{array}{l} x_1 = .56616082 \\ x_2 = .10274465 \\ x_3 = .32053807 \\ x_4 = .01055646 \end{array}\right\} \text{i.f.p.}$$

TABLE III (cont.)

II.3.d - ext - 2

(55) + 2(12) + 2(13) + 2(14) + 2(15) + 2(23) + 2(24) + 2(25)
+ 2(34) + 2(35) + 2(45)

(33)

(11)

(22)

(44)

i.f.p. given by ($x = x_1$):

$$x + x^2 + x^4 + x^8 + x^{16} - 1 = 0$$

(cf. Eq. 21) of the text)

m → i.f.p.

$$\left.\begin{array}{l} x_1 = .56612380 \\ x_2 = .10271779 \\ x_3 = .32049615 \\ x_4 = .01055094 \\ x_5 = .00011132 \end{array}\right\} \text{i.f.p.}$$

III.1.c - ext - 1

(11) + (22) + 2(14) + 2(24) + 2(34)

2(13) + 2(23)

2(12) + (44)

(33)

i.f.p. given by ($x = x_3$):

$$1 - x^3 + 4(2x-1)(1-x-x^2)^2 = 0$$

m → i.f.p.

$$\left.\begin{array}{l} x_1 = .35443555 \\ x_2 = .34053827 \\ x_3 = .24500079 \\ x_4 = .06002539 \end{array}\right\} \text{i.f.p.}$$

TABLE III (cont.)

III.1.f - ext - 1

(11) + (44) + 2(14) + 2(24) + 2(34)

2(13) + 2(23)

2(12) + (22)

(33)

i.f.p. given by ($x = x_3$):

$4(1-x-x^2)^2(x-1) + 1 = 0$

m → i.f.p.

$$\left.\begin{aligned} x_1 &= .23278771 \\ x_2 &= .36744075 \\ x_3 &= .30608408 \\ x_4 &= .09368746 \end{aligned}\right\} \text{i.f.p.}$$

EXPLANATION OF TABLES IV AND V

Each of these tables consists in a tabulation of the convergence behavior of random initial points under 26 transformations which are simple modifications of I.3.g. These 26 systems are generated as follows (see Section VIII.2 of text):

In Table IV, we retain the "skeleton"

$$x_1' = 1/2x_2^2 + 1/2\, x_3^2 + 2x_1x_2 + \ldots$$
$$x_2' = 2x_1x_3 + 2x_2x_3 + \ldots$$
$$x_3' = 1/2\, x_1^2 + \ldots$$

To this we add in all possible ways the missing terms $1/2\, x_1^2$, $1/2\, x_2^2$, $1/2\, x_3^2$. Twenty-six new systems result (the 27th is identical with I.3.g). The notation is the same as in the previous tables, but we give only the numerical results, omitting comments and i.f.p. equations.

Table V is similar, except that the skeleton is

$$x_1' = x_2^2 + x_3^2 + x_1x_2 + \ldots$$
$$x_2' = x_1x_3 + x_2x_3 + \ldots$$
$$x_1' = x_1^2 + \ldots$$

and the missing terms are x_1x_2, x_1x_3, x_2x_3.

TABLE IV

MODIFICATIONS OF SYSTEM I.3.g

Case 1.

1/2(11) + (22) + (33) + 2(12)	$x_1 = .56804585$	i.f.p.
2(13) + 2(23)	$x_2 = .27061611$	
1/2(11)	$x_3 = .16133804$	
m → i.f.p.		

Case 2.

1/2(22) + 1/2(33) + 2(12)		
2(13) + 2(23) + 1/2(11) + 1/2(22) + 1/2(33)		
1/2(11)	$x_1 = .47868722$	i.f.p.
m → i.f.p.	$x_2 = .40674205$	
	$x_3 = .11457073$	

Case 3.

1/2(22) + 1/2(33) + 2(12)	$x_1 = .38326262$	i.f.p.
2(13) + 2(23)	$x_2 = .37085176$	
(11) + 1/2(22) + 1/2(33)	$x_3 = .24588562$	
m → i.f.p.		

TABLE IV (cont.)

Case 4.

1/2(11) + (22) + 1/2(33) + 2(12)	$x_1 = .56267887$	i.f.p.
2(13) + 2(23) + 1/2(33)	$x_2 = .27901737$	
1/2(11)	$x_3 = .15830376$	
m → i.f.p.		

Case 5.

1/2(11) + 1/2(22) + (33) + 2(12)	$x_1 = .54858377$	i.f.p.
2(13) + 2(23) + 1/2(22)	$x_2 = .30094415$	
1/2(11)	$x_3 = .15047208$	
m → i.f.p.		

Case 6.

(22) + (33) + 2(12)	$x_1 = .51130605$	i.f.p.
2(13) + 2(23) + 1/2(11)	$x_2 = .35797701$	
1/2(11)	$x_3 = .13071694$	
m → i.f.p.		

Case 7.

1/2(11) + (22) + 1/2(33) + 2(12)	$x_1 = .55386328$	i.f.p.
2(13) + 2(23)	$x_2 = .27874434$	
1/2(11) + 1/2(33)	$x_3 = .16739238$	
m → i.f.p.		

TABLE IV (cont.)

Case 8.

1/2(11) + 1/2(22) + (33) + 2(12)	x_1 = .52319512	i.f.p.
2(13) + 2(23)	x_2 = .29610055	
1/2(11) + 1/2(22)	x_3 = .18070433	

m → i.f.p.

Case 9.

1/2(22) + 1/2(33) + 2(12)	x_1 = .47452268	i.f.p.
2(13) + 2(23) + 1/2(11) + 1/2(22)	x_2 = .40572059	
1/2(11) + 1/2(33)	x_3 = .11975673	

m → i.f.p.

Case 10.

1/2(22) + 1/2(33) + 2(12)	x_1 = .43437288	i.f.p.
2(13) + 2(23) + 1/2(11) + 1/2(33)	x_2 = .39376269	
1/2(11) + 1/2(22)	x_3 = .17186443	

m → i.f.p.

Case 11.

1/2(22) + 1/2(33) + 2(12)	x_1 = .42681726	i.f.p.
2(13) + 2(23) + 1/2(22) + 1/2(33)	x_2 = .39100977	
(11)	x_3 = .18217297	

m → i.f.p.

TABLE IV (cont.)

Case 12.

1/2(22) + 1/2(33) + 2(12)	x_1 = .48181136	i.f.p.
2(13) + 2(23) + 1/2(11) + 1/2(22)	x_2 = .40211755	
1/2(11) + 1/2(33)	x_3 = .11607109	

m → i.f.p.

Case 13.

(22) + 1/2(33) + 2(12)	x_1 = .50829830	i.f.p.
2(13) + 2(23) + 1/2(11) + 1/2(33)	x_2 = .36251813	
1/2(11)	x_3 = .12918357	

m → i.f.p.

Case 14.

1/2(11) + 1/2(22) + 1/2(33) + 2(12)	x_1 = .54269064	i.f.p.
2(13) + 2(23) + 1/2(22) + 1/2(33)	x_2 = .31005278	
1/2(11)	x_3 = .14725658	

m → i.f.p.

Case 15.

1/2(22) + 1/2(33) + 2(12)	x_1 = .39529957	i.f.p.
2(13) + 2(23) + 1/2(33)	x_2 = .37727171	
(11) + 1/2(22)	x_3 = .22742872	

m → i.f.p.

TABLE IV (cont.)

Case 16.

1/2(22) + 1/2(33) + 2(12)	x_1 = .41841444	i.f.p.
2(13) + 2(23) + 1/2(22)	x_2 = .38772371	
(11) + 1/2(33)	x_3 = .19386185	

m → i.f.p.

Case 17.

1/2(22) + 1/2(33) + 2(12)	x_1 = .42566975	i.f.p.
2(13) + 2(23) + 1/2(11)	x_2 = .39057539	
1/2(11) + 1/2(22) + 1/2(33)	x_3 = .18375486	

m → i.f.p.

Case 18.

1/2(22) + (33) + 2(12)	x_1 = .41080271	i.f.p.
2(13) + 2(23)	x_2 = .35678917	
(11) + 1/2(22)	x_3 = .23240812	

m → i.f.p.

Case 19.

(22) + 1/2(33) + 2(12)	x_1 = .44081956	i.f.p.
2(13) + 2(23)	x_2 = .34107310	
(11) + 1/2(33)	x_3 = .21810734	

m → i.f.p.

(Very slow convergence; $|\text{Jacobian}|$ = .99411052 at i.f.p.)

TABLE IV (cont.)

Case 20.

$1/2(11) + 1/2(22) + 1/2(33) + 2(12)$

$2(13) + 2(23)$

$1/2(11) + 1/2(22) + 1/2(33)$

$m \rightarrow$ i.f.p.

$$\left.\begin{aligned} x_1 &= 1/2 \\ x_2 &= \frac{\sqrt{5}-1}{4} = .30901699 \\ x_3 &= \frac{3-\sqrt{5}}{4} = .19098301 \end{aligned}\right\} \text{ i.f.p.}$$

Case 21.

$1/2(11) + 1/2(22) + 1/2(33) + 2(12)$

$2(13) + 2(23) + 1/2(22)$

$1/2(11) + 1/2(33)$

$m \rightarrow$ i.f.p.

$$\left.\begin{aligned} x_1 &= .53484692 \\ x_2 &= .31010205 \\ x_3 &= .15505103 \end{aligned}\right\} \text{ i.f.p.}$$

Case 22.

$1/2(11) + 1/2(22) + 1/2(33) + 2(12)$

$2(13) + 2(23) + 1/2(33)$

$1/2(11) + 1/2(22)$

$m \rightarrow$ i.f.p.

$$\left.\begin{aligned} x_1 &= .51157583 \\ x_2 &= .30963296 \\ x_3 &= .17879121 \end{aligned}\right\} \text{ i.f.p.}$$

Case 23.

$(22) + 1/2(33) + 2(12)$

$2(13) + 2(23) + 1/2(11)$

$1/2(11) + 1/2(33)$

$m \rightarrow$ i.f.p.

$$\left.\begin{aligned} x_1 &= .50306514 \\ x_2 &= .36118335 \\ x_3 &= .13575151 \end{aligned}\right\} \text{ i.f.p.}$$

TABLE IV (cont.)

Case 24.

$$\left.\begin{array}{ll} (22) + 1/2(33) + 2(12) & x_1 = .45127091 \\ 2(13) + 2(23) + 1/2(33) & x_2 = .34508366 \\ (11) & x_3 = .20364543 \end{array}\right\} \text{i.f.p.}$$

i.f.p. given by $(x = x_1)$

$$3x^4 - 6x^2 - 2x + 2 = 0$$

$m \rightarrow$ i.t.p.:

$$\left.\begin{array}{l} x_1^{(n+3)} = x_1^{(n)} = .87777286,\ x_1^{(n+1)} = .00766150,\ x_1^{(n+2)} = .34944206 \\ x_2^{(n+3)} = x_2^{(n)} = .00011739,\ x_2^{(n+1)} = .22185332,\ x_2^{(n+2)} = .65049924 \\ x_3^{(n+3)} = x_3^{(n)} = .12210975,\ x_3^{(n+1)} = .77048518,\ x_3^{(n+2)} = .00005870 \end{array}\right\} \text{i.t.p.}$$

Case 25.

$$\left.\begin{array}{ll} 1/2(22) + (33) + 2(12) & x_1 = .44406835 \\ 2(13) + 2(23) + 1/2(11) & x_2 = .38371478 \\ 1/2(11) + 1/2(22) & x_3 = .17221687 \end{array}\right\} \text{i.f.p.}$$

$m \rightarrow$ i.f.p.

Case 26.

$$\left.\begin{array}{ll} 1/2(22) + (33) + 2(12) & x_1 = .43425854 \\ 2(13) + 2(23) + 1/2(22) & x_2 = .37716097 \\ (11) & x_3 = .18858049 \end{array}\right\} \text{i.f.p.}$$

$m \rightarrow$ i.f.p.

TABLE V

MODIFICATIONS OF SYSTEM I.3.g

Case 1.

$$\left.\begin{array}{l} (22) + (33) + 2(12) + (13) + (23) \\ (13) + (23) \\ (11) \\ m \rightarrow \text{i.f.p.} \end{array}\right. \qquad \left.\begin{array}{l} x_1 = .52488860 \\ x_2 = .19960336 \\ x_3 = .27550804 \end{array}\right\} \text{i.f.p.}$$

Case 2.

$$\left.\begin{array}{l} (22) + (33) + (12) \\ 2(13) + 2(23) + (12) \\ (11) \\ m \rightarrow \text{i.f.p.} \end{array}\right. \qquad \left.\begin{array}{l} x_1 = .39816095 \\ x_2 = .44330691 \\ x_3 = .15853214 \end{array}\right\} \text{i.f.p.}$$

Case 3.

$$\left.\begin{array}{l} (22) + (33) + (12) \\ (13) + (23) \\ (11) + (12) + (13) + (23) \\ m \rightarrow \text{i.f.p.} \end{array}\right. \qquad \left.\begin{array}{l} x_1 = .32471796 \\ x_2 = .24512233 \\ x_3 = .43015971 \end{array}\right\} \text{i.f.p.}$$

Case 4.

$$\left.\begin{array}{l} (22) + (33) + 2(12) + (13) \\ (13) + 2(23) \\ (11) \\ m \rightarrow \text{i.f.p.} \end{array}\right. \qquad \left.\begin{array}{l} x_1 = 1/2 \\ x_2 = 1/4 \\ x_3 = 1/4 \end{array}\right\} \text{i.f.p.}$$

TABLE V (cont.)

Case 5.

(22) + (33) + 2(12) + (23)	$x_1 = .48053382$	i.f.p.
2(13) + (23)	$x_2 = .28855343$	
(11)	$x_3 = .23091275$	
m → i.f.p. (slow convergence)		

Case 6.

(22) + (33) + (12) + (13) + (23)	$x_1 = .46557123$	i.f.p. (cf. Case 9)
(13) + (23) + (12)	$x_2 = .31767220$	
(11)	$x_3 = .21675657$	
m → i.f.p.		

Case 7.

(22) + (33) + 2(12) + (13)	$x_1 = .48787213$	i.f.p.
(13) + (23)	$x_2 = .21060631$	
(11) + (23)	$x_3 = .30152156$	
m → i.f.p.		

Case 8.

(22) + (33) + 2(12) + (23)	$x_1 = .43691113$	i.f.p.
(13) + (23)	$x_2 = .22408140$	
(11) + (13)	$x_3 = .33900747$	
m → i.f.p.		

TABLE V (cont.)

Case 9.

(22) + (33) + (12) + (13) + (23)	$x_1 = .46557123$	i.f.p. (cf. Case 6)
(13) + (23)	$x_2 = .21675657$	
(11) + (12)	$x_3 = .31767220$	

Case 10.

(22) + (33) + (12)	$x_1 = .36632453$	i.f.p.
2(13) + (23) + (12)	$x_2 = .40727446$	
(11) + (23)	$x_3 = .22640101$	
m → i.f.p.		

Case 11.

(22) + (33) + (12)	$x_1 = .37008112$	i.f.p.
(13) + 2(23) + (12)	$x_2 = .41249400$	
(11) + (13)	$x_3 = .21742488$	
m → i.f.p.		

Case 12.

(22) + (33) + (12)	$x_1 = .35320996$	i.f.p.
2(13) + 2(23)	$x_2 = .38577366$	
(11) + (12)	$x_3 = .26101638$	
m → i.f.p.		

TABLE V (cont.)

Case 13.

(22) + (33) + (12) + (23)

2(13) + (23) + (12)

(11)

m → i.f.p.

$$\left.\begin{array}{l} x_1 = .42578666 \\ x_2 = .39291905 \\ x_3 = .18129429 \end{array}\right\} \text{i.f.p.}$$

Case 14.

(22) + (33) + (12) + (13)

(13) + 2(23) + (12)

(11)

m → i.f.p.

$$\left.\begin{array}{l} x_1 = .42878157 \\ x_2 = .38736479 \\ x_3 = .18385364 \end{array}\right\} \text{i.f.p.}$$

Case 15.

(22) + (33) + (12)

(13) + 2(23)

(11) + (12) + (13)

m → i.f.p.

$x_1 = x_2 = x_3 = 1/3$ i.f.p.

Case 16.

(22) + (33) + (12)

2(13) + (23)

(11) + (12) + (23)

m → i.f.p.

$x_1 = x_2 = x_3 = 1/3$ i.f.p.

TABLE V (cont.)

Case 17.

(22) + (33) + (12)	$x_1 = x_2 = x_3 = 1/3$ i.f.p.
(13) + (23) + (12)	
(11) + (13) + (23)	
m → i.f.p.	

Case 18.

(22) + (33) + (12) + (23)	$x_1 = x_3 = .38196601$ } i.f.p.
(13) + (23)	$x_2 = .23606798$
(11) + (12) + (13)	(cf. Case 20)
m → i.f.p.	

Case 19.

(22) + (33) + (12) + (23)	$x_1 = .41964338$ } i.f.p.
(13) + (23)	$x_2 = .22815549$
(11) + (12) + (13)	$x_3 = .35220113$
m → i.f.p.	

Case 20.

(22) + (33) + 2(12)	$x_1 = x_3 = .38196601$ } i.f.p.
(13) + (23)	$x_2 = .23606798$
(11) + (13) + (23)	(cf. Case 18)
m → i.f.p.	

TABLE V (cont.)

Case 21.

(22) + (33) + 2(12)	$x_1 = .42788476$
2(13) + (23)	$x_2 = .30766792$ i.f.p.
(11) + (23)	$x_3 = .26444732$
m → i.f.p.	

Case 22.

(22) + (33) + 2(12)	$x_1 = .41421356$
(13) + 2(23)	$x_2 = x_3 = .29289322$ i.f.p.
(11) + (13)	(cf. Case 24)
m → i.f.p.	

Case 23.

(22) + (33) + (12) + (13)	$x_1 = .41594131$
(13) + (23) + (12)	$x_2 = .32699283$ i.f.p.
(11) + (23)	$x_3 = .25706586$
m → i.f.p.	

Case 24.

(22) + (33) + (12) + (13)	$x_1 = .41421356$
(13) + 2(23)	$x_2 = x_3 = .29289322$ i.f.p.
(11) + (12)	(cf. Case 22)

TABLE V (cont.)

Case 25.

(22) + (33) + (12) + (23)	$x_1 = .40157157$	i.f.p.
(13) + (23) + (12)	$x_2 = .32895639$	
(11) + (13)	$x_3 = .26947204$	
m → i.f.p		

Case 26.

(22) + (33) + (12) + (23)	$x_1 = .39816095$	i.f.p.
2(13) + (23)	$x_2 = .31706428$	
(11) + (12)	$x_3 = .28477477$	
m → i.f.p.		

TABLE VI

SUPER-SYSTEMS

I.2.e - Super

a. Initial Configuration

$y_1 = .03342810$	$y_4 = .19438836$	$y_7 = .19222349$
$y_2 = .30728575$	$y_5 = .02148296$	$y_8 = .02173869$
$y_3 = .03472276$	$y_6 = .05711904$	$y_9 = .13761085$

b. Initial Configuration

$y_1 = .06578151$	$y_4 = .07528814$	$y_7 = .32057547$
$y_2 = .02301108$	$y_5 = .13873987$	$y_8 = .27225037$
$y_3 = .00731792$	$y_6 = .03090597$	$y_9 = .06612967$

Both of these gave the final periodic configuration ($y_i^{(n+2)} = y_i^{(n)}$):

$y_1^{(n)} = .18055154$	$y_4^{(n)} = 0$	$y_7^{(n)} = .13889692$
$y_2^{(n)} = .38464618$	$y_5^{(n)} = 0$	$y_8^{(n)} = .29590536$
$y_3^{(n)} = 0$	$y_6^{(n)} = 0$	$y_9^{(n)} = 0$

$y_1^{(n+1)} = .18055154$	$y_4^{(n+1)} = .38464618$	$y_7^{(n+1)} = 0$
$y_2^{(n+1)} = 0$	$y_5^{(n+1)} = 0$	$y_8^{(n+1)} = 0$
$y_3^{(n+1)} = .13889692$	$y_6^{(n+1)} = .29590536$	$y_9^{(n+1)} = 0$

TABLE VI (cont.)

For the initial configuration

c.

$y_1 = .14398619$	$y_4 = .06422167$	$y_7 = .04324188$
$y_2 = .10409303$	$y_5 = .02141784$	$y_8 = .07707331$
$y_3 = .40338819$	$y_6 = .06973376$	$y_9 = .07284413,$

the final configuration was $(y_i^{(n+2)} = y_i^{(n)})$:

$y_1^{(n)} = .10204732$	$y_4^{(n)} = .21740114$	$y_7^{(n)} = 0$
$y_2^{(n)} = .21740114$	$y_5^{(n)} = .46315040$	$y_8^{(n)} = 0$
$y_3^{(n)} = 0$	$y_6^{(n)} = 0$	$y_9^{(n)} = 0$

$y_1^{(n+1)} = .31944846$	$y_4^{(n+1)} = 0$	$y_7^{(n+1)} = .24574926$
$y_2^{(n+1)} = 0$	$y_5^{(n+1)} = 0$	$y_8^{(n+1)} = 0$
$y_3^{(n+1)} = .24574926$	$y_6^{(n+1)} = 0$	$y_9^{(n+1)} = .18905302$

TABLE VI (cont.)

I.3.g - Super

Initial Configurations:

a.

$y_1 = .01631507$	$y_4 = .11350670$	$y_7 = .24077161$
$y_2 = .06252989$	$y_5 = .05348809$	$y_8 = .05243129$
$y_3 = .01977135$	$y_6 = .03871014$	$y_9 = .40247586$

b.

$y_1 = .04649838$	$y_4 = .02351925$	$y_7 = .14210424$
$y_2 = .20513993$	$y_5 = .02478259$	$y_8 = .00038505$
$y_3 = .11967344$	$y_6 = .05513905$	$y_9 = .38275807$

These both gave the final configuration ($y_i^{(n+3)} = y_i^{(n)}$):

$y_1^{(n)} = .24781045$	$y_4^{(n)} = .24675180$	$y_7^{(n)} = .00324338$
$y_2^{(n)} = .24675180$	$y_5^{(n)} = .24569765$	$y_8^{(n)} = .00322954$
$y_3^{(n)} = .00324338$	$y_6^{(n)} = .00322954$	$y_9^{(n)} = .00004246$

$y_1^{(n+1)} = .54648124$	$y_4^{(n+1)} = .00957014$	$y_7^{(n+1)} = .18319231$
$y_2^{(n+1)} = .00957014$	$y_5^{(n+1)} = .00016760$	$y_8^{(n+1)} = .00320812$
$y_3^{(n+1)} = .18319231$	$y_6^{(n+1)} = .00320812$	$y_9^{(n+1)} = .06141002$

$y_1^{(n+2)} = .00651538$	$y_4^{(n+2)} = .03009170$	$y_7^{(n+2)} = .04411082$
$y_2^{(n+2)} = .03009170$	$y_5^{(n+2)} = .13898048$	$y_8^{(n+2)} = .20372868$
$y_3^{(n+2)} = .04411082$	$y_6^{(n+2)} = .20372868$	$y_9^{(n+2)} = .29864174$

TABLE VI (cont.)

Initial Configuration:

c.

$y_1 = .11265988$	$y_4 = .09322974$	$y_7 = .03769313$
$y_2 = .25586952$	$y_5 = .00321861$	$y_8 = .00633159$
$y_3 = .16027686$	$y_6 = .09787061$	$y_9 = .23285006$

This gave the final configuration ($y_i^{(n+3)} = y_i^{(n)}$):

$y_1^{(n)} = .04018182$	$y_4^{(n)} = .18558236$	$y_7^{(n)} = .27204143$
$y_2^{(n)} = .04001017$	$y_5^{(n)} = .18478956$	$y_8^{(n)} = .27087928$
$y_3^{(n)} = .00052591$	$y_6^{(n)} = .00242894$	$y_9^{(n)} = .00356053$
$y_1^{(n+1)} = .36799967$	$y_4^{(n+1)} = .36642757$	$y_7^{(n+1)} = .00481645$
$y_2^{(n+1)} = .00644452$	$y_5^{(n+1)} = .00641699$	$y_8^{(n+1)} = .00008435$
$y_3^{(n+1)} = .12336144$	$y_6^{(n+1)} = .12283443$	$y_9^{(n+1)} = .00161458$
$y_1^{(n+2)} = .05967020$	$y_4^{(n+2)} = .00104496$	$y_7^{(n+2)} = .02000274$
$y_2^{(n+2)} = .27559068$	$y_5^{(n+2)} = .00482623$	$y_8^{(n+2)} = .09238395$
$y_3^{(n+2)} = .40398281$	$y_6^{(n+2)} = .00707467$	$y_9^{(n+2)} = .13542376$

APPENDIX

1. In this appendix we summarize the results for a class of homogeneous quadratic transformations quite distinct from the class we have called Binary Reaction Systems. This class arises in a natural way in the study of a certain crude model of the "evolutionary process," which will be described below. It also has some mathematical interest, owing to the fact that the limiting behavior of all systems belonging to this class can be explicitly predicted.

2. <u>The Evolutionary Model</u>

Consider a large population in which each distinct "type" of individual is labelled by an index pair (i,j), $i,j = 1,2, \ldots, N$. Let the fraction of the male population which is of type (i,j) be denoted by x_{ij}. We shall assume that $x_{ij} = x_{ji}$; also, we take the number of females of type (i,j) to be equal to the number of males of this type--hence there is no need to denote the fraction of females by a separate letter.

We now impose a mating rule (random mating is assumed) which states, in effect, that if individuals of type (k,ℓ) mate with individuals of type (m,n), the progeny will be of all types (i,j) such that

$$\min(k,m) \leq i \leq \max(k,m)$$
$$\min(\ell,n) \leq j \leq \max(\ell,n) \qquad 1)$$

Loosely speaking, we may call each index of a pair a "characteristic." A given mating will produce all possible children such that 1) is satisfied, the distribution of the two indices determining the progeny being the product of two identical distributions. The <u>number</u> of children will, of course, be proportional to the number of parents of each type. Mathematically:

$$x_{ij}^{(n+1)} = \sum_{k,\ell,m,n} \gamma_i^{km} \gamma_j^{\ell n} x_{k\ell}^{(n)} x_{mn}^{(n)} \qquad 2)$$

The sum in 2) is to be carried out under the restriction 1). We specify the system further by postulating:

$$\gamma_i^{km} = \gamma_i^{mk} > 0, \quad \min(k,m) \leq i \leq \max(k,m)$$
$$= 0 \quad \text{otherwise;} \qquad 3)$$

$$\sum_{i=m}^{k} \gamma_i^{km} = 1 \qquad 4)$$

$$\sum_{i=m}^{k} i\, \gamma_i^{km} = \frac{m+k}{2} \qquad 5)$$

In addition, we normalize by taking:

$$\sum_{i,j=1}^{N} x_{ij}^{(o)} = 1, \quad 0 \leq x_{ij}^{(o)} \leq 1, \quad \text{all } i,j \qquad 6)$$

It is then evident that we have:

$$\sum_{i,j}^{N} x_{ij}^{(n)} = 1$$

for all n.

The class of systems defined above is the one we shall actually discuss. However, it may be of interest to at least mention the actual evolutionary model in connection with which equation 2) arises. Evolution is assumed to take place by mutation. A type (i,j) can give rise to two new types (i+1,j) and (i,j+1) with some small probability. When we include this (linear) effect, we get a set of equations:

$$x_{ij} = -\epsilon\, x_{ij} + \epsilon/2(x_{i-1,j} + x_{i,j-1}) + \sum \gamma_i^{km}\, \gamma_j^{\ell n}\, x_{k\ell}\, x_{mn} \qquad 7)$$

Here ϵ is taken as some small number. We actually performed many numerical experiments on systems of the form 7) with special values of the γ_i^{km} satisfying 3), 4), 5). Two particularly convenient choices are:

$$\gamma_i^{jk} = \frac{1}{2^{|j-k|}} \binom{|j-k|}{i-\min(j,k)} \qquad 8)$$

and

$$\gamma_i^{jk} = \frac{1}{|j-k| + 1}, \quad \min(j,k) \leq i \leq \max(j,k) \qquad 9)$$

$$= 0 \quad \text{otherwise}$$

Note that with our definition 3) we may take the sum in 2) as unrestricted, i.e., over-all $k,\ell,m,n = 1,2, \ldots, N$.

We shall not discuss the behavior of the "mutating" system 7) in this report, but rather restrict ourselves to the pure "mating" system 2).[1]

3. If we sum over one of the indices in 2), we obtain the system:

$$C_i^{(n+1)} = \sum_{k,m=1}^{N} \gamma_i^{km} C_k^{(n)} C_m^{(n)} \qquad i = 1, \ldots, N \qquad 10)$$

$$C_i \equiv \sum_{j=1}^{N} x_{ij} \qquad 11)$$

and, of course,

$$\sum_{i=1}^{N} C_i^{(n)} = 1 \qquad \text{all } n. \qquad 12)$$

By virtue of condition 5) on the γ_i^{km}, the system 10) possesses a <u>linear invariant</u> (distinct from $\sum C_i = 1$); in fact:

$$\sum_j {}_j\, C_j^{(n+1)} = \sum_{\ell,m} C_\ell^{(n)} C_m^{(n)} \sum_j {}_j\, \gamma_j^{\ell m} = \sum_{\ell,m} C_\ell^{(n)} C_m^{(n)} \left(\frac{\ell+m}{2}\right)$$

$$= \sum_\ell {}_\ell\, C_\ell^{(n)} \qquad 13)$$

---

(1) It may be objected that our mating rules have nothing to do with Mendel's laws. This is intentionally the case. The Mendelian case has been treated in great generality in a series of papers by Hilda Geiringer; cf., e.g., <u>Annals of Mathematical Statistics</u>, <u>15</u> (1944), pp. 25-57, <u>Genetics</u>, <u>33</u> (1948), pp. 548-564, etc. See also C. C. Li: <u>Population Genetics</u>, University of Chicago, Chicago, 1955.

The consequences of this property are very interesting. It turns out that the existence of this linear invariant enables one to predict explicitly the limiting behavior of any initial vector $(C_1^{(o)}, C_2^{(o)}, \ldots C_N^{(o)})$ when iterated according to 10). Using the fact that $\sum C_i = 1$ is also an invariant, we may define an invariant:

$$\sigma \equiv \sum_{i=1}^{N-1} (N-i)C_i \qquad 14)$$

σ is, of course, explicitly determined by the initial vector. It can then be proved that every initial vector will converge to a definite fixed point which is determined as follows:

For the given value of σ, there is one index j such that:

$$N - j \geq \sigma > N - j - 1 \qquad 15)$$

The f.p. is then explicitly given by:

$$\left.\begin{array}{l} C_j = \sigma - (N-j-1) \\ C_{j+1} = N - j - \sigma \\ \text{all other } C_i = 0 \end{array}\right\} \qquad 16)$$

The f.p. is independent of the actual values of the coefficients

γ_i^{km} providing they satisfy 3), 4), 5).[2]

These results can easily be referred back to the original variables x_{ij}. Defining a quantity α from 16) by:

$$c_j \equiv \frac{1}{1+\alpha}, \qquad 17)$$

we find that the corresponding symmetric tensor $(x_{mn}^{(o)})$ will converge to the final state:

$$x_{jj} = \frac{1}{(1+\alpha)^2}$$

$$x_{j+1,j} = x_{j,i+1} = \frac{\alpha}{(1+\alpha)^2} \qquad 18)$$

$$x_{j+1,j+1} = \frac{\alpha^2}{(1+\alpha)^2}$$

---

[2] Of course the <u>rate</u> of convergence will, in general, depend on the actual values of the γ_i^{km}. In one case it is actually possible to solve explicitly for the iterates as functions of n. This is the case N = 2 where we find:

$$x_{11}^{(n)} = \frac{1}{2^n}\left[x_{11}^{(o)} + \sigma^2(2^n-1)\right]$$

$$\sigma = x_{11}^{(o)} + x_{12}^{(o)}.$$

This case can be viewed as a generalization of the Mendelian law for a single gene. The population has the same limiting configuration in both cases, but for Mendel's rules equilibrium is reached in a single step ("Hardy's Law").

The results can also be extended to the case of M "characteristics" $i_1, i_2, \ldots i_M$.

The fraction of the population of type $(i_1, \ldots, i_M)$ is then denoted by $x_{i_1 \ldots i_M}$. This is taken to be symmetric in all M indices; hence, there are:

$$\binom{N+M-1}{M}$$

types of individuals in the population. The corresponding mating rule is:

$$x^{(n+1)}_{i_1 \ldots i_M} = \sum_{S_1, S_2} \sum_{S_3, S_4} \ldots \gamma_{i_1}^{S_1 S_2} \gamma_{i_2}^{S_3 S_4} \ldots \gamma_{i_M}^{S_{2M-1} S_{2M}}$$

$$x^{(n)}_{S_1 S_3 \ldots S_{2M-1}} x^{(n)}_{S_2 S_4 \ldots S_{2M}} \qquad 19)$$

If we sum over M-1 indices, we can again define:

$$C_i \equiv \sum_{i_2, \ldots, i_M = 1}^{N} x_{i_1 \ldots i_M} \qquad 20)$$

and we again get the system 10). In terms of the tensor $x_{i_1 \ldots i_M}$, the final state is:

$$x_{j,j\ldots j} = \frac{1}{(1+\alpha)^M}$$

$$x_{j+1,j,j\ldots j} = \frac{\alpha}{(1+\alpha)^M} \qquad 21)$$

$$\vdots$$

$$x_{j+1,j+1,\ldots j+1} = \frac{\alpha^M}{(1+\alpha)^M}$$

Here j and α are determined as in 16) and 17).

4. The explicit nature of these results is due to the existence of the linear invariant σ, which in turn is a consequence of the "mean-preserving" property 5) assumed for the coefficients γ_i^{jk}. However, the actual convergence properties are probably more closely connected with the "index-limiting" condition 3). (Our Binary Reaction systems do not, in general, have this property; the existence of oscillating final states may well be a consequence of allowing such mating rules as $j \oplus j \rightarrow k \neq j$.) Some cases have been investigated in which the conditions on the coefficients were relaxed. In particular, we have considered systems for which we no longer require:

$$\gamma_i^{jk} > 0 \qquad \text{all } j,k.$$

As an example, we chose the case:

$$\sum_{i=1}^{N} \gamma_i^{jk} = \delta_{j,k} + \delta_{j+1,k} + \delta_{j-1,k} \qquad 22)$$

This corresponds to a sort of "selective mating scheme" in which only individuals with "nearby" indices can mate. Of course we then no longer have $\sum C_i = 1$. In order to secure this property, we must "renormalize" at every step, i.e., set:

$$\frac{C_i'}{\sum C_i'} \rightarrow C_i'$$

If this is formally written out, the system is no longer quadratic. There are then many types of fixed points possible. So far it has not been possible to give an analytical treatment of such systems.[3]

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[3] Note, however, that if we simply take $\gamma_i^{jk} = \frac{\delta_{ij} + \delta_{ik}}{2}$, we find that every initial vector is a fixed point.