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Monotonicity conditions in the nodal mimetic finite difference method for diffusion problems on quadrilateral meshes

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Abstract

A set of sufficient and necessary conditions for the local stiffness matrix of the nodal mimetic finite difference method on quadrilaterals is derived under which such matrix is a singular M-matrix. These conditions are directly related to the shape of the quadrilaterals and to the value of the off-diagonal elements and they are illustrated by the new concept, here introduced, of *monotonicity region*. When such conditions are satisfied, the global matrix for the Poisson problem with Dirichlet boundary conditions is a singular M-matrix. After setting the boundary condition, yields a monotone matrix operator and we prove that the final discretization possesses a Discrete Maximum Principle.

Key words: Hourglass instability, diffusion problem, Poisson equation, mimetic finite difference method, polygonal mesh, generalized mesh

1. Introduction

Mimetic discretization schemes like the Support-Operator Method (SOM) [13, 18, 19] and the Mimetic Finite Difference (MFD) method proposed in mixed and mixed-hybrid form [3, 5, 6, 9–11] and very recently in nodal form [4, 8] have been successfully employed for solving elliptic problems such as linear diffusion and convection-diffusion problems [1, 11], also with domain decomposition [17], and the steady Stokes model [2, 4]. Mimetic discretization method has also relations with other recent numerical formulations [23], and those extending the finite element method to more general meshes of polygons and polyhedra [16, 20–22].

Moreover, the MFD method combines the flexibility of the finite volume formulations on general polygonal and polyhedral meshes, even with non-matching and non-convex elements, with the possibility of incorporating important properties of physical and mathematical models into the discrete framework. For example, mimetic methods may include in the numerical framework discrete versions of the integration by parts formulas, conservation laws and, also, provide solution symmetries, positivity and monotonicity. This latter property is related to the existence of maximum and minimum principles, which are among the major characteristics of the solution of an elliptic equation. For the mixed-hybrid formulation, monotonicity criteria that lead to schemes characterized by a Discrete Maximum (and Minimum) Principle (DMP) are proposed in [15].

The paper is organized as follows. In Section 2, we review the formulation of the model problem and some background material concerning the issue of maximum/minimum principles and the monotonicity property. In Section 3 we review the construction of the nodal mimetic formulation [8]. In Section 4, we discuss necessary and sufficient monotonicity conditions on the construction of the elemental stiffness matrix that forces the M-matrix property. In Section 5, we offer our final remarks, perspectives for future developments and conclusions.

2. Background Material

2.1. Model problem

Let $\Omega \subset \mathbb{R}^2$ be the computational domain and $\Gamma = \Gamma^D \cup \Gamma^N$ its boundary. The diffusion of the scalar variable u in Ω is governed by the Poisson problem

$$-\operatorname{div}(\mathbf{K}\nabla u) = f \quad \text{in } \Omega, \quad (1)$$

$$u = g \quad \text{on } \Gamma^D, \quad (2)$$

$$\mathbf{n} \cdot \nabla u = g^N \quad \text{on } \Gamma^N, \quad (3)$$

where $\mathbf{K}$ is the diffusion tensor coefficient, f is the volumetric source, g is the Dirichlet condition defined on $\Gamma^D = \bar{\Gamma}^D \subseteq \Gamma$, $\mathbf{n}$ is the unit normal vector to Γ pointing out of Ω , and g^N is the Neumann condition defined on $\Gamma^N \subset \Gamma$. We require that $\Gamma = \Gamma^D \cup \Gamma^N$. To guarantee the uniqueness of the solution to (1)-(3) and to the discrete problem that will be defined in the next section, we also require a positive measure of Γ^D , i.e. $|\Gamma^D| > 0$. Moreover, we assume that f , g and g^N are given functions that belong to $L^2(\Omega)$, $H^{1/2}(\Gamma^D)$ and $H^{-1/2}(\Gamma^N)$, respectively. We also assume that $\mathbf{K}$ is a *strongly elliptic* tensor on Ω , i.e., $\mathbf{K}(\mathbf{x})$ is a symmetric and positive definite (SPD) matrix for every point $\mathbf{x} \in \Omega$. From this assumption it follows immediately that $\mathbf{K}(\mathbf{x})$ is a non-singular matrix for every $\mathbf{x} \in \Omega$, and that $\mathbf{K}^{-1}(\mathbf{x})$ is also a symmetric and positive definite matrix for every $\mathbf{x} \in \Omega$. We consider the functional subset of $H^1(\Omega)$

$$X_{\mathbf{g}} = \{v \in H^1(\Omega) \mid v|_{\Gamma^D} = g\},$$

and the linear subspace X_0 for $g = 0$. The variational form of problem (1)-(3) reads as

find $u \in X_{\mathbf{g}}(\Omega)$ such that

$$\int_{\Omega} \mathbf{K}\nabla u \cdot \nabla v \, dV = \int_{\Omega} f v \, dV + \int_{\Gamma^N} g^N v \, dS \quad \forall v \in X_0(\Omega). \quad (4)$$

Under the regularity assumptions discussed above, problem (4) is well-posed [12]. Existence and uniqueness of the weak solutions of the variational formulation follows from the coerciveness and boundedness of the bilinear form in the left-hand side of (4) [12].

2.2. Maximum and minimum principles and monotonicity

In the theory and applications of linear and nonlinear elliptic problems, the Maximum Principles play a fundamental role. We recall the following two famous results.

Theorem 2.1 (Strong Maximum Principle) *Let us suppose that u satisfies*

$$-\operatorname{div}(\mathbf{K}\nabla u) \leq 0 \quad \text{in } \Omega$$

under the assumption listed above on $\mathbf{K}$ and Ω . If u attains a nonnegative maximum $\hat{u}$ at an interior point of Ω , then:

$$u = \hat{u} \quad \text{in } \bar{\Omega}.$$

Theorem 2.2 (Weak Maximum Principle) *Let us suppose that u satisfies*

$$-\operatorname{div}(\mathbf{K}\nabla u) \leq 0 \quad \text{in } \Omega$$

under the assumption listed above on $\mathbf{K}$ and Ω . Then,

$$\max_{\mathbf{x} \in \bar{\Omega}} u(\mathbf{x}) \leq \max \left(0, \max_{\mathbf{x} \in \Gamma} u(\mathbf{x}) \right).$$

From the Weak Maximum Principle it is immediate to derive a *monotonicity property* for the Dirichlet problem. In case of mixed boundary conditions, the monotonicity property is as follows [14].

Corollary 2.1 (Monotonicity Property) *Let us suppose that u satisfies*

$$\begin{aligned} -\operatorname{div}(\mathbf{K}\nabla u) &\geq 0 \quad \text{in } \Omega, \\ \mathbf{n} \cdot \mathbf{K}\nabla u &\geq 0 \quad \text{on } \Gamma^N, \\ u &\geq 0 \quad \text{on } \Gamma^D, \end{aligned}$$

under the assumption listed above on $\mathbf{K}$ and Ω . Then,

$$u \geq 0 \quad \text{in } \Omega.$$

Many discretization schemes have been developed that try to reproduce this important property at the numerical level. In this work, we investigate how a discrete version of this property, the *Discrete Maximum Principle* (DMP), can be incorporated into the nodal mimetic formulation.

2.3. Algebraic monotonicity conditions

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{x} = (x_i)$ be a generic matrix and a generic vector. Following the notation used in [7], we say that:

- (i) $\mathbf{A}$ and $\mathbf{x}$ are a *nonnegative matrix* and a *nonnegative vector*, respectively, if $a_{ij} \geq 0$ and if $x_i \geq 0$ for any i, j . We denote this property by $\mathbf{A} \geq 0$ and $\mathbf{x} \geq 0$;
- (ii) $\mathbf{A}$ and $\mathbf{x}$ are a *positive matrix* and a *positive vector*, respectively, if $\mathbf{A} \geq 0$ and $a_{ij} > 0$ for some i, j and if $\mathbf{x} \geq 0$ and $x_i > 0$ for some i . We denote this property by $\mathbf{A} > 0$ and $\mathbf{x} > 0$.

The two sets of matrices defined by $\{\mathbf{A} \geq 0\}$ and $\{\mathbf{A} > 0\}$ differ by the zero matrix $\mathbf{A} = 0$, which belongs to the former set. Similarly, the two sets of vectors $\{\mathbf{x} \geq 0\}$ and $\{\mathbf{x} > 0\}$, differ by the zero vector $\mathbf{x} = 0$, which belongs to the former set.

A matrix $\mathbf{A}$ is called *irreducible* if its graph is connected, that is, for any pair of indices $i \neq j$ there exists a sequence of nonzero entries $a_{i,k_1}, a_{k_1,k_2}, \dots, a_{k_s,j}$. It is called *weakly diagonally dominant* (with respect to its rows) if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad \forall i \tag{5}$$

and the inequality is strong for at least one row. Finally, a matrix $\mathbf{A}$ is called *irreducibly weakly diagonally dominant* if it is both irreducible and weakly diagonally dominant.

We list below some other fundamental definitions, which are also taken from [7], that will be useful throughout the paper. Let $\mathbf{A} = (a_{ij})_{i,j=1}^n$.

- (i) A nonsingular matrix $\mathbf{A}$ is called a *monotone matrix* if $\mathbf{A}^{-1} > 0$. If $\mathbf{A}\mathbf{x} = \mathbf{b}$, then

$$\mathbf{b} \geq 0 \quad \text{implies} \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \geq 0;$$

- (ii) A matrix $\mathbf{A}$ is called a *Z-matrix* if $a_{ij} \leq 0$ for $i \neq j$.
- (iii) A nonsingular matrix $\mathbf{A}$ is called an *M-matrix* if $a_{ij} \leq 0$ for $i \neq j$ and $\mathbf{A}^{-1} > 0$.
- (iv) A symmetric positive definite M-matrix $\mathbf{A}$ is called a *Stieltjes matrix*.
- (v) A matrix $\mathbf{A}$ is called a *singular M-matrix* if $\epsilon \mathbf{I} + \mathbf{A}$ is an M-matrix for any $\epsilon \in (0, \epsilon_0)$ with some sufficiently small ϵ_0 .

Fig. 1. The left plot illustrates the diagonal vector $\mathbf{D}_j$ and the local subregions T_j and $\hat{\mathsf{T}}_j$ associated with the vertex $\mathbf{v}_j$ in a convex-shaped cell. The right plot illustrates the diagonal vectors $\mathbf{D}_1$ and $\mathbf{D}_2$ and the local subregions T_2 , T_3 , and T_4 in a nonconvex-shaped cell. The subregion T_1 (not marked in the picture) is the triangle with vertices 1, 2, and 4. Note that the area of T_3 is negative. In both pictures, the subscript j runs from 1 to 4 following a local counterclockwise ordering.

We collect three fundamental properties of nonnegative matrices that we will use later. For their proof we refer the reader to [7].

Proposition 2.1 *Let $\mathbf{A} = (a_{ij})_{i,j=1}^n$.*

- (i) *If $\mathbf{A}$ is an irreducibly weakly diagonally dominant Z-matrix with positive diagonal entries, then $\mathbf{A}$ is an M-matrix.*
- (ii) *If $\mathbf{A}$ is an irreducibly weakly diagonally dominant M-matrix, then*

$$a_{ii} > 0 \quad \forall i \quad \text{and} \quad \mathbf{A}\mathbf{e} > 0.$$

- (iii) *If $\mathbf{A}$ is a Stieltjes matrix, then it is an M-matrix and, therefore, it is a monotone matrix.*

3. Mimetic setting on quadrilateral grids

3.1. Notation

We suppose that the polygonal domain Ω is partitioned into a mesh of quadrilaterals Ω_h . We denote a generic vertex of the mesh by $\mathbf{v}$, its coordinate vector by $\mathbf{x}_\mathbf{v}$, and the set of all mesh vertices by $\mathcal{V}$; we denote a generic face of the mesh by $\mathbf{f}$, its length by $|\mathbf{f}|$, and the set of all mesh faces by $\mathcal{F}$; we denote a generic quadrilateral cell of the mesh by $\mathbf{Q}$, its area by the same symbol $|\mathbf{Q}|$ and the coordinate vector of its center of gravity by $\mathbf{x}_\mathbf{Q}$.

We denote the set of all the mesh faces by $\mathcal{F}$ and the subset of the faces containing the vertex $\mathbf{v}$ by $\mathcal{F}^\mathbf{v}$; we also denote the set of the mesh cells containing the vertex $\mathbf{v}$ by $\Omega_h^\mathbf{v}$.

For each quadrilateral cell $\mathbf{Q}$ we consider the following geometric objects: the four oriented diagonal vectors $\mathbf{D}_j$ for $j = 1, \dots, 4$, the four triangular subregions T_j for $j = 1, \dots, 4$ delimited by $\mathbf{D}_j$ and the two segments incident in the j -th vertex and their complement $\hat{\mathsf{T}}_j$ with respect to $\mathbf{Q}$, i.e., $\hat{\mathsf{T}}_j = \mathbf{Q} \setminus \mathsf{T}_j$. Observe that $\mathbf{D}_3 = -\mathbf{D}_1$ and $\mathbf{D}_4 = -\mathbf{D}_2$, so that we will express all results in terms of $\mathbf{D}_1$ and $\mathbf{D}_2$ only. Furthermore, $\hat{\mathsf{T}}_1 = \mathsf{T}_3$, $\hat{\mathsf{T}}_2 = \mathsf{T}_4$, $\hat{\mathsf{T}}_3 = \mathsf{T}_1$, and $\hat{\mathsf{T}}_4 = \mathsf{T}_2$. The oriented areas of the subcells T_j and $\hat{\mathsf{T}}_j$ are denoted by the same symbols T_j and $\hat{\mathsf{T}}_j$, respectively. Note that $|\mathbf{Q}| = \hat{\mathsf{T}}_1 + \hat{\mathsf{T}}_3 = \hat{\mathsf{T}}_2 + \hat{\mathsf{T}}_4$; also, note that one (and only one) of the four oriented areas $\hat{\mathsf{T}}_i$ is always negative if $\mathbf{Q}$ has a non-convex shape. We also associate the constant tensor $\mathbf{K}_\mathbf{Q}$, which is a suitable approximation of the diffusion tensor $\mathbf{K}$ on $\mathbf{Q}$, with each quadrilateral cell $\mathbf{Q}$. We approximate the scalar fields in $X_\mathbf{g}$ through the set of discrete scalar fields $\mathcal{V}_h$ in accordance with the correspondence

$$u, v \in H^1(\Omega) \cap C^0(\overline{\Omega}) \quad \longrightarrow \quad u_h, v_h \in \mathcal{V}_h. \quad (6)$$

A field v_h in $\mathcal{V}_h$ consists of one real number per mesh vertex, e.g. $v_\mathbf{v} := v_h|_\mathbf{v}$. These numbers are the *degrees of freedom* of the numerical method and the dimension of the affine space $\mathcal{V}_h$ equals the number of mesh vertices. The restriction of the discrete field v_h to cell $\mathbf{Q}$, e.g. $v_\mathbf{Q} := v_h|_\mathbf{Q}$, belongs to the local set $\mathcal{V}_{h,\mathbf{Q}} := \mathcal{V}_h|_\mathbf{Q}$ with dimension equal to the number of vertices of $\mathbf{Q}$.

The degrees of freedom of a function v are the pointwise values given through the interpolant v^I , which is defined as

$$v_v^I = v^I|_v := v(\mathbf{x}_v) \quad \forall v \in \Omega_h. \quad (7)$$

The local interpolation of v is the discrete field $v_Q^I = v^I|_Q = \{v_v^I\}_{v \in \partial Q}$ of $\mathcal{V}_{h,Q}$ and is defined from $H^1(Q) \cap C^0(\bar{Q})$ to $\mathcal{V}_{h,Q}$ for every cell Q of Ω_h . The global interpolation operator is defined from $H^1(\Omega) \cap C^0(\bar{\Omega})$ to $\mathcal{V}_h$ and combines the local ones.

In the nodal MFD method we approximate the left-hand side of (4) through the bilinear form $\mathcal{A}_h : \mathcal{V}_h \times \mathcal{V}_h \rightarrow \mathbb{R}$, so that

$$\mathcal{A}_h(u^I, v^I) \approx \int_{\Omega} \mathbf{K} \nabla u \cdot \nabla v \, dV,$$

and the right-hand side of (4) through the linear functional $\mathcal{L}_h : \mathcal{V}_h \rightarrow \mathbb{R}$, so that

$$\mathcal{L}_h(v^I) \approx \int_{\Omega} f v \, dV + \int_{\Gamma^N} g^N v \, dS.$$

The construction of the bilinear form $\mathcal{A}_h$ and the linear functional $\mathcal{L}_h$ for the degrees of freedom of the linear space $\mathcal{V}_h$ is fully detailed in the next subsections. We numerically treat the Dirichlet boundary conditions of non-homogeneous type by seeking the mimetic solution u_h in $\mathcal{V}_{h,g} \subset \mathcal{V}_h$, the subset of the discrete fields v_h such that:

$$v_v = g(\mathbf{x}_v) \quad \forall v \in \Gamma^D.$$

The subspace $\mathcal{V}_{h,0}$ is defined by setting $g = 0$ in the definition of $\mathcal{V}_{h,g}$.

The mimetic finite difference method that numerically approximates the variational problem (4) reads as:

Find $u_h \in \mathcal{V}_{h,g}$ such that:

$$\mathcal{A}_h(u_h, v_h) = \mathcal{L}_h(v_h) \quad \forall v_h \in \mathcal{V}_{h,0}. \quad (8)$$

The coercivity and the continuity of the bilinear form $\mathcal{A}_h$ imply the well-posedness of the numerical approximation (8). Such properties are discussed in subsection 3.2.

3.2. The mimetic bilinear form $\mathcal{A}_h$

The mimetic bilinear form $\mathcal{A}_h$ assembles the local bilinear forms $\mathcal{A}_{h,Q} : \mathcal{V}_{h,Q} \times \mathcal{V}_{h,Q} \rightarrow \mathbb{R}$ defined for each mesh cell Q and acts on the local degrees of freedom $v_Q = v_h|_Q = \{v_v\}_{v \in \partial Q} \in \mathcal{V}_{h,Q}$ for $v_h \in \mathcal{V}_h$. Formally, it holds that

$$\mathcal{A}_h(u_h, v_h) = \sum_{Q \in \Omega_h} \mathcal{A}_{h,Q}(u_Q, v_Q) \quad \forall u_h, v_h \in \mathcal{V}_h. \quad (9)$$

To get a proper characterization of the *local* bilinear forms $\mathcal{A}_{h,Q}$ we proceed along the following three steps.

In the *first step*, we introduce $\mathcal{I}_f(\{v_v\}_{v \in \partial f})$, a quadrature rule for each face f that integrates the linear polynomials exactly. We assume that:

(Q1) for every face f there exists a set of non-negative weights $\{\omega_v^f\}_{v \in \partial f}$ associated with the vertices v of face f and such that

$$\sum_{v \in \partial f} \omega_v^f = |f| \quad \text{and} \quad \sum_{v \in \partial f} (\mathbf{x}_v - \mathbf{x}_f) \omega_v^f = 0, \quad (10)$$

where $\mathbf{x}_v$ is the position vector of v and $\mathbf{x}_f$ is the center of gravity of f .

In two dimensions, the obvious choice for the numerical integration rule $\mathcal{I}_f$ is given by the trapezoidal rule:

$$\mathcal{I}_f(\{v_v, v_{v'}\}) = \frac{|f|}{2} (v_v + v_{v'}), \quad (11)$$

where v_v and $v_{v'}$ are the nodal values of v_h at the vertices v and v' delimiting the line segment f . In three dimensions, the weights $\{\omega_v^f\}_{v \in \partial f}$ that define $\mathcal{I}_f(\{p_v^I\}_{v \in \partial f})$ are easily constructed by expressing $\mathbf{x}_f$, the center of gravity of face F , as a linear combination of the vectors $\{\mathbf{x}_v\}$ for $v \in \partial f$.

In the *second step*, we approximate $K|_Q$, the restriction of the diffusion tensor K on element Q , by a constant tensor K_Q , e.g., the value of K at the center of gravity of Q or its average over Q . The diffusion tensor K , and, consequently, K_Q , is assumed to be strongly elliptic and this assumption is sufficient to ensure the well-posedness of the numerical formulation in the sense of existence and uniqueness of the numerical solution. However, a stronger regularity assumption on K , as, for example, that each component K_{ij} be locally Lipschitz continuous on Ω_h , allows us to derive the upper bound

$$\max_{i,j=1,d} \sup_{x \in Q} |(K_Q)_{ij} - K_{ij}(x)| \leq C_K^* h_Q, \quad (12)$$

where C_K^* is a non-negative constant independent of h_Q and of Q . To this purpose, it is formally sufficient to require that the components of $K|_Q$ (and, therefore, of K_Q^{-1}) are in $W^{1,\infty}(Q)$ for every $Q \in \Omega_h$. Inequality (12) is used to prove the convergence of the method, see [8].

In the *third step*, we impose that the bilinear form $\mathcal{A}_{h,Q}$ satisfies the two conditions of *spectral stability* and *local consistency*. Space $\mathcal{V}_h$ is endowed with the H^1 -like mesh-dependent norm

$$\|v_h\|_{1,h}^2 = \sum_{Q \in \Omega_h} \|v_h\|_{1,h,Q}^2, \quad (13)$$

where the summation argument is given by

$$\|v_h\|_{1,h,Q}^2 = |Q| \sum_{e \in \partial Q} \frac{|v_e^{v'} - v_e^v|^2}{|e|^2}, \quad (14)$$

and v and v' are the vertices connected by edge e , i.e. $e = \overline{vv'}$ (note that in this definition we must distinguish between edges and faces).

(S1) *spectral stability*: there exists two positive constants σ_* and σ^* such that for every $v_Q \in \mathcal{V}_{h,Q}$ there holds:

$$\sigma_* \|v_Q\|_{1,h,Q}^2 \leq \mathcal{A}_{h,Q}(v_Q, v_Q) \leq \sigma^* \|v_Q\|_{1,h,Q}^2;$$

(S2) *local consistency*: for every $p \in \mathbb{P}_1(Q)$ and every $v_h \in \mathcal{V}_h$ there holds:

$$\mathcal{A}_{h,Q}(v_Q, p^I) = K_Q \nabla p \cdot \sum_{f \in \partial Q} \mathbf{n}_{Q,f} \mathcal{I}_f(v_Q).$$

Assumptions (S1)-(S2) are used in the practical implementation of the method, cf. subsection 3.5.

3.3. Discretization of the load term $\mathcal{L}_h$

The discretization of the load term $\mathcal{L}_h$ is based on the existence of a volume integration rule, denoted by $\mathcal{I}_Q(\cdot)$, that integrates the linear polynomials exactly. We assume that

(Q2) for every polyhedron $Q \in \Omega_h$ there exists a set of non-negative weights $\{\omega_Q^v\}_{v \in \partial Q}$ associated with the vertices of Q and such that

$$\sum_{v \in \partial Q} \omega_Q^v = |Q| \quad \text{and} \quad \sum_{v \in \partial Q} (\mathbf{x}_v - \mathbf{x}_Q) \omega_Q^v = 0, \quad (15)$$

where $\mathbf{x}_v$ is the position vector of the vertex v of ∂Q and $\mathbf{x}_Q$ is the center of gravity of Q .

Assumption (Q2) is stronger than the corresponding assumption for the cell integration rule considered in [8]. Indeed, in [8] only exact integration of constant fields is required and a possible choice of weights is given by $\omega_Q^v = |Q|/m_Q$ for every $v \in \partial Q$ where m_Q is the number of vertices of Q . The exactness on affine functions is required to have an $\mathcal{O}(h^2)$ approximation of the cell average value. Let $\bar{f}_Q$ be either the cell average of f on Q , or, when f is sufficiently regular, the pointwise value $f(\mathbf{x}_Q)$. Using the weights of (Q2), we approximate the forcing term by using the linear functional $(f, \cdot)_h : \mathcal{V}_h \rightarrow \mathbb{R}$ that is defined by

$$(f, v_h)_h = \sum_{Q \in \Omega_h} [f, v_h]_Q \quad \text{where} \quad [f, v_h]_Q = \bar{f}_Q \sum_{v \in \mathcal{V}_Q} v_Q^v \omega_Q^v. \quad (16)$$

Using the weights of the quadrature rule in (Q1), we approximate the Neumann boundary term by using the linear functional $\langle g^N, \cdot \rangle_h : \mathcal{V}_h \rightarrow \mathbb{R}$ that is defined by

$$\langle g^N, v_h \rangle_h = \sum_{f \in \Gamma^N} \mathcal{I}_f(\{g^N(\mathbf{x}_v)v_v\}_{v \in \partial f}). \quad (17)$$

Eventually, the right-hand side of (4) is approximated by the linear functional $\mathcal{L}_h : \mathcal{V}_h \rightarrow \mathbb{R}$ that is defined by:

$$\mathcal{L}_h(\mathbf{v}_h) = (f, v_h)_h + \langle g^N, v_h \rangle_h.$$

3.4. Convergence theorem

We recall the convergence result in the mesh-dependent norm defined by (13)-(14). in the theorem below. The proof is found in [8].

Theorem 3.1 *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of problem (1)-(2) (with $g = 0$ and $\Gamma^N = \emptyset$) in variational form, $u^I = (u(\mathbf{x}_v))_{v \in \mathcal{V}}$ the restriction of u to the mesh vertices in $\mathcal{V}$, and u_h the numerical solution provided by (8). Let h be the maximum length of the quadrilateral faces of the meshes, and assume that the aspect ratio of the cells is uniformly bounded from above and below for each mesh considered in the approximation process for $h \rightarrow 0$. Then, there exists a positive constant C independent of h such that*

$$\|u_h - u^I\|_{1,h} \leq Ch(|u|_{H^2(\Omega)} + \|f\|_{L^2(\Omega)}).$$

3.5. Implementation of the method on quadrilateral cells

Each bilinear form $\mathcal{A}_{h,Q}$ is implemented through a symmetric and positive definite matrix $\mathbf{M}$ (the super-script “m” stands for *mimetic*), which is such that

$$\mathcal{A}_{h,Q}(u_Q, v_Q) = (v_Q)^T \mathbf{M} u_Q \quad \forall u_Q, v_Q \in \mathcal{V}_{h,Q},$$

where u_Q and v_Q are here identified with algebraic vectors collecting the local degrees of freedom of u_Q and v_Q , respectively. To derive a formula for matrix $\mathbf{M}$, we use the two additional matrices $\mathbf{N}$ and $\mathbf{R}$ with size 4×2 , which are defined as follows. The columns of matrix $\mathbf{N}$ are the interpolated fields $1^I, x^I, y^I$. The columns of matrix $\mathbf{R}$ are correspondingly obtained from (S2) by taking p in $\{1, x, y\}$. Accordingly, the i -th row of $\mathbf{N}$ is the row vector $(1, (\mathbf{x}_{v_i}^T)^I)$ where v_i is the i -th vertex of Q and the i -th row of $\mathbf{R}$ is given by

$$\left(0, \sum_{f \in \partial Q \cap \mathcal{F}^v} (\mathbf{K}_Q \mathbf{n}_{Q,f})^T \mathcal{I}_f(v_h^{v_i})\right),$$

where the summation index is running over the two faces of ∂Q that contain vertex v . Using a local enumeration of the vertices and faces of Q , e.g., v_i and f_i for $i = 1, \dots, m_Q$, we have the explicit formulas:

$$\mathbf{N} = \begin{pmatrix} 1 & \mathbf{x}_{v_1}^T \\ 1 & \mathbf{x}_{v_2}^T \\ \vdots & \vdots \\ 1 & \mathbf{x}_{v_{m_Q}}^T \end{pmatrix} = (\mathbf{1}, \widehat{\mathbf{N}}) \quad (18)$$

and

$$\mathbf{R} = \frac{1}{2} \begin{pmatrix} 0 & |\mathbf{f}_{m_Q}| \mathbf{n}_{\mathbf{f}_{m_Q}}^T + |\mathbf{f}_1| \mathbf{n}_{\mathbf{f}_1}^T \\ 0 & |\mathbf{f}_1| \mathbf{n}_{\mathbf{f}_1}^T + |\mathbf{f}_2| \mathbf{n}_{\mathbf{f}_2}^T \\ \vdots & \vdots \\ 0 & |\mathbf{f}_{m_Q-1}| \mathbf{n}_{\mathbf{f}_{m_Q-1}}^T + |\mathbf{f}_{m_Q}| \mathbf{n}_{\mathbf{f}_{m_Q}}^T \end{pmatrix} \mathbf{K}_Q = (0, \widehat{\mathbf{R}}), \quad (19)$$

where $\mathbb{1} = (1, 1, 1, 1)^T$; $\widehat{\mathbf{N}}$ and $\widehat{\mathbf{R}}$ are two suitably defined $m_Q \times d$ dimensional matrices. The local consistency condition (S2) can be reformulated as $\mathbf{M}\mathbf{N} = \mathbf{R}$, or, equivalently, as $\mathbf{M}\widehat{\mathbf{N}} = \widehat{\mathbf{R}}$. A straightforward calculation shows that

$$\mathbf{R}^T \mathbf{N} = \begin{pmatrix} 0 & 0^T \\ 0 & \widehat{\mathbf{R}}^T \widehat{\mathbf{N}} \end{pmatrix} = \begin{pmatrix} 0 & 0^T \\ 0 & |\mathbf{Q}| \mathbf{K}_Q \end{pmatrix}. \quad (20)$$

Matrix $\mathbf{M}$ can be written as:

$$\mathbf{M} = \mathbf{M}_0 + \mathbf{M}_1 \quad (21)$$

where matrix $\mathbf{M}_0 = \widehat{\mathbf{R}} \frac{\mathbf{K}_Q^{-1}}{|\mathbf{Q}|} \widehat{\mathbf{R}}^T$ is such that $\mathbf{M}_0 \mathbf{N} = \mathbf{R}$ thanks to (20) and matrix $\mathbf{M}_1$ is such that $\mathbf{M}_1 \mathbf{N} = 0$ according to condition (S2). Matrix $\mathbf{M}_1$ allows us to impose the stability condition (S1), but is not uniquely determined by (S1)-(S2). The formulation of matrix $\mathbf{M}$ that we consider in this work is provided by the proposition below.

Proposition 3.1 *The ij -th element of matrix $\mathbf{M}$ is given by*

$$(\mathbf{M})_{ij} = \frac{1}{4|\mathbf{Q}|} \mathbf{D}_i \cdot \widetilde{\mathbf{K}}_Q \mathbf{D}_j + (-1)^{i+j} \frac{\widehat{\mathbf{T}}_i \widehat{\mathbf{T}}_j}{|\mathbf{Q}|^2} \tau_Q \quad (22)$$

where τ_Q is a non-negative parameter and $\widetilde{\mathbf{K}}_Q$ is the rotated permeability tensor defined by

$$\widetilde{\mathbf{K}}_Q = \mathcal{R}^T \mathbf{K}_Q \mathcal{R} \quad \text{with} \quad \mathcal{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (23)$$

Proof. Let us first see that matrix $\mathbf{M}_0$ can be expressed through the diagonal vectors $\mathbf{D}_i$ as

$$\mathbf{M}_0 = \frac{1}{4|\mathbf{Q}|} \begin{pmatrix} \mathbf{D}_1^T \\ \mathbf{D}_2^T \\ \mathbf{D}_3^T \\ \mathbf{D}_4^T \end{pmatrix} \widetilde{\mathbf{K}}_Q \begin{pmatrix} \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \mathbf{D}_4 \end{pmatrix} \quad (24)$$

where $\widetilde{\mathbf{K}}_Q = \mathcal{R}^T \mathbf{K}_Q \mathcal{R}$. In fact, let us define the rotated diagonal vector $\mathbf{D}_i^{\mathcal{R}} = \mathcal{R} \mathbf{D}_i$, where, we recall that the matrix operator $\mathcal{R}$ formally represent a clockwise rotation of $\pi/2$. A simple geometric argument implies that $\mathbf{D}_i^{\mathcal{R}} = |\mathbf{f}_{i-1}| \mathbf{n}_{\mathbf{f}_{i-1}} + |\mathbf{f}_i| \mathbf{n}_{\mathbf{f}_i}$, and, by comparison with (19), we see that

$$\widehat{\mathbf{R}}^T = \frac{\mathbf{K}_Q}{2} (\mathbf{D}_1^{\mathcal{R}}, \mathbf{D}_2^{\mathcal{R}}, \mathbf{D}_3^{\mathcal{R}}, \mathbf{D}_4^{\mathcal{R}}) = \frac{\mathbf{K}_Q}{2} \mathcal{R} (\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3, \mathbf{D}_4). \quad (25)$$

Now, we substitute the expression of $\widehat{\mathbf{R}}$ provided by (25) into the matrix definition and we have that

$$(\mathbf{M}_0)_{ij} = \left(\widehat{\mathbf{R}} \frac{\mathbf{K}_Q^{-1}}{|\mathbf{Q}|} \widehat{\mathbf{R}}^T \right)_{ij} = \left(\frac{\mathbf{K}_Q \mathbf{D}_i^{\mathcal{R}}}{2} \right)^T \frac{\mathbf{K}_Q^{-1}}{|\mathbf{Q}|} \frac{\mathbf{K}_Q \mathbf{D}_j^{\mathcal{R}}}{2} = \frac{1}{4|\mathbf{Q}|} \mathbf{D}_i^T \mathcal{R}^T \mathbf{K}_Q \mathcal{R} \mathbf{D}_j.$$

On its turn, matrix M_1 is a one-parameter family of matrices such that $M_1 N = 0$, in order that $MN = R$. Indeed, the general form of such matrices for a quadrilateral cell is given by

$$M_1 = \mu_Q D D^T, \quad (26)$$

where D is a column vector in the kernel of N , or, alternatively by

$$M_1 = \mu_Q (I - N(N^T N)^{-1} N^T), \quad (27)$$

where

$$\mu_Q = \gamma \operatorname{trace} \left(\hat{R} \frac{K_Q^{-1}}{|Q|} \hat{R}^T \right) \quad (28)$$

and γ is a non-dimensional constant factor. The choice of the parameter γ returns the members of the matrix family of the nodal mimetic schemes.

A possible choice of matrix D such that $N^T D = 0$ is given by $D^T = (-\hat{T}_1, \hat{T}_2, -\hat{T}_3, \hat{T}_4)$. In fact,

$$N^T D = \begin{pmatrix} -\hat{T}_1 + \hat{T}_2 - \hat{T}_3 + \hat{T}_4 \\ -x_1 \hat{T}_1 + x_2 \hat{T}_2 - x_3 \hat{T}_3 + x_4 \hat{T}_4 \\ -y_1 \hat{T}_1 + y_2 \hat{T}_2 - y_3 \hat{T}_3 + y_4 \hat{T}_4 \end{pmatrix}.$$

The first entry is zero because $\hat{T}_1 + \hat{T}_3 = \hat{T}_2 + \hat{T}_4 = |Q|$. The second entry is zero because it can be rewritten as:

$$-x_1 \hat{T}_1 + x_2 \hat{T}_2 - x_3 \hat{T}_3 + x_4 \hat{T}_4 = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}.$$

The third entry is also zero for the same argument because it can be rewritten as a determinant of a matrix that has the third and the fourth row equal.

Moreover, the connection between matrix M_1 and matrix M_1 in the form (27) also follows from the identity

$$I - N(N^T N)^{-1} N^T = \frac{1}{\hat{T}_1^2 + \hat{T}_2^2 + \hat{T}_3^2 + \hat{T}_4^2} \begin{pmatrix} -\hat{T}_1 \\ \hat{T}_2 \\ -\hat{T}_3 \\ \hat{T}_4 \end{pmatrix} (-\hat{T}_1, \hat{T}_2, -\hat{T}_3, \hat{T}_4) \quad (29)$$

that can be proved by a straightforward calculation. We substitute the identity

$$\operatorname{trace} \left(\hat{R} \frac{K_Q^{-1}}{|Q|} \hat{R}^T \right) = \frac{D_1 \cdot K_Q D_1 + D_2 \cdot K_Q D_2}{Q},$$

in (28), we use (29) and we obtain the relation:

$$M_1 = \gamma \left(\frac{D_1 \cdot K_Q D_1 + D_2 \cdot K_Q D_2}{Q} \right) \frac{1}{\tau_Q} \left(\frac{|Q|^2}{\sum_{i=1}^4 \hat{T}_i^2} \right) M_1. \quad (30)$$

It also holds that $|Q|^2 / \sum_{i=1}^4 \hat{T}_i^2 = 1$ if and only if Q is a parallelogram. This fact suggests us to interpret such factor as a measure of the deviation of Q from a parallelogram. $\square$

Proposition 3.2 *Let $M = M_0 + M_1$, cf. (21), where matrix $M_0 = \hat{R} \frac{K_Q^{-1}}{|Q|} \hat{R}^T$ is such that $M_0 N = R$ thanks to (20) and matrix M_1 is such that $M_1 N = 0$ according to condition (S2). Then, it holds that*

$$\ker(M) = \operatorname{span}\{\mathbb{1}\}.$$

Proof. By a straightforward calculation we have that $\mathbf{M}\mathbb{1} = \mathbf{M}_0\mathbb{1} + \mathbf{M}_1\mathbb{1} = \mathbf{M}_0\mathbb{1}$ because $\mathbf{M}_1\mathbf{N} = 0$ and $\mathbb{1}$ is the first column of $\mathbf{N}$. Moreover, let us note that $\mathbf{R}\mathbb{1} = [0, \widehat{\mathbf{R}}\mathbb{1}]$ and

$$\widehat{\mathbf{R}}\mathbb{1} = \sum_{f \in \partial Q} |f| \mathbf{n}_{Q,f} = 0,$$

which implies that $\mathbf{M}_0\mathbb{1} = 0$, and, hence, that $\text{span}\{\mathbb{1}\} \subseteq \ker(\mathbf{M})$. The proof is completed by noting that $\text{rank}(\mathbf{M}) = 3$, which implies that $\dim(\ker(\mathbf{M})) = 1$. $\square$

4. Discrete monotonicity and maximum principle

Under a few assumptions, which depends only on the geometric shape of Q and the cell approximation $\mathbf{K}_Q$ of the permeability tensor $\mathbf{K}$, there exists a range of values for each τ_Q such that the global matrix that is originated by the assembly process (9) is a non-singular M-matrix. The stiffness matrix $\mathbf{M}_0$ of (24) takes the very peculiar block-structured form:

$$\mathbf{M}_0 = \frac{1}{4|Q|} \begin{pmatrix} \mathbf{S} & -\mathbf{S} \\ -\mathbf{S} & \mathbf{S} \end{pmatrix} \quad \text{where} \quad \mathbf{S} = \begin{pmatrix} \mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_1 & \mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2 \\ \mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2 & \mathbf{D}_2 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2 \end{pmatrix}.$$

Proposition 4.1 *Matrix $\mathbf{M}$ is an M-matrix if Q is a convex quadrilateral cell and the parameter τ_Q satisfies the inequalities*

$$\max \left(\frac{-\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\min(\widehat{\mathbf{T}}_1 \widehat{\mathbf{T}}_4, \widehat{\mathbf{T}}_2 \widehat{\mathbf{T}}_3)}, \frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\min(\widehat{\mathbf{T}}_1 \widehat{\mathbf{T}}_2, \widehat{\mathbf{T}}_3 \widehat{\mathbf{T}}_4)} \right) \leq \frac{4\tau_Q}{|Q|} \leq \min \left(\frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_1}{\widehat{\mathbf{T}}_1 \widehat{\mathbf{T}}_3}, \frac{\mathbf{D}_2 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\widehat{\mathbf{T}}_2 \widehat{\mathbf{T}}_4} \right).$$

A sufficient condition for Q to ensure that the set of possible values of τ_Q is non-empty is given by:

$$|\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2| \leq 4 \min(\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_1, \mathbf{D}_2 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2). \quad (31)$$

If the quadrilateral cell Q has a non-convex shape, it does not exist any parameter τ_Q for which $\mathbf{M}$ is an M-matrix.

Proof. A straightforward calculation yields:

$$\mathbf{M} = \begin{pmatrix} \frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_1}{4|Q|} + \frac{\tau_Q \widehat{\mathbf{T}}_1^2}{|Q|^2} & \frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{4|Q|} - \frac{\tau_Q \widehat{\mathbf{T}}_1 \widehat{\mathbf{T}}_2}{|Q|^2} & -\frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_1}{4|Q|} + \frac{\tau_Q \widehat{\mathbf{T}}_1 \widehat{\mathbf{T}}_3}{|Q|^2} & -\frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{4|Q|} - \frac{\tau_Q \widehat{\mathbf{T}}_1 \widehat{\mathbf{T}}_4}{|Q|^2} \\ & \frac{\mathbf{D}_2 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{4|Q|} + \frac{\tau_Q \widehat{\mathbf{T}}_2^2}{|Q|^2} & -\frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{4|Q|} - \frac{\tau_Q \widehat{\mathbf{T}}_2 \widehat{\mathbf{T}}_3}{|Q|^2} & -\frac{\mathbf{D}_2 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{4|Q|} + \frac{\tau_Q \widehat{\mathbf{T}}_2 \widehat{\mathbf{T}}_4}{|Q|^2} \\ & & \frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_1}{4|Q|} + \frac{\tau_Q \widehat{\mathbf{T}}_3^2}{|Q|^2} & \frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{4|Q|} - \frac{\tau_Q \widehat{\mathbf{T}}_3 \widehat{\mathbf{T}}_4}{|Q|^2} \\ \text{symmetric} & & & \frac{\mathbf{D}_2 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{4|Q|} + \frac{\tau_Q \widehat{\mathbf{T}}_4^2}{|Q|^2} \end{pmatrix} \quad (32)$$

First, we note that the diagonal entries are all of the form $(\mathbf{M})_{ii} = \mathbf{D}_i \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_i / 4|Q| + \tau_Q \widehat{\mathbf{T}}_i^2 / |Q|^2$ and are strictly positive quantities for every $\tau_Q \geq 0$. For $\mathbf{M}$ being an M-matrix, we impose that its off-diagonal terms are non-positive quantities, i.e., $(\mathbf{M})_{ij} \leq 0$ for $i \neq j$. Since $\mathbf{M}$ is a symmetric matrix, we need to impose that

$$M_{12} = +\frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{4|\mathbf{Q}|} - \frac{\tau_Q \hat{\mathbf{T}}_1 \hat{\mathbf{T}}_2}{|\mathbf{Q}|^2} \leq 0, \quad (33)$$

$$M_{13} = -\frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{4|\mathbf{Q}|} + \frac{\tau_Q \hat{\mathbf{T}}_1 \hat{\mathbf{T}}_3}{|\mathbf{Q}|^2} \leq 0, \quad (34)$$

$$M_{14} = -\frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{4|\mathbf{Q}|} - \frac{\tau_Q \hat{\mathbf{T}}_1 \hat{\mathbf{T}}_4}{|\mathbf{Q}|^2} \leq 0, \quad (35)$$

$$M_{23} = -\frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{4|\mathbf{Q}|} - \frac{\tau_Q \hat{\mathbf{T}}_2 \hat{\mathbf{T}}_3}{|\mathbf{Q}|^2} \leq 0, \quad (36)$$

$$M_{24} = -\frac{\mathbf{D}_2 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{4|\mathbf{Q}|} + \frac{\tau_Q \hat{\mathbf{T}}_2 \hat{\mathbf{T}}_4}{|\mathbf{Q}|^2} \leq 0, \quad (37)$$

$$M_{34} = +\frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{4|\mathbf{Q}|} - \frac{\tau_Q \hat{\mathbf{T}}_3 \hat{\mathbf{T}}_4}{|\mathbf{Q}|^2} \leq 0. \quad (38)$$

We distinguish the following three cases.

(i) **Case $\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2 < 0$.** Inequalities (33) and (38) are always satisfied. Inequalities (34) and (37) provides the upper bound and are respectively satisfied by taking $\tau_Q \leq \mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_1 |\mathbf{Q}| / 4\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_3$ and $\tau_Q \leq \mathbf{D}_2 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2 |\mathbf{Q}| / 4\hat{\mathbf{T}}_3 \hat{\mathbf{T}}_4$. Inequalities (35) and (36) provides the lower bound and are respectively satisfied by taking $\tau_Q \geq -\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2 |\mathbf{Q}| / 4\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_4$ and $\tau_Q \geq -\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2 |\mathbf{Q}| / 4\hat{\mathbf{T}}_2 \hat{\mathbf{T}}_3$. These conditions are simultaneously satisfied by taking:

$$\frac{-\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\min(\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_4, \hat{\mathbf{T}}_2 \hat{\mathbf{T}}_3)} \leq \frac{4\tau_Q}{|\mathbf{Q}|} \leq \min \left(\frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_1}{\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_3}, \frac{\mathbf{D}_2 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\hat{\mathbf{T}}_2 \hat{\mathbf{T}}_4} \right). \quad (39)$$

(ii) **Case $\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2 > 0$.** Inequalities (35) and (36) are always satisfied. Inequalities (34) and (37) provide the same upper bound of the previous case. Inequalities (33) and (38) provide the lower bound and are respectively satisfied by taking $\tau_Q \geq \mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2 |\mathbf{Q}| / 4\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_2$ and $\tau_Q \geq \mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2 |\mathbf{Q}| / 4\hat{\mathbf{T}}_3 \hat{\mathbf{T}}_4$. These conditions are simultaneously satisfied by taking:

$$\frac{-\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\min(\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_2, \hat{\mathbf{T}}_3 \hat{\mathbf{T}}_4)} \leq \frac{4\tau_Q}{|\mathbf{Q}|} \leq \min \left(\frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_3}, \frac{\mathbf{D}_2 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\hat{\mathbf{T}}_2 \hat{\mathbf{T}}_4} \right). \quad (40)$$

(iii) **Case $\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2 = 0$.** Inequalities (33), (35), (36), and (38) are always satisfied. Inequalities (34) and (37) still provide the same upper bound of the previous cases.

Eventually, the proposition follows from noting that the upper bound is the same for the three cases and taking the maximum of the two lower bounds given by the cases $\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2 < 0$ and $\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2 > 0$.

For the range of τ_Q to be non-empty, we must obviously require that:

$$\max \left(\frac{-\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\min(\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_4, \hat{\mathbf{T}}_2 \hat{\mathbf{T}}_3)}, \frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\min(\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_2, \hat{\mathbf{T}}_3 \hat{\mathbf{T}}_4)} \right) \leq \min \left(\frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_1}{\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_3}, \frac{\mathbf{D}_2 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\hat{\mathbf{T}}_2 \hat{\mathbf{T}}_4} \right). \quad (41)$$

We note that $\hat{\mathbf{T}}_i \hat{\mathbf{T}}_j \leq |\mathbf{Q}|^2$ for every combination of the indices i and j and we bound from above the left-hand side and from below the right-hand side of the previous inequality. It follows that

$$\max \left(\frac{-\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\min(\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_4, \hat{\mathbf{T}}_2 \hat{\mathbf{T}}_3)}, \frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\min(\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_2, \hat{\mathbf{T}}_3 \hat{\mathbf{T}}_4)} \right) \leq \frac{|\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2|}{|\mathbf{Q}|^2}$$

Fig. 2. Sketch of the monotonicity region (in grey) when $K_Q = I$ for the quadrilateral cell formed by the three points A, B, C and the fourth point P or p' . When the fourth point is chosen inside the monotonicity region.

Fig. 3. Monotonicity regions (shown in grey) for $K_Q = I$ and different angles θ , which are equal to 147 degrees, 119 degrees, 98 degrees on the top row, and 90 degrees, 47 degrees, 20 degrees on the bottom row.

and

$$\frac{4 \min(\mathbf{D}_1 \cdot \tilde{K}_Q \mathbf{D}_1, \mathbf{D}_2 \cdot \tilde{K}_Q \mathbf{D}_2)}{|\mathbf{Q}|^2} \leq \min\left(\frac{\mathbf{D}_1 \cdot \tilde{K}_Q \mathbf{D}_1}{\hat{T}_1 \hat{T}_3}, \frac{\mathbf{D}_2 \cdot \tilde{K}_Q \mathbf{D}_2}{\hat{T}_2 \hat{T}_4}\right).$$

We combine the two inequalities above and we see that inequality (41) surely holds if we require that $\mathbf{Q}$ be such that

$$\frac{|\mathbf{D}_1 \cdot \tilde{K}_Q \mathbf{D}_2|}{|\mathbf{Q}|^2} \leq \frac{4 \min(\mathbf{D}_1 \cdot \tilde{K}_Q \mathbf{D}_2, \mathbf{D}_2 \cdot \tilde{K}_Q \mathbf{D}_2)}{|\mathbf{Q}|^2},$$

and the second statement of the proposition immediately follows.

Finally, we note that if $\mathbf{Q}$ is a non-convex cell, at least one of the factor $\hat{T}_i$ must be a negative number. Let us suppose, without loss of generality, that $\hat{T}_1 < 0$. A rapid inspection of the set of inequalities (33)-(38) reveals that inequality (35) is never satisfied when $\mathbf{D}_1 \cdot \tilde{K}_Q \mathbf{D}_2 < 0$, inequality (33) is never satisfied when $\mathbf{D}_1 \cdot \tilde{K}_Q \mathbf{D}_2 > 0$, and neither inequality (33) nor inequality (35) are satisfied when $\mathbf{D}_1 \cdot \tilde{K}_Q \mathbf{D}_2 = 0$. $\square$

Remark 4.1 A sufficient condition for $\mathbf{Q}$ and $\tilde{K}_Q$ to ensure that the set of possible values of the related τ_Q is non-empty is given by

Fig. 4. Monotonicity regions (shown in grey) for different tensor $\mathbf{K}_Q$ with the same starting initial angle. The effect of the full tensor $\mathbf{K}_Q$ is to rotate the monotonicity region. Note the existence of a limiting beyond which there is no monotonicity region.

$$|\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2| \leq 4 \min(\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_1, \mathbf{D}_2 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2).$$

Furthermore, if Q is a non-convex cell, either $\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_3 < 0$ or $\hat{\mathbf{T}}_2 \hat{\mathbf{T}}_4 < 0$. In such a case, the left-most term of inequality (42) is non-negative while its right-most term is strictly negative since otherwise Q would be degenerate. Thus, there does not exist any value of τ_Q satisfying condition (42).

Figure 2 shows the concept of *monotonicity region* that practically illustrates the major consequence of the previous proposition. Let us take $\mathbf{K}_Q$ equal to the identity matrix and consider the three vertices A , B and C as in Figure 2. If we choose the fourth vertex of Q inside the grey region, e.g., point P of the picture, we obtain the quadrilateral cell $ABCP$ for which the set of values for τ_Q specified by condition (42) is non-empty. On the contrary, if we choose the fourth vertex outside the grey region, e.g., point P' of the picture, we obtain the quadrilateral cell $ABCP'$ for which condition (42) cannot be satisfied. The shape of the monotonicity region strongly depends on the angle $\theta = \widehat{ABC}$, as shown by Figure-3. In particular, in the sequence from obtuse to acute angles it is worth noting that the portion of the line originated from point B , orthogonal to segment AC and outside the triangle ABC is always inside the monotonicity region. In fact, on such line the left-most term of inequality (42) is always zero while the right-most term is always strictly positive; hence, the set of values of τ_Q that satisfies (42) is never empty. It is also worth noting that for very acute angles the monotonicity region tends to a strip whose base is the segment AC .

In Figure 4 we show the effect of a full tensor $\mathbf{K}_Q$ on the monotonicity region (still shaded in gray), starting from the identity tensor in the top-left-most picture. The angle at the leading vertex is fixed. The presence of a full diffusion tensor with increasing off-diagonal elements rotate the monotonicity region until a limit configuration is reached beyond which it is no longer possible to build monotone quadrilaterals (see the bottom-right-most picture).

Proposition 4.2 (Monotonicity) *If for every quadrilateral cell Q of mesh Ω_h the associated parameter denoted by τ_Q satisfies the inequalities*

$$\max \left(\frac{-\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\min(\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_4, \hat{\mathbf{T}}_2 \hat{\mathbf{T}}_3)}, \frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\min(\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_2, \hat{\mathbf{T}}_3 \hat{\mathbf{T}}_4)} \right) \leq \frac{4\tau_Q}{|Q|} \leq \min \left(\frac{\mathbf{D}_1 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_1}{\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_3}, \frac{\mathbf{D}_2 \cdot \tilde{\mathbf{K}}_Q \mathbf{D}_2}{\hat{\mathbf{T}}_2 \hat{\mathbf{T}}_4} \right), \quad (42)$$

then, the matrix operator is a (singular) M-matrix.

Proof. Proposition 4.1 implies that each elemental matrix is a (singular) M-matrix because it is a Z-matrix by construction and $M_\varepsilon = \varepsilon I + M$ is an M-matrix for every $\varepsilon > 0$. This fact follows from Proposition 2.1 and the connectivity of the mesh after noting that M_ε is a Z-matrix with positive diagonal entries, i.e., $M_\varepsilon|_{ii} \geq 0$, and that $M_\varepsilon \mathbf{1} = \varepsilon \mathbf{1}$. Thus, $M_\varepsilon \mathbf{1}$ is a diagonally dominant matrix with positive diagonal entries. This property readily extends to the global matrix originated by the assembly process (9). In fact, the diagonal entries of each elemental matrix M are accumulated to the diagonal entries of the global matrix. Likewise, the off-diagonal entries of each elemental matrix M are accumulated to the off-diagonal entries of the global matrix. Thus, the global matrix is also a (singular) M-matrix. Imposing the Dirichlet boundary conditions in strong form remove the singularity without altering the Z-matrix structure. $\square$

The immediate consequence of Proposition 4.2 is the Discrete Maximum Principle stated by the theorem below.

Theorem 4.1 *Under the conditions of Proposition 4.2, let u_h be the numerical solution of (8) that approximates the exact solution of problem (1)-(3) with $f \leq 0$ on Γ^D and $g^N = 0$ on Γ^N . Then, it holds that*

$$\max_{v \in \mathcal{V}} u_v \leq \min(0, \max_{v \in \Gamma^D} (g(\mathbf{x}_v))).$$

5. Conclusions

We derived a set of sufficient and necessary conditions for the local stiffness matrix the Poisson problem provided by the nodal mimetic finite difference method on quadrilaterals under which the stiffness matrix is a singular M-matrix. This property depends on the shape of the quadrilateral cell and the value of the off-diagonal elements of the diffusion tensor and may be effectively illustrated by the concept of monotonicity regions, which is here introduced. When each local stiffness matrix satisfy these conditions, the assembled global matrix the Poisson problem turns out to be a singular M-matrix, the kernel being spanned by the constant vectors. Setting the Dirichlet boundary conditions eventually yields a monotone matrix operator and we proved that the final discretization possesses a Discrete Maximum Principle.

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References

- [1] L. Beirão da Veiga, J. Droniou, and G. Manzini. A unified approach to handle convection terms in Finite Volumes and Mimetic Discretization Methods for elliptic problems. 31(4):1537–1541, 2011.
- [2] L. Beirão da Veiga, V. Gyrya, K. Lipnikov, and G. Manzini. Mimetic finite difference method for the Stokes problem on polygonal meshes. *J. Comput. Phys.*, 228(19):7215–7232, 2009.
- [3] L. Beirão da Veiga, K. Lipnikov, and G. Manzini. Convergence analysis of the high-order mimetic finite difference method. *Numer. Math.*, 113(3):325–356, 2009.
- [4] L. Beirão da Veiga, K. Lipnikov, and G. Manzini. Arbitrary order nodal mimetic discretizations of elliptic problems on polygonal meshes. *SIAM Journal on Numerical Analysis*, 49(5):1737–1760, 2011.
- [5] L. Beirão da Veiga and G. Manzini. An a posteriori error estimator for the mimetic finite difference approximation of elliptic problems. *Int. J. Numer. Meth. Engrg.*, 76(11):1696–1723, 2008.
- [6] L. Beirão da Veiga and G. Manzini. A higher-order formulation of the mimetic finite difference method. *SIAM Journal on Scientific Computing*, 31(1):732–760, 2008.
- [7] Abraham Berman and Robert J. Plemmons. *Nonnegative matrices in the mathematical sciences*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1979. Computer Science and Applied Mathematics.
- [8] F. Brezzi, A. Buffa, and K. Lipnikov. Mimetic finite differences for elliptic problems. *M2AN Math. Model. Numer. Anal.*, 43(2):277–295, 2009.

- [9] F. Brezzi, K. Lipnikov, and M. Shashkov. Convergence of the mimetic finite difference method for diffusion problems on polyhedral meshes. *SIAM J. Numer. Anal.*, 43(5):1872–1896, 2005.
- [10] A. Cangiani and G. Manzini. Flux reconstruction and pressure post-processing in mimetic finite difference methods. *Comput. Methods Appl. Mech. Engrg.*, 197/9-12:933–945, 2008.
- [11] A. Cangiani, G. Manzini, and A. Russo. Convergence analysis of the mimetic finite difference method for elliptic problems. *SIAM J. Numer. Anal.*, 47(4):2612–2637, 2009.
- [12] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman, Boston, 1985.
- [13] J. Hyman, M. Shashkov, and S. Steinberg. The numerical solution of diffusion problems in strongly heterogeneous non-isotropic materials. *J. Comput. Phys.*, 132:130–148, 1997.
- [14] J. Karátson, S. Korotov, and M. Krizek. On discrete maximum principles for nonlinear elliptic problems. *Mathematics and Computers in Simulation*, 76(1-3):99–108, 2007.
- [15] K. Lipnikov, G. Manzini, and D. Svyatskiy. Analysis of the monotonicity conditions in the mimetic finite difference method for elliptic problems. *J. Comput. Phys.*, 230(7):2620 – 2642, 2011.
- [16] S. Mousavi and N. Sukumar. Numerical integration of polynomials and discontinuous functions on irregular convex polygons and polyhedrons. *Computational Mechanics*, 47(5):535–554, 2011.
- [17] L. Portero, A. Arrarás, and J. C Jorge. Domain decomposition operator splitting for mimetic finite difference discretizations of non-stationary problems. *Computers & Fluids*, 46(1):398–403, 2011. 10th ICFD Conference Series on Numerical Methods for Fluid Dynamics (ICFD 2010).
- [18] M. Shashkov and S. Steinberg. Support-operator finite-difference algorithms for general elliptic problems. *J. Comput. Phys.*, 118(1):131 – 151, 1995.
- [19] M. Shashkov and S. Steinberg. Solving diffusion equations with rough coefficients in rough grids. *Journal of Computational Physics*, 129(2):383 – 405, 1996.
- [20] N. Sukumar, B. Moran, and T. Belytschko. The natural element method in solid mechanics. *Int. J. Numer. Meth. Engrg.*, 43(5):839–887, 1998.
- [21] N. Sukumar and A. Tabarraei. Conforming polygonal finite elements. *Int. J. Numer. Meth. Engrg.*, 61:2045–2066, 2004.
- [22] A. Tabarraei and N. Sukumar. Extended finite element method on polygonal and quadtree meshes. *Comput. Methods Appl. Mech. Engrg.*, 197(5):425–438, 2007.
- [23] T. Yaguchi, T. Matsuo, and M. Sugihara. An extension of the discrete variational method to nonuniform grids. *J. Comput. Phys.*, 229(11):4382–4423, 2010.