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Mimetic Constraints

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The goal is to derive discrete Lagrangian staggered grid hydrodynamic (SGH) equations that mimic the exact equations on arbitrary finite meshes

governing equations

Exact operators

Discrete operators

$$\dot{M} = 0 \rightarrow \dot{M}_z = 0$$

$$\rho \dot{\gamma} = \nabla \mathbf{u} \rightarrow M_z \dot{\gamma}_z = \sum_i^z \mathbf{S}^i \mathbf{u}_p^i$$

$$\rho \dot{\mathbf{u}} = \nabla \cdot \boldsymbol{\sigma} \rightarrow M_p \dot{\mathbf{u}}_p = \sum_i^z \mathbf{S}^i \cdot \underline{\boldsymbol{\sigma}_z^{*i}}$$

$$\rho \dot{\mathbf{j}} = \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}) \rightarrow M_c \dot{\mathbf{j}}_c = \sum_i^c \mathbf{S}^i \cdot \underline{\boldsymbol{\sigma}_z^{*i}} \cdot \mathbf{u}_p^i$$

$$\boldsymbol{\sigma}_z^* = \boldsymbol{\sigma}_z + \mathbf{q}_z$$

Riemann stress includes a dissipation term

curl & divergence identities

Exact operators

Discrete operators

$$\nabla \times (\nabla \mathbf{u}) = 0 \rightarrow \nabla_p \times \sum_i^z \mathbf{S}^i \underline{\mathbf{u}_z^{*i}} = 0$$

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0 \rightarrow \nabla_p \cdot \sum_i^z \mathbf{S}^i \times \underline{\mathbf{u}_z^{*i}} = 0$$

$$\nabla \times (\nabla \cdot \boldsymbol{\sigma}) = 0 \rightarrow \nabla_p \times \sum_i^z \mathbf{S}^i \cdot \underline{\boldsymbol{\sigma}_z^{*i}} = 0$$

$$\nabla \cdot (\nabla \times \boldsymbol{\sigma}) = 0 \rightarrow \nabla_p \cdot \sum_i^z \mathbf{S}^i \times \underline{\boldsymbol{\sigma}_z^{*i}} = 0$$

Used in CCH

• Burton et. al. 2012

Additional constraints

Angular momentum

$$\dot{\mathbf{L}}_p = \sum_i^z \mathbf{x}_{pz}^i \times (\mathbf{S}^i \cdot \underline{\boldsymbol{\sigma}_z^{*i}})$$

Second law of thermodynamics

$$(\mathbf{S}^i \cdot \underline{\mathbf{q}_z}) \cdot \delta^i \mathbf{u}_{pz} \geq 0$$

Symmetry preservation

$$\mathbf{u}_p = -\|\mathbf{u}_p\| (\cos(\theta) \hat{\mathbf{i}} + \sin(\theta) \hat{\mathbf{j}})$$

volume & divergence identities

Exact operators

Discrete operators

$$\nabla \cdot \mathbf{x} = 3 \rightarrow \frac{1}{V_z} \sum_i^z \mathbf{N}^i \cdot \mathbf{x}_p^i = 3$$

$$\nabla \mathbf{x} = \mathbf{I} \rightarrow \frac{1}{V_z} \sum_i^z \mathbf{N}^i \mathbf{x}_p^i = \mathbf{I}$$

$$\frac{1}{V} \dot{V} = \nabla \cdot \mathbf{u} \rightarrow \dot{V}_z = \sum_i^z \mathbf{N}^i \cdot \mathbf{u}_p^i$$

The GCL condition

- Trulio & Trigger 1961
- Whalen 1996
- Maire et. al. 2007
- Despres et. al. 2010
- Burton et. al. 2012

Strain control volume

Momentum control volume

The mimetic approach eliminates numerical errors that result from incompatibilities in the numerical operators

Consider the case with a globally uniform gradient in stress on a distorted mesh. The rate of change of angular momentum must be zero! This is a 1D problem on a 2D mesh.

Uniform stress gradient in this direction

$$S^c = S^{i_1} + S^{i_2}$$

Angular momentum

$$\dot{L}_p = \sum_i^p \left[\mathbf{x}_{pz}^i \times \left(\mathbf{S}^i \cdot \boldsymbol{\sigma}_z^{*i} \right) \right] \rightarrow 0$$

A way to ensure the change in angular momentum is equal to zero is for this term to vanish in each zone corner

$$0 = \mathbf{x}^c \times \left[\left(\mathbf{S}^{i_1} \cdot \boldsymbol{\sigma}_z^{*i_1} \right) + \left(\mathbf{S}^{i_2} \cdot \boldsymbol{\sigma}_z^{*i_2} \right) \right]$$

$$0 = \mathbf{x}^c \times \left[\boldsymbol{\sigma}_z^{*c} \cdot \left(\mathbf{S}^c + \mathbf{T}^c \right) \right]$$

$$0 = \mathbf{x}^c \times \left(\mathbf{S}^c \cdot \boldsymbol{\sigma}_z^{*c} \right) + \mathbf{r}^c \times \left(\mathbf{T}^c \cdot \boldsymbol{\sigma}_z^{*c} \right)$$

=0 if

=0 if

Two conditions must be satisfied

$$\boldsymbol{\sigma}_z^{*} = \text{symmetric}$$

$$\boldsymbol{\sigma}_z^{*i_1} = \boldsymbol{\sigma}_z^{*i_2}$$

Lesson learned:

The Riemann stress must be equal on both faces of the momentum control volume corner (i.e. a corner stress) and is symmetric

Angular momentum analysis will constrain the discretization

From previous slide

$$0 = \mathbf{x}^c \times (\mathbf{S}^c \cdot \boldsymbol{\sigma}_z^{*c}) + \mathbf{x}^c \times (\mathbf{T}^c \cdot \boldsymbol{\sigma}_z^{*c})$$

=0 if

=0 if

$$\boldsymbol{\sigma}_z = \text{symmetric}$$

$$\sigma_z^{i1} = \sigma_z^{i2}$$

Proof:

$$0 = \varepsilon_{ijk} x_j S_m^c \sigma_{mk}^{*c}$$

We will look
at the case
where $\mathbf{x}$ and $\mathbf{s}$
are parallel

$$x_j S_m = \delta_{jm}$$

$$0 = \varepsilon_{ijk} x_j S_m \sigma_{mk}^* = \varepsilon_{ijk} \sigma_{jk}^*$$

$$= (\sigma_{jk}^* - \sigma_{kj}^*)_i$$

$$\sigma_{jk}^* = \sigma_{kj}^*$$

Proof:

$$(\mathbf{S}^{i_1} \cdot \boldsymbol{\sigma}_z^{*i_1}) + (\mathbf{S}^{i_2} \cdot \boldsymbol{\sigma}_z^{*i_2}) = (\mathbf{S}^{i_1} + \mathbf{S}^{i_2}) \cdot \boldsymbol{\sigma}_z^{*c} + 0 = \mathbf{S}^c \cdot \boldsymbol{\sigma}_z^{*c} + 0$$

$$\sigma_z^{*i_1} = \sigma_z^{*i_2} = \sigma_z^{*c}$$

$$\mathbf{T}^c \cdot \boldsymbol{\sigma}_z^{*c} = 0$$

The mimetic approach constrains the Riemann solution

- Consider the vorticity equation, some terms vanish analytically but may not vanish numerically, which produces spurious vorticity

Exact equation

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla) \mathbf{u} - \omega (\nabla \cdot \mathbf{u}) + \nabla v \times (\nabla \cdot \sigma) + \underbrace{\left\{ v \nabla \times (\nabla \cdot \sigma) - \nabla \times \nabla (\mathbf{u}^2/2) + \mathbf{u} \nabla \cdot (\nabla \times \mathbf{u}) \right\}}_{=0}$$

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla) \mathbf{u} - \omega (\nabla \cdot \mathbf{u}) + \nabla v \times (\nabla \cdot \sigma)$$

$$\nabla \cdot (\nabla \times \sigma) = 0 \rightarrow \nabla_p \cdot \sum_i^z \mathbf{S}^i \times \sigma_z^{*i} = 0$$

$$\nabla \times \nabla (\mathbf{u} \cdot \mathbf{u}) = 0 \rightarrow \nabla_p \times \sum_i^z \mathbf{S}^i \mathbf{u}_p^{*i} \cdot \mathbf{u}_p^{*i} = 0$$

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0 \rightarrow \nabla_p \cdot \sum_i^z \mathbf{S}^i \times \mathbf{u}_p^{*i} = 0$$

Lesson learned:

The Riemann velocity must be a unique, single value in the zone center. Next, the Riemann stress must be equal on both faces of the momentum control volume corner

Two conditions must be satisfied

$$\sigma_z^{*i_1} = \sigma_z^{*i_2}$$

$$\mathbf{u}_z^{*i_1} = \mathbf{u}_z^{*i_2} = \mathbf{u}_z^{*i_3} = \mathbf{u}_z^{*i_4}$$

* Burton 2012

The divergence and curl vector identities constrain the Riemann solution at the zone center

Proof: $\nabla \cdot (\nabla \times \sigma) = 0 \rightarrow \sum_i^p S^i \cdot \left(\sum_i^z T^i \cdot \sigma_z^{*i} \right) = 0$

$$\sum_i^z T^i \cdot \sigma_z^{*i} = 0 \rightarrow (S^{i_1} \cdot \sigma_z^{*i_1}) + (S^{i_2} \cdot \sigma_z^{*i_2}) = (S^{i_1} + S^{i_2}) \cdot \sigma_z^{*c} + 0 = S^c \cdot \sigma_z^{*c} + 0$$

$T^c \cdot \sigma_z^{*c} = 0$

$\sigma_z^{*i_1} = \sigma_z^{*i_2} = \sigma_z^{*c}$

Proof: $\nabla \cdot (\nabla \times u) = 0 \rightarrow \sum_i^p S^i \cdot \left(\sum_i^z S^i \times u_z^{*i} \right) = 0$

$$\sum_i^z S^i \times u_z^{*i} = \left(\sum_i^z S^i \right) \times u_z^{*i} = 0$$

$= 0$

$u_z^{*i_1} = u_z^{*i_2} = u_z^{*i_3} = u_z^{*i_4}$

Proof: $\nabla \times \nabla (u \cdot u) = 0 \rightarrow \sum_i^p S^i \times \left(\sum_i^z S^i u_p^i \cdot u_p^i \right) = 0$

$$\left(\sum_i^z S^i \right) \times u_z^{*i} \cdot u_z^{*i} = 0$$

$= 0$

$u_z^{*i_1} = u_z^{*i_2} = u_z^{*i_3} = u_z^{*i_4}$

Null modes can be damped with schemes that satisfy ancillary constraints

- Null modes are unconstrained deformations of the zone
 - Zero volume change during the null mode deformation
 - The average velocity field is constant during the null mode deformation
- Designing a scheme per the requirements below can damp null modes
 - The Riemann stress must be equal on both faces of the momentum control volume corner (i.e. a corner stress) and the stress must be symmetric
 - The Riemann velocity must be a unique, single value at the vertex of the momentum control volume (i.e. located at the center of the zone)
- By solving the Riemann problem at the zone center, a viscous force is created between the zone center and the node (black arrows below).

Null mode examples