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THEORETICAL AND NUMERICAL ANALYSIS OF FREE-ELECTRON LASERS

LECTURE 4

LINEAR THEORY OF STIMULATED EMISSION IN ONE-DIMENSION FOR PLANAR WIGGLER FIELDS

SIDEBAND GENERATION

OUTLINE

- The intent of this lecture is to discuss the gain in the linear regime in 1-D
 - Planar wigglers
 - Low-Gain Regime
 - Gain Formula
 - High-Gain Regime
 - Compton & Raman Regimes
 - Efficiency Estimates
 - Thermal Effects

REVIEW OF SINGLE-PARTICLE ORBITS

- The planar wiggler representation in 1D is

$$\mathbf{B}_w = B_w \hat{\mathbf{e}}_y \sin k_w z \quad \mathbf{A}_w = -\frac{B_w}{k_w} \hat{\mathbf{e}}_y \cos k_w z$$

- The 1D particle orbits are given by

$$v_x = \frac{P_x}{\gamma m_e} + v_w \cos k_w z \quad v_y = \frac{P_y}{\gamma m_e}$$

$$v_z = v_{\parallel} \sqrt{1 - 2 \frac{P_x v_w}{\gamma m_e v_{\parallel}^2} \cos k_w z - \frac{v_w^2}{2 v_{\parallel}^2} \cos 2k_w z}$$

$$\text{where } v_{\parallel} = \sqrt{1 - \frac{1}{\gamma^2} - \frac{v_w^2}{2 v_{\parallel}^2} - \frac{P_x^2 + P_y^2}{\gamma^2 m_e^2 c^2}}$$

- Integration of the form for v_z yields the Lagrangian time coordinate is approximately

$$\tau(z) \approx t_0 + \frac{z}{v_{\parallel}} + \frac{P_x v_w}{\gamma m_e v_{\parallel}^2} \frac{1}{k_w v_{\parallel}} \sin k_w z + \frac{v_w^2}{8 v_{\parallel}^2} \frac{1}{k_w v_{\parallel}} \sin 2k_w z$$

THE PERTURBED DISTRIBUTION

- As in the case of the helical wiggler, the perturbed distribution function is

$$\delta f_b(z, \mathbf{p}, \tau(z)) = \int_0^z \frac{dz'}{v_z(z')} \left[\delta \mathbf{E}(z', \tau(z')) + \frac{1}{c} \mathbf{v} \times \delta \mathbf{B}(z', \tau(z')) \right] \cdot \nabla_{\mathbf{p}} F_b$$

- Since we deal with a planar wiggler, the interaction is with a plane polarized electric field and we write the vector and scalar potentials as

$$\delta \mathbf{A}(z, t) = \frac{1}{2} (\delta \hat{A}(z) e^{-i\omega t} + c.c.) \hat{\mathbf{e}}_x \quad \delta \varphi(z, t) = \frac{1}{2} \delta \hat{\varphi}(z) e^{-i\omega t} + c.c.$$

- This produces the following electric and magnetic fields

$$\delta \mathbf{E}(z, t) = \frac{1}{2} \left(\delta \hat{A} \hat{\mathbf{e}}_x - \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \delta \hat{\varphi} \right) \hat{\mathbf{e}}_x e^{-i\omega t} + c.c.$$

$$\delta \mathbf{B}(z, t) = \frac{1}{2} \hat{\mathbf{e}}_y \frac{\partial}{\partial z} \delta \hat{A} e^{-i\omega t} + c.c.$$

- We transform from the kinetic momenta (p_x, p_y, p_z) to the canonical variables (P_x, P_y, p) where

$$\frac{\partial}{\partial p_x} = \frac{\partial}{\partial P_x} + \frac{P_x}{p} \frac{\partial}{\partial p} \quad \frac{\partial}{\partial p_y} = \frac{\partial}{\partial P_y} + \frac{P_y}{p} \frac{\partial}{\partial p} \quad \frac{\partial}{\partial p_z} = \frac{P_z}{p} \frac{\partial}{\partial p}$$

THE PERTURBED DISTRIBUTION

- As in the case of a helical wiggler, we obtain the following (similar) solution for the perturbed distribution function

$$\begin{aligned}
 \delta \hat{f}_b &= \frac{e}{2c} \int_0^z dz' \frac{e^{-i\omega\tau(z,z')}}{v_z(z')} \left[\left(i\omega - v_z \frac{\partial}{\partial z} \right) \delta \hat{A} \frac{\partial}{\partial P_x} + \left(i\omega p_x \delta \hat{A} - c p_z \frac{\partial}{\partial z} \delta \hat{\varphi} \right) \frac{1}{p} \frac{\partial}{\partial p} \right] F_b \\
 &= -\frac{e}{2c} \left(\delta \hat{A}(z) - \delta \hat{A}(0) e^{-i\omega\tau(z,0)} \right) \frac{\partial}{\partial P_x} F_b \\
 &\quad + \frac{e}{2c} \int_0^z dz' \frac{e^{-i\omega\tau(z,z')}}{v_z(z')} \left(i\omega p_x \delta \hat{A} - c p_z \frac{\partial}{\partial z} \delta \hat{\varphi} \right) \frac{1}{p} \frac{\partial}{\partial p} F_b \\
 &= D_{\perp} \frac{\partial}{\partial P_x} F_b + D_{\parallel} \frac{1}{p} \frac{\partial}{\partial p} F_b
 \end{aligned}$$

- As in the case of a helical wiggler, we assume an equilibrium distribution function of the form

$$F_b(P_x, P_y, p) = n_b \delta(P_x) \delta(P_y) G_b(p)$$

THE SOURCE CURRENTS

- The source current is given in the following expression, where (as in the case of the helical wiggler) we transform from the kinetic to the canonical variables

$$\delta \hat{\mathbf{J}} = - \frac{e}{m_e} \int d^3 p \frac{1}{\gamma} (p_x \hat{\mathbf{e}}_x + p_z \hat{\mathbf{e}}_z) \delta \hat{f}_b = - \frac{e}{m_e} \int dP_x dP_y dp \frac{p}{\gamma} \left(\frac{p_x}{p_z} \hat{\mathbf{e}}_x + \hat{\mathbf{e}}_z \right) \delta \hat{f}_b$$

- As a consequence,

$$\delta \hat{J}_x = \frac{e}{m_e} \int dP_x dP_y dp \frac{p}{\gamma} \left[\frac{\partial}{\partial P_x} \left(\frac{p_x}{p_z} D_{\perp} \right) - \frac{p_x}{p_z} D_z \frac{1}{p} \frac{\partial}{\partial p} \right] F_b$$

$$\delta \hat{J}_z = \frac{e}{m_e} \int dP_x dP_y dp \frac{p}{\gamma} \left[\frac{\partial}{\partial P_x} D_{\perp} - D_z \frac{1}{p} \frac{\partial}{\partial p} \right] F_b$$

where

$$e^{i\omega\tau(z,0)} \approx e^{i\omega z/v_{\parallel}} e^{ib_1 \sin k_w z} e^{ib_2 \sin 2k_w z}$$

$$b_1 = \frac{\omega}{k_w v_{\parallel}} \frac{v_w}{v_{\parallel}} \frac{P_x}{\gamma m_e v_{\parallel}}$$

$$\frac{\partial}{\partial P_x} e^{i\omega\tau(z,0)} \approx \frac{ib_1}{P_x} \sin k_w z e^{i\omega z/v_{\parallel}} e^{ib_1 \sin k_w z} e^{ib_2 \sin 2k_w z}$$

$$\frac{\partial}{\partial P_x} D_{\perp} \approx i \frac{e}{2c} \frac{b_1}{P_x} \delta \hat{A}(0) \sin k_w z e^{i\omega z/v_{\parallel}} e^{ib_1 \sin k_w z} e^{ib_2 \sin 2k_w z}$$

$$b_2 = \frac{\omega}{k_w v_{\parallel}} \frac{v_w^2}{8v_{\parallel}^2}$$

$$\frac{\partial}{\partial P_x} \left(\frac{p_x}{p_z} D_{\perp} \right) \approx - \frac{e}{2c p_z} \left(1 + \frac{p_x^2}{p_z^2} \right) \frac{b_1}{P_x} \delta \hat{A}(z) + \frac{e}{2c p_z} \delta \hat{A}(0) \left(1 + \frac{p_x^2}{p_z^2} + \frac{i p_x}{P_x} b_1 \sin k_w z \right) e^{i\omega z/v_{\parallel}} e^{ib_1 \sin k_w z} e^{ib_2 \sin 2k_w z}$$

THE LOW-GAIN REGIME

- We expand the vector potential using a Floquet analysis

$$\delta\hat{A}(z) = \sum_n \delta\hat{A}_n(z) = \sum_n \delta\hat{A}_n(0) e^{i \int_0^z dz' k_n(z')}$$

$$\longrightarrow k_n(z) = k(z) + nk_w$$

- We now make a slowly-varying phase approximation

$$k_n(z) = (k_0 + nk_w) + \delta k(z) \quad | \delta k(z)/k_0 | \ll 1$$

$$\longrightarrow k_{0n} = k_0 + nk_w$$

- Neglecting the 2nd order terms in δk , the wave equation becomes

$$\begin{aligned} \sum_n \left(\frac{\omega^2}{c^2} - k_{0n}^2 - 2k_{0n}\delta k \right) \delta\hat{A}_n(z) &= - \frac{8\pi}{c} \delta\hat{J}_x \\ &= - \frac{8\pi e}{m_e c} \int dP_x dP_y dp \frac{p}{\gamma} \left[\frac{\partial}{\partial P_x} \left(\frac{p_x}{p_z} D_{\perp} \right) - \frac{p_x}{p_z} D_z \frac{1}{p} \frac{\partial}{\partial p} \right] F_b(P_x, P_y, p) \end{aligned}$$

- We now must determine the D_z coefficient

THE D_z COEFFICIENT

- Rewriting the D_z coefficient, we have that

$$D_z = \frac{e}{2c} \int_0^z dz' \frac{e^{i\omega\tau(z,z')}}{v_z(z')} \left[-c p_z(z') \frac{\partial}{\partial z'} \delta\hat{\varphi}(z') + i\omega p_x(z') \delta\hat{A}(z') \right]$$

neglect

$$= \frac{ie\omega}{2c} \gamma m_e \int_0^z dz' \frac{v_x(z')}{v_z(z')} \delta\hat{A}(z') e^{i\omega\tau(z,z')}$$

- We can manipulate the field amplitude as follows

$$\delta\hat{A}(z') = \sum_n \delta\hat{A}_n(z') = \sum_n \delta\hat{A}_n(0) e^{i \int_0^{z'} dz'' k_n(z'')} = \sum_n \delta\hat{A}_n(z) e^{-i \int_0^z dz' k_n(z')} e^{i \int_0^{z'} dz'' k_n(z'')}$$

- As a result, the D_z coefficient can be cast into the form

$$D_z = \frac{ie\omega}{2c} \gamma m_e \sum_n \delta\hat{A}_n(z) e^{-i \int_0^z dz' k_n(z')} \int_0^z dz' \frac{v_x(z')}{v_z(z')} e^{i \int_0^{z'} dz'' k_n(z'')} e^{i\omega\tau(z,z')}$$

$$= \frac{ie\omega}{2c} \gamma m_e \sum_n \delta\hat{A}_n(z) e^{-i\psi_n(z)} \int_0^z dz' \frac{v_x(z')}{v_z(z')} e^{i\psi_n(z')}$$

- Here ψ_n is the ponderomotive phase for the n^{th} harmonic

$$\psi_n(z) = -\omega t_0 + \int_0^z dz' \left(k_n(z') - \frac{\omega}{v_z(z')} \right)$$

THE D_z COEFFICIENT

- Making the approximations $k_n \approx k_{0n}$ in ψ_n , $P_x/\gamma m_e v_{||} \ll 1$ and

$$\psi_n(z) \approx -\omega t_0 + \left(k_{0n} - \frac{\omega}{v_{||}}\right)z - \frac{v_w P_x}{\gamma m_e v_{||}^2} \frac{\omega}{k_w v_{||}} \sin k_w z - \frac{v_w^2}{8 v_{||}^2} \frac{\omega}{k_w v_{||}} \sin 2k_w z$$

$$\frac{v_x(z)}{v_z(z)} \approx \frac{v_w}{v_{||}} \cos k_w z + \frac{P_x}{\gamma m_e v_{||}}$$

- We can show that

$$\int_0^z dz' \frac{v_x(z')}{v_z(z')} e^{i\psi_n(z')} \approx \frac{i v_w}{v_{||}} e^{-i\omega t_0} \sum_{l,m} (-1)^{l+m} \frac{l}{b_1} J_l(b_1) J_m(b_2) \frac{e^{-i\Delta k_{n+l+2m} z} - 1}{\Delta k_{n+l+2m}}$$

$$\longrightarrow \Delta k_n = \omega/v_{||} - k_{0n}$$

- We finally obtain

$$\frac{p_x}{p_z} D_z \approx -\frac{e\omega}{2c} \frac{v_w^2}{v_{||}^2} \gamma m_e \sum_{\substack{n,l,m, \\ l',m'}} (-1)^{l+m+l'+m'} \delta \hat{A}_n(z) \frac{l}{b_1} J_l(b_1) \frac{l'}{b_1} J_{l'}(b_1) J_m(b_2) J_{m'}(b_2)$$

$$\times \frac{e^{-i\Delta k_{l+2m-l'-2m'} z} - e^{-i\Delta k_{n+l'+2m'} z}}{\Delta k_{n+l+2m}}$$

THE SOURCE CURRENT

- The term in $D_{\perp} \ll D_z$ so that we need only consider

$$\frac{p_x}{p_z} D_z \approx - \frac{e\omega}{2c} \frac{v_w^2}{4v_{\parallel}^2} \gamma m_e \sum_{n,l,m} \delta \hat{A}_n(z) J_l^2(b_1) J J_m^{(\pm)}(b_2) \frac{1 - e^{-i\Delta k_{n+l+2m\pm 1} z}}{\Delta k_{n+l+2m\pm 1}}$$

where the JJ factor appears as

$$J J_m^{(\pm)}(b_2) = [J_m(b_1) - J_{m\pm 1}(b_1)]^2$$

and we have used the Bessel function identity

$$\frac{l}{x} J_l(x) = \frac{1}{2} [J_{l-1}(x) + J_{l+1}(x)]$$

- As a result, the source current is

$$\delta \hat{J}_x(z) \approx \frac{\omega_b^2 \omega}{8\pi c} \frac{v_w^2}{4v_{\parallel}^2} m_e \sum_{n,l,m} \delta \hat{A}_n(z) \int dP_x dP_y dp \frac{\partial F_b}{\partial p} J_l^2(b_1) J J_m^{(\pm)}(b_2) \frac{1 - e^{-i\Delta k_{n+l+2m\pm 1} z}}{\Delta k_{n+l+2m\pm 1}}$$

THE SMALL-SIGNAL GAIN

- As a result, the imaginary component of δk is

$$2\text{Im } \delta k(z) \approx -\frac{\omega_b^2 \omega m_e}{2k_{0n} c^2} \frac{v_w^2}{2v_{||}^2} \sum_{n,l,m} \int dP_x dP_y dp \frac{\partial F_b}{\partial p} J_l^2(b_1) J J_m^\pm(b_2) \frac{\sin \Delta k_{n+l+2m\pm 1} z}{\Delta k_{n+l+2m\pm 1}}$$

- Using this expression, the gain becomes

$$G(L) = 2 \int_0^L dz \text{Im } \delta k(z) \\ \approx \frac{\omega_b^2 L^2 \omega m_e}{4k_{0n} c^2} \frac{v_w^2}{2v_{||}^2} \sum_{n,l,m} \int dP_x dP_y dp \frac{\partial F_b}{\partial p} J_l^2(b_1) J J_m^\pm(b_2) \left(\frac{\sin \Theta_{n+l+2m\pm 1}}{\Theta_{n+l+2m\pm 1}} \right)^2$$

where $\Theta_{n+l+2m\pm 1} = \Delta k_{n+l+2m\pm 1} L/2$

$$G(L) \approx \frac{\omega_b^2 L^3 k_{0n} m_e}{8\gamma_{||}^2 v_{||}^2} \frac{v_w^2}{2v_{||}^2} \sum_{n,l,m} \int dP_x dP_y dp \frac{p}{p_{||}} F_b J_l^2(b_1) J J_m^\pm(b_2) \frac{\partial}{\partial \Theta_{n+l+2m\pm 1}} \left(\frac{\sin \Theta_{n+l+2m\pm 1}}{\Theta_{n+l+2m\pm 1}} \right)^2$$

$$\frac{\partial}{\partial p} = \frac{\partial \Theta_{n+l+2m\pm 1}}{\partial p} \frac{\partial}{\partial \Theta_{n+l+2m\pm 1}} \quad \frac{\partial \Theta_{n+l+2m\pm 1}}{\partial p} = -\frac{\omega L}{2v_{||}^2} \frac{1}{\gamma \gamma_{||}^2 m_e} \frac{p}{p_{||}}$$

THE SMALL-SIGNAL GAIN

- If we now choose a cold beam distribution, as in the case of the helical wiggler, of the form

$$F_b(P_x, P_y, p) = \delta(P_x) \delta(P_y) \frac{p_{\parallel}}{p} \delta(p - p_0)$$

- We also choose $n = l = 0$ and the small-signal gain in the low gain regime can be written as

$$G_m(L) \approx \frac{\omega_b^2 L^3 k_{0n}}{8 \gamma_0 \gamma_{\parallel}^2 v_{\parallel}^2} \frac{v_w^2}{2 v_{\parallel}^2} J J_m^+(b_2) \frac{\partial}{\partial \Theta_{2m+1}} \left(\frac{\sin \Theta_{2m+1}}{\Theta_{2m+1}} \right)^2$$

$$\approx K_{rms}^2 / \gamma_0^2$$

$$J J_m^+(b_2) = [J_m(b_2) - J_{m+1}(b_2)]^2$$

$$b_2 \approx \frac{K_{rms}^2 / 2}{1 + K_{rms}^2}$$

harmonics

$$\Theta_{2m+1} = \frac{L}{2} \left[\frac{\omega}{v_{\parallel}} - k_0 - (2m + 1) k_w \right]$$

THE HIGH-GAIN REGIME

- In the high-gain regime, we assume that

$$\delta\hat{A}(z) = \sum_n \delta\hat{A}_n e^{ik_n z}$$

$$\delta\hat{\varphi}(z) = \sum_n \delta\hat{\varphi}_n e^{ik_n z}$$

- We also neglect the initial value terms, so that

$$\frac{\partial}{\partial P_x} D_{\perp} = 0$$

$$\frac{\partial}{\partial P_x} \left(\frac{p_x}{p_z} D_{\perp} \right) \approx - \frac{e}{2cp_z} \left(1 + \frac{p_x^2}{p_z^2} \right) \delta\hat{A}(z)$$

$$D_z \approx \frac{ie\gamma m_e}{2} \left\{ \sum_n k_n \delta\hat{\varphi}_n(z) \int_0^z dz' e^{ik_n z' + i\omega\tau(z,z')} + \frac{\omega}{c} \sum_n \delta\hat{A}_n(z) \int_0^z dz' \frac{v_x(z')}{v_z(z')} e^{ik_n z' + i\omega\tau(z,z')} \right\}$$

 Two orbit integrals need to be evaluated

THE ORBIT INTEGRALS

- From the single-particle orbits, we know that

$$\frac{v_x(z')}{v_z(z')} = \frac{P_x}{\gamma m_e v_{\parallel}} \left(1 + \frac{v_w^2}{2v_{\parallel}^2} \right) + \frac{v_w}{v_{\parallel}} \left(1 + \frac{v_w^2}{8v_{\parallel}^2} + \frac{P_x^2}{\gamma^2 m_e^2 v_{\parallel}^2} \right) \cos k_w z' \\ + \frac{3v_w^2}{4v_{\parallel}^2} \frac{P_x}{\gamma m_e v_{\parallel}} \cos 2k_w z' + \frac{v_w^3}{8v_{\parallel}^3} \cos 3k_w z'$$

$$\tau(z, z') = \frac{z - z'}{v_{\parallel}} + \frac{v_w}{v_{\parallel}} \frac{P_x}{\gamma m_e v_{\parallel}} \frac{1}{k_w v_{\parallel}} (\sin k_w z - \sin k_w z') \\ + \frac{v_w^2}{8v_{\parallel}^2} \frac{1}{k_w v_{\parallel}} (\sin 2k_w z - \sin 2k_w z')$$

- We also have the Bessel function identities

$$e^{ib \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(b) e^{in\theta}$$

$$\cos \theta e^{ib \sin \theta} = \sum_{n=-\infty}^{\infty} \frac{n}{b} J_n(b) e^{in\theta}$$

THE ORBIT INTEGRALS

- As a result

$$\begin{aligned}
 \int_0^z dz' e^{ik_n z' + i\omega\tau(z,z')} &= e^{i\omega\tau(z)} \int_0^z dz' e^{ik_n z' - i\omega\tau(z')} \\
 &= e^{i\omega\tau(z) - i\omega t_0} \sum_{l,m} (-1)^{l+m} J_l(b_1) J_m(b_2) \int_0^z dz' e^{i\Delta k_{n+l+2m} z' - i\omega\tau(z')} \\
 &= e^{i\omega\tau(z) - i\omega t_0} \sum_{l,m} (-1)^{l+m} J_l(b_1) J_m(b_2) \frac{e^{-i\Delta k_{n+l+2m} z} - 1}{-i\Delta k_{n+l+2m}} \quad \text{neglect}
 \end{aligned}$$

- By a similar procedure

$$\begin{aligned}
 \int_0^z dz' \frac{v_x(z')}{v_z(z')} e^{ik_n z' + i\omega\tau(z,z')} &= e^{i\omega\tau(z) - i\omega t_0} \sum_{l,m} (-1)^{l+m} J_l(b_1) J_m(b_2) \\
 &\quad \times \left(\frac{P_x}{\gamma m_e v_{\parallel}} - \frac{v_w}{v_{\parallel}} \frac{l}{b_1} \right) \frac{e^{-i\Delta k_{n+l+2m} z} - 1}{-i\Delta k_{n+l+2m}} \quad \text{neglect}
 \end{aligned}$$

THE D_z COEFFICIENT

- As a result,

$$D_z = \frac{e\gamma m_e}{2c} e^{i\omega\tau(z) - i\omega t_0} \sum_{n,l,m} \frac{(-1)^{l+m}}{\Delta k_{n+l+2m}} J_l(b_1) J_m(b_2) e^{-i\Delta k_{n+l+2m} z} \left(ck_n \delta \hat{\varphi}_n + \omega \frac{v_w}{v_{||}} \frac{l}{b_1} \delta \hat{A}_n \right)$$

- We can show that

$$e^{i\omega\tau(z)} = e^{i\omega t_0} e^{i\omega z/v_{||}} \sum_{l,m} (-1)^{l+m} J_l(b_1) J_m(b_2) e^{-i(l+2m)k_w z}$$

$$\frac{p_x}{p_z} e^{i\omega\tau(z)} = -\frac{v_w}{v_{||}} e^{i\omega t_0} e^{i\omega z/v_{||}} \sum_{l,m} (-1)^{l+m} \frac{l}{b_1} J_l(b_1) J_m(b_2) e^{-i(l+2m)k_w z}$$

- It can now be shown that

$$D_z = \frac{e\gamma m_e}{2c} \sum_{\substack{n,l,m, \\ l',m'}} (-1)^{l+m+l'+m'} e^{ik_n z} e^{i(l+2m-l'-2m')k_w z} J_l(b_1) J_{l'}(b_1) J_{m'}(b_2) \\ \times \left\{ \frac{ck_n \delta \hat{\varphi}_n}{\Delta k_{n+l+2m}} J_m(b_2) - \frac{v_w}{2v_{||}} \frac{\omega \delta \hat{A}_n}{\Delta k_{n+l+2m \pm 1}} e^{\pm ik_w z} [J_m(b_2) - J_{m \pm 1}(b_2)] \right\}$$

$$\frac{p_x}{p_z} D_z = \frac{e\gamma m_e}{4c} \frac{v_w}{v_{||}} \sum_{\substack{n,l,m, \\ l',m'}} (-1)^{l+m+l'+m'} e^{ik_n z} e^{i(l+2m-l'-2m')k_w z} J_l(b_1) J_{l'}(b_1) [J_{m'}(b_2) - J_{m' \pm 1}(b_2)] \\ \times \left\{ \frac{ck_n \delta \hat{\varphi}_n}{\Delta k_{n+l+2m}} e^{\mp ik_w z} J_m(b_2) - \frac{v_w}{2v_{||}} \frac{\omega \delta \hat{A}_n}{\Delta k_{n+l+2m \pm 1}} [J_m(b_2) - J_{m \pm 1}(b_2)] \right\}$$

THE WAVE EQUATIONS

- In the Coulomb Gauge ($\nabla \cdot \mathbf{A} = 0$) the wave equation can be expressed as

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \delta A_x = - \frac{4\pi}{c} \delta J_x \longrightarrow \sum_n \left(\frac{\omega^2}{c^2} - k_n^2 \right) \delta \hat{A}_n e^{ik_n z} = - \frac{8\pi}{c} \delta \hat{J}_x$$

$$\frac{\partial}{\partial z} \frac{\partial}{\partial t} \delta \varphi = 4\pi \delta J_z \longrightarrow \sum_n k_n \delta \hat{\varphi}_n e^{ik_n z} = - \frac{8\pi}{\omega} \delta \hat{J}_z$$

- We now take the cold beam limit in which $b_1 \rightarrow 0$, and we have that $l = l' = 0$. In this regime we find that

$$D_z = \frac{e\gamma m_e}{2c} \sum_{n,m} e^{ik_n z} J_m(b_2) \left\{ \frac{ck_n \delta \hat{\varphi}_n}{\Delta k_{n+2m}} J_m(b_2) - \frac{v_w}{2v_{||}} \frac{\omega \delta \hat{A}_n e^{\pm ik_w z}}{\Delta k_{n+2m \pm 1}} [J_m(b_2) - J_{m \pm 1}(b_2)] \right\}$$

$$\frac{p_x}{p_z} D_z = \frac{e\gamma m_e}{4c} \frac{v_w}{v_{||}} \sum_{n,m} e^{ik_n z} [J_m(b_2) - J_{m \pm 1}(b_2)] \\ \times \left\{ \frac{ck_n \delta \hat{\varphi}_n e^{\mp ik_w z}}{\Delta k_{n+2m}} J_m(b_2) - \frac{v_w}{2v_{||}} \frac{\omega \delta \hat{A}_n}{\Delta k_{n+2m \pm 1}} [J_m(b_2) - J_{m \pm 1}(b_2)] \right\}$$

THE SOURCE CURRENTS

- As a result, the source currents can be expressed as

$$\begin{aligned}
 \delta \hat{J}_z &\approx -\frac{\omega_b^2}{8\pi} \sum_{n,m} \delta \hat{\varphi}_n e^{ik_n z} \int_0^\infty dp J_m^2(b_2) \frac{k_n v_{||} m_e}{\omega - k_{n+2m} v_{||}} \frac{\partial G_b}{\partial p} \\
 &\quad + \frac{\omega_b^2}{16\pi c} \sum_{n,m} \delta \hat{A}_{n\mp 1} e^{ik_n z} \int_0^\infty dp J_m(b_2) [J_m(b_2) - J_{m\pm 1}(b_2)] \frac{\omega v_w m_e}{\omega - k_{n+2m} v_{||}} \frac{\partial G_b}{\partial p} \\
 \delta \hat{J}_x &\approx -\frac{\omega_b^2}{8\pi c} \sum_n \delta \hat{A}_n e^{ik_n z} \int_0^\infty dp G_b(p) \frac{p}{\gamma p_z} \left(1 + \frac{p_x^2}{p_z^2} \right) \\
 &\quad + \frac{\omega_b^2}{16\pi c} \sum_{n,m} \delta \hat{A}_n e^{ik_n z} \int_0^\infty dp [J_m(b_2) - J_{m\pm 1}(b_2)]^2 \frac{v_w^2}{2v_{||}^2} \frac{\omega m_e}{\omega - k_{n+2m\pm 1} v_{||}} \frac{\partial G_b}{\partial p} \\
 &\quad - \frac{\omega_b^2}{16\pi} \sum_{n,m} \delta \hat{\varphi}_n e^{ik_{n\mp 1} z} \int_0^\infty dp J_m(b_2) [J_m(b_2) - J_{m\pm 1}(b_2)] \frac{\omega v_w m_e}{\omega - k_{n+2m} v_{||}} \frac{\partial G_b}{\partial p}
 \end{aligned}$$

- We now apply the cold beam distribution

$$G_b = \frac{p_z}{p} \delta(p - p_0)$$

THE AXIAL CURRENT

- We integrate by parts noting that

$$\frac{\partial}{\partial p} \left(\frac{k_n v_{||} m_e}{\omega - k_{n+2m} v_{||}} \right) = \frac{p}{\gamma p_{||}} \frac{k_n}{\gamma_{||}^2} \frac{\omega - 2mk_w v_{||}}{(\omega - k_{n+2m} v_{||})^2}$$

$$\frac{\partial}{\partial p} \left(\frac{v_w m_e}{\omega - k_{n+2m} v_{||}} \right) = \frac{p}{\gamma p_{||}} \frac{v_w (k_{n+2m} - \omega v_{||}/c)}{(\omega - k_{n+2m} v_{||})^2}$$

- So that the axial source current becomes

$$\delta \hat{J}_z \approx \frac{\omega_b^2}{8\pi\gamma_0\gamma_{||}^2} \sum_{n,m} k_n \delta \hat{\varphi}_n e^{ik_n z} J_m^2(b_2) \frac{\omega}{(\omega - k_{n+2m} v_{||})^2}$$

$$- \frac{\omega_b^2}{8\pi\gamma_0} \frac{v_w}{c} \sum_{n,m} \delta \hat{A}_{n\mp 1} e^{ik_n z} J_m(b_2) [J_m(b_2) - J_{m\pm 1}(b_2)] \frac{\omega (k_{n+2m} - \omega v_{||}/c^2)}{(\omega - k_{n+2m} v_{||})^2}$$

THE TRANSVERSE CURRENT

- Using Bessel function identities, we can show that

$$\sum_m [J_m(b_2) - J_{m\pm 1}(b_2)]^2 = \sum_m \overset{=1}{J_m(b_2)^2} + \sum_m \overset{=1}{J_{m\pm 1}(b_2)^2} - 2 \sum_m \overset{=0}{J_m(b_2)J_{m\pm 1}(b_2)} = 2$$

$$\longrightarrow \frac{v_w^2}{2v_{||}^2} = \frac{v_w^2}{4v_{||}^2} \sum_m [J_m(b_2) - J_{m\pm 1}(b_2)]^2$$

- As a result, the transverse source current can be expressed as

$$\begin{aligned} \delta \hat{J}_x \approx & -\frac{\omega_b^2}{8\pi\gamma_0 c} \sum_n \delta \hat{A}_n e^{ik_n z} \left\{ 1 - \frac{v_w^2}{4v_{||}^2} \sum_m [J_m(b_2) - J_{m\pm 1}(b_2)]^2 \frac{(\omega^2 - k_{n+2m\pm 1}^2 c^2)}{(\omega - k_{n+2m\pm 1} v_{||})^2} \right\} \\ & + \frac{\omega_b^2}{16\pi\gamma_0} \sum_{n,m} \delta \hat{\varphi}_n e^{ik_{n\mp 1} z} J_m(b_2) [J_m(b_2) - J_{m\pm 1}(b_2)] \frac{v_w (k_{n+2m} - \omega v_{||}/c^2)}{(\omega - k_{n+2m} v_{||})^2} \end{aligned}$$

THE COUPLED MODE EQUATIONS

- We now obtain the coupled mode equations

$$\left[\left(\omega - k_{n+2m} v_{\parallel} \right)^2 - \frac{\omega_b^2}{\gamma_0 \gamma_{\parallel}^2} \right] \delta \hat{\varphi}_n J_m(b_2) \approx$$

$$\approx - \frac{\omega_b^2}{\gamma_0 \gamma_{\parallel}^2} \frac{v_w}{2c} \delta \hat{A}_{n \mp 1} \left(k_{n+2m} - \omega v_{\parallel} / c^2 \right) [J_m(b_2) - J_{m \pm 1}(b_2)]$$

This term vanishes in the absence of a wiggler and yields the uncoupled space-charge wave

$$\left[\omega^2 - c^2 k_{n \mp 1}^2 - \frac{\omega_b^2}{\gamma_0} \left(1 - \frac{v_w^2}{4 v_{\parallel}^2} J J_m^{(\pm)}(b_2) \frac{\omega^2 - c^2 k_{n+2m}^2}{\left(\omega - k_{n+2m} v_{\parallel} \right)^2} \right) \right] \delta \hat{A}_{n \mp 1} J_m(b_2) \approx$$

$$\approx - \frac{\omega_b^2}{2 \gamma_0} \frac{k_n v_w}{\left(\omega - k_{n+2m} v_{\parallel} \right)^2} \delta \hat{\varphi}_n \left(k_{n+2m} - \omega v_{\parallel} / c^2 \right) J_m(b_2) [J_m(b_2) - J_{m \pm 1}(b_2)]$$

This term vanishes in the absence of a wiggler and yields the uncoupled E&M wave

THE DISPERSION EQUATION

- The coupled mode equations lead to the quartic dispersion equation

$$\left[\left(\omega - k_{n+2m} v_{\parallel} \right)^2 - \frac{\omega_b^2}{\gamma_0 \gamma_{\parallel}^2} \right] \left(\omega^2 - c^2 k_{n-1}^2 - \frac{\omega_b^2}{\gamma_0} \right) \approx$$

$$\approx - \frac{v_w^2}{4 v_{\parallel}^2} \frac{\omega_b^2}{\gamma_0} J J_m^{(\pm)}(b_2) \left(\omega^2 - c^2 k_{n+2m}^2 - \frac{\omega_b^2}{\gamma_0} \right)$$

- As in the case of the helical wiggler, this reduces to the cubic Pierce dispersion equation

$$\delta k (\delta k + 2\kappa) (\delta k - \Delta k) \approx - \frac{v_w^2}{4 v_{\parallel}^2} \frac{\omega_b^2}{\gamma_0 c^2 k_w^2} k_w^3 J J_m^{(\pm)}(b_2)$$

Not present with a helical wiggler

In comparison with the helical wiggler, this describes K_{rms}

THERMAL EFFECTS

- Using the same pitch-angle distribution as described for the helical wiggler, we can show that the dispersion equation for a planar wiggler becomes Freund & Antonsen, *Principles of Free-Electron Lasers*, Chapman & Hall, London, 1996, 2nd ed., p. 129]

$$\left[\left(\omega - k_{n+l+2m} v_{\parallel} \right)^2 - \frac{\omega_b^2}{\gamma_0 \gamma_{\parallel}^2} T_l(\xi_{n+l+2m}) \right] \left(\omega^2 - c^2 k_{n-1}^2 - \frac{\omega_b^2}{\gamma_0} \right) \approx$$

$$\approx - \frac{v_w^2 \omega_b^2}{4c^2 \gamma_0} J J_m^{(+)}(b_2) T_l(\xi_{n+l+2m}) \left(\omega^2 - c^2 k_{n+l+2m}^2 - \frac{\omega_b^2}{\gamma_0} T_l(\xi_{n+l+2m}) \right)$$

where the thermal function is

$$T_l(\xi) = \frac{\xi^2}{2\pi} \int_0^{2\pi} d\phi \int_0^{\infty} dz \frac{e^{-z}}{(z - \xi)^2} J_l^2(b_2)$$

 typo in book “+”

$$b_1 = \frac{\omega}{k_w v_{\parallel}} \frac{v_w}{v_{\parallel}} \frac{\Delta P}{\gamma m_e v_{\parallel}} z^{1/2} \cos \phi$$

$$\xi = \frac{\gamma_0^2 m_e^2}{\Delta P^2} \left(\frac{\omega^2}{k^2} - v_{\parallel}^2 \right)$$

THE HELICAL WIGGLER LIMIT

- The thermal function in for the helical wiggler is recovered in the limit in which

$$b_2 \rightarrow 0 \quad \longrightarrow \quad l = 0$$

- As a result T_0 can be expressed as

$$\begin{aligned} T_0(\xi) &\approx \xi^2 e^{-\xi} \int_{-\xi}^{\infty} dz \frac{e^{-z}}{z^2} \\ &= -\xi^2 e^{-\xi} \int_{-\xi}^{\infty} dz e^{-z} \frac{d}{dz} \left(\frac{1}{z} \right) \\ &= -\xi^2 e^{-\xi} \left\{ \frac{e^{-z}}{z} \Big|_{-\xi}^{\infty} + \int_{-\xi}^{\infty} dz \frac{e^{-z}}{z} \right\} \\ &= -\xi \left[1 + \xi e^{-\xi} E_1(-\xi) \right] \end{aligned}$$

ORBITS WITH AN AXIAL GUIDE FIELD

- The transverse velocity of single-particle orbits in a combined planar wiggler and axial guide field are

$$\mathbf{v}_{\perp} = \alpha_x \hat{\mathbf{e}}_x \cos k_w z + \alpha_y \hat{\mathbf{e}}_y \sin k_w z$$

$$\alpha_x = \frac{\Omega_w k_w v_{\parallel}^2}{\Omega_0^2 - k_w^2 v_{\parallel}^2} \quad \alpha_y = \frac{\Omega_w \Omega_0 v_{\parallel}}{\Omega_0^2 - k_w^2 v_{\parallel}^2}$$

- From conservation of energy, we require that the average axial velocity satisfies the equation

$$\beta_{\parallel}^2 \left[1 + \frac{\Omega_w^2 (\Omega_0^2 + k_w^2 v_{\parallel}^2)}{2(\Omega_0^2 - k_w^2 v_{\parallel}^2)^2} \right] = 1 - \frac{1}{\gamma^2}$$

- As for a helical wiggler, there are positive and negative energy regimes

$$\frac{dv_{\parallel}}{d\gamma} = \frac{c^2}{\gamma \gamma_{\parallel}^2 v_{\parallel}} \Phi_p$$

$$\Phi_p = 1 - \frac{\gamma_{\parallel}^2 \Omega_w^2 \Omega_0^2 (\Omega_0^2 + 3k_w^2 v_{\parallel}^2)}{2(\Omega_0^2 - k_w^2 v_{\parallel}^2)^3 + \Omega_w^2 \Omega_0^2 (\Omega_0^2 + 3k_w^2 v_{\parallel}^2)}$$

DISPERSION EQUATION - AXIAL FIELD

- When an axial guide field is included, the dispersion equation for the RHCP wave becomes [Freund & Antonsen, Principles of Free-Electron Lasers, Chapman & Hall, London, 1996, 2nd ed., Eq. (4.203)]

$$\left[\left(\omega - k_n v_{\parallel} \right)^2 - \frac{\omega_b^2}{\gamma_0 \gamma_{\parallel}^2} \Phi_p \right] \left(\omega^2 - c^2 k_{n-1}^2 - \frac{\omega_b^2}{\gamma_0} \frac{\omega - k_{n-1} v_{\parallel}}{\omega - \Omega_0 - k_{n-1} v_{\parallel}} \right) \approx$$

typo in book

$$\approx \frac{\omega_b^2}{4\gamma_0} \frac{\Omega_w^2 k_w k_{n-1} v_{\parallel}^2}{\left(\Omega_0 - k_w v_{\parallel} \right)^2} \Psi$$

typo in book

where

$$\Psi = 1 - \frac{\alpha_y^2 \left(\Omega_0^2 + 3k_w^2 v_{\parallel}^2 \right)}{2v_{\parallel}^2 \left(\Omega_0^2 - k_w^2 v_{\parallel}^2 \right)^3 + \alpha_y^2 \left(\Omega_0^2 + 3k_w^2 v_{\parallel}^2 \right)}$$

- This exhibits the same type of resonance behavior as in the case of a helical wiggler, but there is the $\mathbf{B}_0 \times \nabla \mathbf{B}_w$ drift that cause the beam to expand when there is a planar wiggler

STABILITY OF SPACE-CHARGE WAVES

- The dispersion of the space-charge waves in a planar and helical wiggler with axial guide fields is given by

Planar Wiggler

$$\left(\omega - k v_{\parallel}\right)^2 = \frac{\omega_b^2}{\gamma \gamma_{\parallel}^2} \Phi_p \quad \Phi_p = 1 - \frac{\gamma_{\parallel}^2 \Omega_w^2 \Omega_0^2 \left(\Omega_0^2 + 3k_w^2 v_{\parallel}^2\right)}{2\left(\Omega_0^2 - k_w^2 v_{\parallel}^2\right)^3 + \Omega_w^2 \Omega_0^2 \left(\Omega_0^2 + 3k_w^2 v_{\parallel}^2\right)}$$

Helical Wiggler

$$\left(\omega - k v_{\parallel}\right)^2 = \frac{\omega_b^2}{\gamma \gamma_{\parallel}^2} \Phi \quad \Phi = 1 - \frac{\gamma_{\parallel}^2 \beta_w^2 \Omega_0^2}{\left(1 + \beta_w^2\right) \Omega_0 - k_w v_{\parallel}}$$

- As a result, the space-charge waves becomes unstable in the negative-mass regime

SIDEBAND GENERATION

- Once electrons become trapped in the pondermotive potential, they oscillate in the trough of the wave. This oscillatory motion acts back on the radiation field to generate upper and lower beat waves about the radiation wavelength called sidebands.

Shifts in wavenumber & frequency due to sideband

$$[(k + \Delta k) + k_w]z - (\omega + \Delta\omega)t \approx \pm Kz$$

Sideband period

$$\omega = (k + k_w)v_{||}$$

$$\Delta\omega/\Delta k = v_{gr}$$

$$\Delta\omega \approx \mp \frac{Kv_{||}}{1 - v_{||}/v_{gr}}$$

- Sideband generation causes a broadening of the spectral bandwidth
 - The oscillation period varies with the specific trapped orbit, and this also causes a broadening of the spectral bandwidth

THE FIELD EQUATIONS

- We treat the case of a helical wiggler and consider fields corresponding to the vector potentials of the primary radiation field, $\delta A(z,t)$ and the sideband field, $\delta A_s(z,t)$

$$\delta A(z,t) = \delta A(z) [\hat{\mathbf{e}}_x \cos \alpha(z,t) - \hat{\mathbf{e}}_y \sin \alpha(z,t)]$$

$$\alpha(z,t) = \int_0^z dz' k(z') - \omega t$$

$$\delta A_s(z,t) = \delta A_s(z) [\hat{\mathbf{e}}_x \cos \alpha_s(z,t) - \hat{\mathbf{e}}_y \sin \alpha_s(z,t)]$$

$$\alpha_s(z,t) = \int_0^z dz' k_s(z') - \omega_s t + \theta_s$$

- The field equations are as derived previously in the low gain limit

$$2k \frac{d}{dz} \delta a \approx - \frac{\omega_b^2}{c^2} v_{z0} a_w \left\langle \frac{\cos \psi}{\gamma |v_z|} \right\rangle$$

$$\psi = \psi_0 + \int_0^z dz' \left(k + k_w - \frac{\omega}{v_z} \right)$$

$$2k_s \frac{d}{dz} \delta a_s \approx - \frac{\omega_b^2}{c^2} v_{z0} a_w \left\langle \frac{\cos \psi_s}{\gamma |v_z|} \right\rangle$$

$$\psi_s = \frac{\omega_s}{\omega} \psi_0 + \int_0^z dz' \left(k_s + k_w - \frac{\omega_s}{v_z} \right) + \theta_s$$

where

$$\langle (\dots) \rangle = \frac{1}{2\pi N} \int_{-N\pi}^{N\pi} d\psi_0 (\dots)$$

$$N = \frac{\omega}{|\omega - \omega_s|}$$

THE PENDULUM EQUATION

- The pendulum equation subject to the neglect of space-charge forces can be shown to be [Freund & Antonsen, *Principles of Free-Electron Lasers*, Chapman & Hall, London, 2nd ed. , chap. 6]

$$\frac{d^2}{dz^2} \psi \approx - \frac{\mu^2}{\cos \psi_{res}} \left(\sin \psi - \sin \psi_{res} + \frac{\delta a_s}{\delta a} \frac{\omega_s}{\omega} \sin \psi_s \right)$$

$$\psi_s \approx \frac{\omega_s}{\omega} \psi + \Delta k z + \theta_s \quad \longleftarrow \quad \Delta k = k_w (\omega - \omega_s) / \omega$$

$$\mu^2 = \frac{4k_w^2 a_w \delta a}{1 + a_w^2} \cos \psi_{res}$$

$$\sin \psi_{res} = - \frac{1}{2a_w \delta a} \left(\frac{a_w}{k_w} \frac{da_w}{dz} - \frac{1 + a_w^2}{2k_w^2} \frac{dk_w}{dz} \right)$$

- Here we use the notation $a_w = \Omega_w / ck_w$ which is equivalent to the wiggler strength parameter (K) which symbol is used for the sideband period here

THE TRAPPED TRAJECTORIES

- We expand $\psi = \psi_{res} + \delta\psi$, where

$$\frac{d^2}{dz^2}\delta\psi \approx -\mu^2\delta\psi + \frac{\mu^2}{2}\delta\psi^2\tan\psi_{res} + \frac{\mu^2}{6}\delta\psi^3 - \frac{\delta a_s}{\delta a} \frac{\mu^2}{\cos\psi_{res}} \sin\left(\frac{\omega_s}{\omega}\psi_{res} + \Delta kz + \theta_s\right)$$

To lowest order the bounce period is $2\pi/\mu$

- Expanding $\delta\psi = \delta\psi^{(0)} + \delta\psi^{(1)}$ we find that to lowest order

$$\delta\psi^{(0)} \approx \alpha \cos \mu_a z + \frac{\alpha^2}{4} \tan \psi_{res} \left(1 - \frac{1}{3} \cos 2\mu_a z\right)$$

- Where α is chosen so that $\delta\psi^{(0)}(z=0) = \psi_0$; hence

$$\psi_0 - \psi_{res} = \alpha + \frac{\alpha^2}{6} \tan \psi_{res}$$

- Where the anharmonic bounce period is given by

$$\mu_a = \mu \left[1 - \frac{\alpha^2}{16} \left(1 + \frac{5}{3} \tan^2 \psi_{res} \right) \right]$$

THE TRAPPED TRAJECTORIES

- The first order solution can be shown to be

$$\begin{aligned} \delta\psi^{(1)} \approx & \frac{\delta a_s}{\delta a} \frac{\mu}{\Delta\mu_{\pm}} \frac{1}{2 \cos \psi_{res}} \left[1 \pm \frac{\alpha^2 \mu}{16 \Delta\mu_{\pm}} \left(1 + \frac{5}{3} \tan^2 \psi_{res} \right) \right] \\ & \times \left\{ \left[1 \pm \frac{\alpha^2 \mu}{16 \Delta\mu_{\pm}} \left(1 + \frac{5}{3} \tan^2 \psi_{res} \right) \right] [\cos (\Delta\mu_{\pm} z + \phi_{res}) - \cos \phi_{res}] \sin \mu z \right. \\ & \mp [\sin (\Delta\mu_{\pm} z + \phi_{res}) - \sin \phi_{res}] \cos \mu z \\ & \left. \pm \frac{\alpha^2 \mu}{8 \Delta\mu_{\pm}} \left(1 + \frac{5}{3} \tan^2 \psi_{res} \right) \mu z \sin \phi_{res} \sin \mu z \right\} \end{aligned}$$

- Here

$$\phi_{res} = \frac{\omega_{res}}{\omega} \psi_{res} + \theta_s \qquad \Delta\mu_{\pm} = \Delta k \pm \mu$$

- This solution satisfies the following initial conditions

$$\delta\psi^{(1)}(z=0) = 0 \qquad \frac{d}{dz} \delta\psi^{(1)}(z=0) = 0$$

THE GAIN AT THE PRIMARY

- As we have seen, the small-signal power gain can be expressed as

$$G(L) = \frac{\omega_b^2}{\omega \gamma_0 c} \frac{a_w}{\delta a} \int_0^L dz \langle \sin \psi \rangle$$

- We choose a flat-top phase distribution with width 2Δ of electrons near the bottom of the ponderomotive well

$$\sigma(\psi_0) = \frac{\Delta}{\pi} H(\psi_0 + \Delta - \psi_{res}) H(\psi_{res} + \Delta - \psi_0)$$

$$\longrightarrow \langle \sin \psi \rangle \approx \sin \psi_{res} \left(1 - \frac{1}{2} \langle \delta \psi^{(0)2} \rangle - \langle \delta \psi^{(0)} \delta \psi^{(1)} \rangle \right)$$

- The small-signal gain can now be expressed as

$$G(L) \approx \frac{\omega_b^2 L}{\omega \gamma_0 c} \frac{a_w}{\delta a} \sin \psi_{res} \left\{ 1 + \frac{\Delta^2}{12} \left[\frac{\sin \mu L}{\mu L} + \frac{2 \sin 2\mu L}{3\mu L} \right] \right\}$$

THE SMALL-SIGNAL GAIN

- The small-signal gain for the sidebands is given by

$$G_s(L) = \frac{\omega_b^2}{\omega \gamma_0 c} \frac{a_w}{\delta a_s} \int_0^L dz \langle \sin \psi_s \rangle$$

where

$$\langle \sin \psi_s \rangle \approx \frac{\omega_s}{\omega} \langle \delta \psi_s^{(1)} \cos \psi_s^{(0)} \rangle$$

- After evaluation of these phase averages, the sideband gain becomes

$$G_s(L) \approx \frac{\omega_b^2 \mu L^2}{8 \omega_s \gamma_0 c} \frac{a_w}{\delta a_s} \frac{1}{\cos \psi_{res}} \left\{ 1 \pm \frac{\mu \Delta^2}{48 \Delta \mu_{\pm}} \left(1 + \frac{5}{e} \tan^2 \psi_{res} \right) \right. \\ \left. \times \left(1 - \frac{\Delta \mu_{\pm} L}{2} \frac{d}{d \Theta_{\pm}} \right) \right\} \left(\frac{\sin \Theta_{\pm}}{\Theta_{\pm}} \right)^2$$

$$\Theta_{\pm} = \Delta \mu_{\pm} L / 2$$

- It is the lower sideband that has the positive gain