LA-UR-13-29036

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Title: Benchmark Solutions in Solid Mechanics and Implicit-Explicit Schemes
in radiation Hydrodynamics

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Intended for: Job interview.

Issued: 2013-11-26

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Benchmark Solutions in Solid Mechanics and Implicit-Explicit Schemes in Radiation Hydrodynamics

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December 3, 2013

Previous scientific work

- University of Minnesota (UMN)
 - Developed and analyzed new constitutive models of polymer gel swelling
 - Devised and implemented optimally convergent mixed finite element method (FEM) for linearized viscous polymer gels
 - Numerically simulated stress buildup along gel boundary due to solid-fluid mixing.
- Pennsylvania State University (PSU)
 - Devised new explicit multi-rate time integration scheme for differential-algebraic equations (DAEs)
 - Developed a *posteriori* error estimator; Demonstrated feasibility of using time integrator and estimator to solve multi-rate problems in computational ocean modeling.

Objectives of current research

- (LANL Computational Physics Division) Develop benchmark solutions for solid mechanics problems to verify physics simulation code.
- (LANL Theoretical Division) Develop an efficient operator splitting algorithm for solving hyperbolic problems in radiation hydrodynamics.

Part I: Benchmark Solution of the Dynamic Sphere Problem

Motivation for this research

- Research institutions (such as LANL) develop large-scale general-purpose physics codes that must be verified for correctness.
- One way to do this is to develop test problems with analytic solutions to serve as computational benchmarks.

Goals accomplished in this research

- Devise series of test problems for physics code developers.
- Used convergence analyses to obtain converged benchmark solutions.
- Improved LANL benchmark solution code (fixed bugs, implemented higher order quadrature schemes to remove oscillations, improved memory usage).
- Devise and implement consistent initial and boundary conditions.

Schematic of dynamic sphere problem

- Analytic solution includes:
 - Dirichlet BC: Displacement
 - Neumann BC: Strain
 - Robin BC: Stress
- Time-varying BCs applied at r_i , r_o
- This talk focuses only on Robin (stress) BCs

Radially symmetric dynamic sphere IBVP

In spherical coordinates and Lagrangian reference frame, the governing equations are:

$$c_r^{-2} u_{tt} = u_{rr} + \frac{2}{r} u_r - \frac{\eta^2}{r^2} u + c_r^{-2} f$$

$$f = \frac{1}{\rho} \left[\frac{\partial \lambda_{rr}^p}{\partial r} + \frac{2}{r} (\lambda_{rr}^p - \lambda_{\theta\theta}^p) \right]$$

$$\lambda^p = -C \varepsilon^p$$

$$u(r, 0) = 0, \quad u_t(r, 0) = 0,$$

$$\alpha_B u_r(r_B, t) + \beta_B u(r_B, t) + \lambda_{rr}^p(r_B, t) = \text{BC}_B(t); \quad B = i, o$$

Plastic stress tensor λ^p represents part of stress arising from plastic strain.

Analytic solution for u

- Assume $\lambda_{rr}^p, \lambda_{\theta\theta}^p$ are known.
- u has the form $u(r, t) = r^{-1/2}[\bar{w}(r, t) + \tilde{w}(r, t)]$.
- $\bar{w}(r, t) = \gamma_0(t) + \gamma_1(t)r$ contains BCs.
- $\tilde{w}(r, t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(r)$ satisfies homogeneous BCs.
- $a_n(t)$ contains plasticity effects.

Numerical implementation

- Choose number of grid points nr through shell, number of time steps nt , and number of eigenmodes nl .
- Numerical quadrature in time and space.
- Take $\bar{w}(r, t) = \gamma_0(t) + \gamma_1(t)r$ and $\tilde{w}(r, t) = \sum_{n=1}^{nl} a_n(t)\psi_n(r)$ at each time step and grid point.
- At each time step, iterate to compute λ^p and u if necessary.
- Salient numerical error arises from space and time quadratures and series truncation.

Objective: Demonstrate converged benchmark solution.

Testing convergence of solution for different materials

We demonstrate convergence of benchmark solution for:

- Elastic transverse isotropic material
- Bodner-Partom elastic-viscoplastic isotropic material

Effectively converged benchmark solution

An effectively converged benchmark solution is a solution whose error with respect to a fixed reference solution is much smaller than the difference between a computational physics code solution and the benchmark solution.

Convergence for test problems

- Consider convergence of analytic solution in nr , nt , nl .
- At inner radius r_i , impose void stress BC: $BC_i(t) = 0$.
- At outer radius r_o , impose time-varying smooth jump Robin (stress) BC.
- All integrals are computed using sixth-order Newton-Cotes quadrature.

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Elastic transverse isotropic material

$\lambda^p = 0$, so IBVP to solve is:

$$c_r^{-2} u_{tt} = u_{rr} + \frac{2}{r} u_r - \frac{\eta^2}{r^2} u$$

$$u(r, 0) = 0$$

$$u_t(r, 0) = 0$$

$$\alpha_i u_r(r_i, t) + \beta_i u(r_i, t) = BC_i(t)$$

$$\alpha_o u_r(r_o, t) + \beta_o u(r_o, t) = BC_o(t)$$

Here, $\eta^2 = -2(C_{rr\theta\theta} - C_{\theta\theta\varphi\varphi} - C_{\theta\theta\theta\theta})/C_{rrrr}$.

Convergence results for elastic transverse isotropic problem

- Eigenmode convergence of u .
- Temporal convergence of $\sigma_{\theta\theta}$.
- Spatial convergence of u_t .

Elastic transverse isotropic: Self-convergence in eigenmode I

- We take very small time steps and grid sizes to reduce the errors caused by time and space discretization.
- Take $nt = 14000$, $nr = 14000$ and compute up to $T = 5 \mu s$.

Figure: Left: nI convergence plot of displacement u vs. t at r_i . Right: Zoomed in plot of u .

Elastic transverse isotropic: Self-convergence in eigenmode II

- We compute $L^\infty(0, T)$ norm of percent errors of u relative to reference solution computed with a very large number of eigenmodes ($nl = 1200$) to test eigenmode convergence.
- At $nl = 400$, percent error is $\approx 10^{-7} \%$.
- Slope of line (i.e. rate of convergence) is ≈ 3 .
- We choose $nl = 400$ for our eigenmode converged solution.

Figure: Log-log plot of $L^\infty(0, T)$ norm of percent error of u at r_i vs. nl .
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Elastic transverse isotropic: Self-convergence in time I

- We take the eigenmode number for self-converged solution ($nl = 400$) and very small grid size to reduce spatial discretization error.
- Take $nr = 14000$ and compute up to $T = 5 \mu s$.

Figure: Left: nt convergence plot of tangential stress $\sigma_{\theta\theta}$ vs. r at T . Right: Zoomed in plot of $\sigma_{\theta\theta}$.

Elastic transverse isotropic: Self-convergence in time II

- We compute $L^\infty(r_i, r_o)$ norm of percent errors of $\sigma_{\theta\theta}$ relative to reference solution computed with a very large number of time steps ($nt = 14000$) to test temporal convergence.
- At $nt = 4000$, percent error is $\approx 10^{-9}$ %.
- Slope of line (i.e. rate of convergence) is ≈ 6 .
- We choose $nt = 4000$ for our temporally converged solution.

Figure: Log-log plot of $L^\infty(r_i, r_o)$ norm of percent error of $\sigma_{\theta\theta}$ at T vs. nt .
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Elastic transverse isotropic: Self-convergence in space I

- We take the eigenmode and time step number for self-converged solution ($nr = 400$, $nt = 4000$).
- Compute up to $T = 5 \mu\text{s}$.

Figure: Left: nr convergence plot of velocity u_t vs. t at r_o . Right: Zoomed in plot of u_t .

Elastic transverse isotropic: Self-convergence in space II

- We compute $L^\infty(0, T)$ norm of percent errors of u_t relative to reference solution computed with a very large number of zones ($nr = 14000$) to test spatial convergence.
- At $nr = 500$, percent error is $\approx 10^{-8}$ %.
- Slope of line (i.e. rate of convergence) is ≈ 6 .
- We choose $nr = 500$ for our spatially converged solution.

Figure: Log-log plot of $L^\infty(0, T)$ norm of percent error of u_t at r_o vs. nr .
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Summary: Elastic transverse isotropic benchmark solution

Figure: Plots of benchmark u vs. t at r_i , $\sigma_{\theta\theta}$ vs. r at T , and u_t vs. t at r_o .

All benchmark fields presented have space, time, and eigenmode errors of $< 10^{-6}$ %.

Future use of elastic transverse isotropic benchmark

This benchmark can be used to test any physics code that contains elastic transverse isotropic models ([Chabaud et. al., 2012] "The dynamic sphere test problem", LA-UR-12-21452).

Bodner-Partom viscoplastic constitutive model

The constitutive model variables are

- ϵ^P : plastic strain tensor
- Z : isotropic hardening
- W^P : plastic work

and the equations of the model are

$$\frac{d\epsilon^P}{dt} = G(S, Z, W^P)S$$

$$Z = Z_1 + (Z_0 - Z_1) \exp(-mW^P/Z_0)$$

$$\frac{dW^P}{dt} = \sigma_{ij} \frac{d\epsilon_{ij}^P}{dt}$$

where σ is the stress and $S = \sigma - \frac{1}{3}\text{tr}\sigma I$
is the stress deviator.

Convergence results for Bodner-Partom problem

- Eigenmode convergence of ε_{rr} .
- Temporal convergence of λ_{rr}^p .
- Spatial convergence of u_t .

Bodner-Partom: Self-convergence in eigenmode I

- We take very small time steps and grid sizes to reduce the errors caused by time and space discretization.
- Take $nt = 14000$, $nr = 14000$ and compute up to $T = 3.5 \mu s$.

Figure: Left: nI convergence plot of radial strain ε_{rr} vs. t at r_o . Right: Zoomed in plot of ε_{rr} .

Bodner-Partom: Self-convergence in eigenmode II

- We compute $L^\infty(0, T)$ norm of percent errors of ε_{rr} relative to reference solution computed with a very large number of eigenmodes ($nl = 500$) to test eigenmode convergence.
- At $nl = 300$, percent error is $\approx 2.5 \times 10^{-7} \%$.
- Slope of line (i.e. rate of convergence) is ≈ 3 .
- We choose $nl = 300$ for our eigenmode converged solution.

Figure: Log-log plot of $L^\infty(0, T)$ norm of percent error of ε_{rr} at r_o vs. nl .
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Bodner-Partom: Self-convergence in time I

- We take the eigenmode number for self-converged solution ($nl = 300$) and very small grid size to reduce spatial discretization error.
- Take $nr = 14000$ and compute up to $T = 3.5 \mu s$.

Figure: Left: nt convergence plot of radial plastic stress λ_{rr}^p vs. r at T . Right: Zoomed in plot of λ_{rr}^p .

Bodner-Partom: Self-convergence in time II

- We compute $L^\infty(r_i, r_o)$ norm of percent errors of λ_{rr}^p relative to reference solution computed with a very large number of time steps ($nt = 14000$) to test temporal convergence.
- At $nt = 9000$, percent error is $\approx 4.5 \times 10^{-3} \%$.
- Slope of line (i.e. rate of convergence) is ≈ 3 .
- We choose $nt = 9000$ for our temporally converged solution.

Figure: Log-log plot of $L^\infty(r_i, r_o)$ norm of percent error of λ_{rr}^p at T vs. nt .
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Bodner-Partom: Self-convergence in space I

- We take the eigenmode and time step number for converged solution ($nr = 300$, $nt = 9000$).
- Compute up to $T = 3.5 \mu\text{s}$.

Figure: Left: nr convergence plot of velocity u_t vs. t at r_i . Right: Zoomed in plot of u_t .

Bodner-Partom: Self-convergence in space II

- We compute $L^\infty(0, T)$ norm of percent errors of u_t relative to reference solution computed with a very large number of zones ($nr = 14000$) to test spatial convergence.
- At $nr = 4000$, percent error is $\approx 5 \times 10^{-9} \%$.
- Slope of line (i.e. rate of convergence) is ≈ 6 .
- We choose $nr = 4000$ for our spatially converged solution.

Figure: Log-log plot of $L^\infty(0, T)$ norm of percent error of u_t at r_i vs. nr .
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Summary: Bodner-Partom benchmark solution

Figure: Plots of benchmark ε_{rr} vs. t at r_o , λ_{rr}^p vs. r at T , and u_t vs. t at r_i .

All benchmark fields presented have space, time, and eigenmode errors of $< 5 \times 10^{-3} \%$.

Future use of Bodner-Partom benchmark

This benchmark can be used to test any physics code that contains Bodner-Partom visco-plastic models.

Connection to larger scientific community

These analytic solutions will be published in peer-reviewed journals and used as benchmarks to test accuracy of sophisticated computational physics codes (such as FLAG).

Part II: Computational Radiation Hydrodynamics

Outline

- Describe 1D radiation hydrodynamics problem.
- Extend previous IMEX work to non-quasi-static radiation flux.
- Apply algorithm to radiative shock test case.

Motivation for this research

- Radiation transport and fluid flow exhibit multiple time scales, with radiation transport exhibiting much smaller time scales.
- Radiation effects impose prohibitive stability restriction on time step size for fully explicit scheme to solve coupled problems.
- Faster time to solution for coupled physics: Split the system in a consistent way and use explicit scheme to solve hydrodynamics problem and implicit scheme to solve for radiation effects.

Goals accomplished in this research

- Extended implicit-explicit (IMEX) time integration schemes to purely hyperbolic radiation hydrodynamics problems.
- Established temporal convergence of splitting algorithm.

Previous work on IMEX methods for radiation hydrodynamics

- [Kadiaglu, Knoll 2010] Kadiaglu and Knoll describe second order self-consistent implicit-explicit (IMEX) time integration for low-energy density radiation hydrodynamics (1D Euler equations with energy diffusion term).
- [Kadiaglu et. al. 2010] Kadiaglu et. al. extend IMEX scheme to high-energy density radiation hydrodynamics (1D Euler equations with source terms plus radiation energy equation with diffusion, quasi-static radiation flux).

1D radiation hydrodynamics equations in previous work I

In Eulerian coordinates, for quasi-static radiation flux

($\partial F_r / \partial t = 0$):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) = -PS_F$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}[u(E + p)] = -PCS_E$$

$$\frac{\partial E_r}{\partial t} + C \frac{\partial F_r}{\partial x} = CS_E$$

$$0 + \frac{1}{3}C \frac{\partial E_r}{\partial x} = CS_F \implies F_r = \frac{4}{3} \frac{u}{C} E_r - \frac{1}{3\sigma} \frac{\partial E_r}{\partial x}$$

with source terms

$$S_F = -\sigma(F_r - \frac{4}{3} \frac{u}{C} E_r)$$

$$S_E = \sigma[T^4 - E_r - \frac{u}{C}(F_r - \frac{4}{3} \frac{u}{C} E_r)]$$

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1D radiation hydrodynamics equations in previous work II

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p + \frac{1}{3} P E_r) = 0$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}[u(E + p)] = -PC\sigma(T^4 - E_r) - \frac{1}{3} P u \frac{\partial E_r}{\partial x}$$

$$\frac{\partial E_r}{\partial t} + \frac{\partial}{\partial x}(\frac{4}{3} u E_r) = \frac{\partial}{\partial x}(\frac{C}{3\sigma} \frac{\partial E_r}{\partial x}) + C\sigma(T^4 - E_r) + \frac{1}{3} u \frac{\partial E_r}{\partial x}$$

Note diffusion term in last equation.

Coupling 1D ideal gas flow with radiation transport

New contribution: $\partial F_r / \partial t \neq 0$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) = -PS_F$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}[u(E + p)] = -PCS_E$$

$$\frac{\partial E_r}{\partial t} + C \frac{\partial F_r}{\partial x} = CS_E$$

$$\frac{\partial F_r}{\partial t} + \frac{1}{3}C \frac{\partial E_r}{\partial x} = CS_F$$

with source terms

$$S_F = -\sigma(F_r - \frac{4}{3} \frac{u}{C} E_r)$$

$$S_E = \sigma[T^4 - E_r - \frac{u}{C}(F_r - \frac{4}{3} \frac{u}{C} E_r)]$$

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IMEX splitting: Explicit system

$$\frac{\partial \rho}{\partial t} + \frac{\partial m}{\partial x} = 0$$

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial x} \left(\frac{m^2}{\rho} + p + \frac{1}{3} P E_r \right) = 0$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x} \left[\frac{m}{\rho} \left(E + p + \frac{1}{3} P E_r \right) \right] = 0$$

$$\frac{\partial E_r}{\partial t} + \frac{\partial}{\partial x} \left(\frac{m}{\rho} E_r \right) = 0$$

Note: This is same explicit block as previous work.

IMEX splitting: Implicit system

$$\begin{aligned}\frac{\partial}{\partial t}(\rho u) &= -PS_F + \frac{1}{3}P\frac{\partial E_r}{\partial x} \\ \frac{\partial}{\partial t}\left(c_v\frac{\rho T}{\gamma} + \frac{1}{2}\rho u^2\right) &= -PCS_E + \frac{\partial}{\partial x}\left(\frac{1}{3}PE_ru\right) \\ \frac{\partial E_r}{\partial t} + \frac{\partial}{\partial x}(CF_r - uE_r) &= CS_E \\ \frac{\partial F_r}{\partial t} + \frac{1}{3}C\frac{\partial E_r}{\partial x} &= CS_F\end{aligned}$$

Time discretization: Explicit block

Use explicit predictor-corrector second order TVD Runge-Kutta method (strong stability preserving, good near shocks) to solve explicit problem.

$$\mathbf{U}^*(x) = \mathbf{U}^{n-1}(x) - \Delta t^n \frac{\partial \mathbf{F}}{\partial x}(\mathbf{U}^{n-1})$$

$$\bar{\mathbf{U}}^k(x) = \frac{1}{2}\mathbf{U}^{n-1}(x) + \frac{1}{2}\mathbf{U}^*(x) - \frac{1}{2}\Delta t^n \frac{\partial \mathbf{H}}{\partial x}(\mathbf{V}^{n,k-1})$$

where

$$\mathbf{U}^* = \begin{bmatrix} \rho^* \\ m^* \\ E^* \\ E_r^* \end{bmatrix}, \quad \mathbf{V}^{n,k-1} = \begin{bmatrix} \rho^* \\ m^{n,k-1} \\ \tau^{n,k-1} \\ E_r^{n,k-1} \end{bmatrix}, \quad \bar{\mathbf{U}}^k = \begin{bmatrix} \rho^{n,k} \\ \bar{m}^k \\ \bar{E}^k \\ \bar{E}_r^k \end{bmatrix}$$

IMEX algorithm

At time step n ,

- Explicit solve: Predict solution of explicit block using solution from previous time step.
- For $k = 1, 2, \dots$
 - Explicit solve: Correct solution of explicit block using information from implicit block.
 - Implicit solve: Solve Newton iteration system $\mathbf{J}(W^{n,k})\delta W^{n,k} = -\mathcal{G}(W^{n,k})$, $W^{n,k+1} = W^{n,k} + \delta W^{n,k}$ for implicit block.
- Update solution for next time step.

Time discretization: Implicit block I

Use implicit Crank-Nicolson scheme.

$$\begin{aligned} \frac{2}{\Delta t^n}(\rho^{n,k} u^{n,k} - \bar{m}^k) &= -PS_F^{n,k} + \frac{1}{3}P \frac{\partial E_r^{n,k}}{\partial x} - PS_F^{n-1} \\ &+ \frac{1}{3}P \frac{\partial E_r^{n-1}}{\partial x} \\ \frac{2}{\Delta t^n} [c_v \frac{\rho^{n,k} T^{n,k}}{\gamma} + \frac{1}{2} \rho^{n,k} (u^{n,k})^2 - \bar{E}^k] &= -PCS_E^{n,k} \\ &+ \frac{\partial}{\partial x} (\frac{1}{3} P E_r^{n,k} u^{n,k}) - PCS_E^{n-1} + \frac{\partial}{\partial x} (\frac{1}{3} P E_r^{n-1} u^{n-1}) \end{aligned}$$

Time discretization: Implicit block II

$$\begin{aligned}
 & \frac{2}{\Delta t^n} (E_r^{n,k} - \bar{E}_r^k) + \frac{\partial}{\partial x} (CF_r^{n,k} - u^{n,k} E_r^{n,k}) \\
 & + \frac{\partial}{\partial x} (CF_r^{n-1} - u^{n-1} E_r^{n-1}) = CS_E^{n,k} + CS_E^{n-1} \\
 & \frac{2}{\Delta t^n} (F_r^{n,k} - F_r^{n-1}) + \frac{1}{3} C \frac{\partial E_r^{n,k}}{\partial x} + \frac{1}{3} C \frac{\partial E_r^{n-1}}{\partial x} = \\
 & CS_F^{n,k} + CS_F^{n-1}
 \end{aligned}$$

Spatial discretization: Explicit block

Use centered difference scheme with local Lax-Friedrichs fluxing procedure. Compute explicit quantities (ρ, m, E, E_r) at cell centers. At cell center x_i ,

$$\begin{aligned}\mathbf{U}_i^* &= \mathbf{U}_i^{n-1} - \frac{\Delta t^n}{\Delta x} (\hat{\mathbf{F}}_{i+1/2}^{n-1} - \hat{\mathbf{F}}_{i-1/2}^{n-1}) \\ \overline{\mathbf{U}}_i^k &= \frac{1}{2} \mathbf{U}_i^{n-1} + \frac{1}{2} \mathbf{U}_i^* - \frac{1}{2} \frac{\Delta t^n}{\Delta x} (\hat{\mathbf{H}}_{i+1/2}^{n,k-1} - \hat{\mathbf{H}}_{i-1/2}^{n,k-1})\end{aligned}$$

Spatial discretization: Implicit block I

Use staggered grid (needed for stability of hyperbolic problem) centered difference. Compute (ρ, u, T, E_r) at cell centers, F_r at cell faces.

$$\begin{aligned}
 \frac{2}{\Delta t^n}(\rho_i^{n,k} u_i^{n,k} - \bar{m}_i^k) &= -PS_{F,i}^{n,k} + \frac{1}{3}P \frac{E_{r,i+1}^{n,k} - E_{r,i-1}^{n,k}}{2\Delta x} \\
 &\quad - PS_{F,i}^{n-1} + \frac{1}{3}P \frac{E_{r,i+1}^{n-1} - E_{r,i-1}^{n-1}}{2\Delta x} \\
 \frac{2}{\Delta t^n} \left[c_v \frac{\rho_i^{n,k} T_i^{n,k}}{\gamma} + \frac{1}{2} \rho_i^{n,k} (u_i^{n,k})^2 - \bar{E}_i^k \right] &= -PCS_{E,i}^{n,k} \\
 &\quad + \frac{1}{3}P \frac{E_{r,i+1}^{n,k} u_{i+1}^{n,k} - E_{r,i-1}^{n,k} u_{i-1}^{n,k}}{2\Delta x} - PCS_{E,i}^{n-1} \\
 &\quad + \frac{1}{3}P \frac{E_{r,i+1}^{n-1} u_{i+1}^{n-1} - E_{r,i-1}^{n-1} u_{i-1}^{n-1}}{2\Delta x}
 \end{aligned}$$

Spatial discretization: Implicit block II

$$\begin{aligned}
 & \frac{2}{\Delta t^n} (E_{r,i}^{n,k} - \bar{E}_{r,i}^k) + C \frac{F_{r,i+1/2}^{n,k} - F_{r,i-1/2}^{n,k}}{\Delta x} \\
 & - \frac{E_{r,i+1}^{n,k} u_{i+1}^{n,k} - E_{r,i-1}^{n,k} u_{i-1}^{n,k}}{2\Delta x} + C \frac{F_{r,i+1/2}^{n-1} - F_{r,i-1/2}^{n-1}}{\Delta x} \\
 & - \frac{E_{r,i+1}^{n-1} u_{i+1}^{n-1} - E_{r,i-1}^{n-1} u_{i-1}^{n-1}}{2\Delta x} = CS_{E,i}^{n,k} + CS_{E,i}^{n-1} \\
 & \frac{2}{\Delta t^n} (F_{r,i+1/2}^{n,k} - F_{r,i+1/2}^{n-1}) + \frac{1}{3} C \frac{E_{r,i+1}^{n,k} - E_{r,i}^{n,k}}{\Delta x} + \\
 & \frac{1}{3} C \frac{E_{r,i+1}^{n-1} - E_{r,i}^{n-1}}{\Delta x} = CS_{F,i+1/2}^{n,k} + CS_{F,i+1/2}^{n-1}
 \end{aligned}$$

Numerical test: Moving radiative shock I

Consider initial shock wave propagating through fluid. We take a uniform spatial mesh of 400 cells.

Figure: Left: Propagation of shock in fluid temperature T in time. Right: Propagation of smooth radiation temperature wave $T_r = E_r^{1/4}$.

Numerical test: Moving radiative shock II

Since moving shock wave problem is highly transient, we carry out time convergence study. T converges at a rate of ≈ 1.2 and T_r converges at a rate of ≈ 1.1 .

Figure: Temporal convergence plots for fluid and radiation temperature at time $t = 0.02$.

Connection to larger scientific community

This work applies an efficient time-stepping algorithm to a new type of radiation hydrodynamics model that leads to faster time to convergence than purely explicit or implicit methods.

Summary for Part II

- We describe a 1D radiation hydrodynamics problem.
- We develop a stable implicit-explicit finite difference algorithm.
- We establish temporal convergence of the method for a moving shock wave.

Summary of scientific work

- (UMN) Mathematical modeling and numerical simulation of polymer gel dynamics.
- (PSU) Development of efficient multi-rate time integration schemes and *a posteriori* error estimators.
- (LANL) Development of benchmark solutions for verification of computational physics codes.
- (LANL) Development of operator-splitting methods to solve coupled radiation/hydrodynamics problems.