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Author(s): Baertschi, Andreas

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interactive examples on https://cnls.lanl.gov/~baertschi/QCA/

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Quantum Computing Algorithms
Tricks and Tools

LLNL CASIS Quantum Sensing and Information Processing Series

Andreas Bärtschi
baertschi@lanl.gov

July 31 / August 1, 2019

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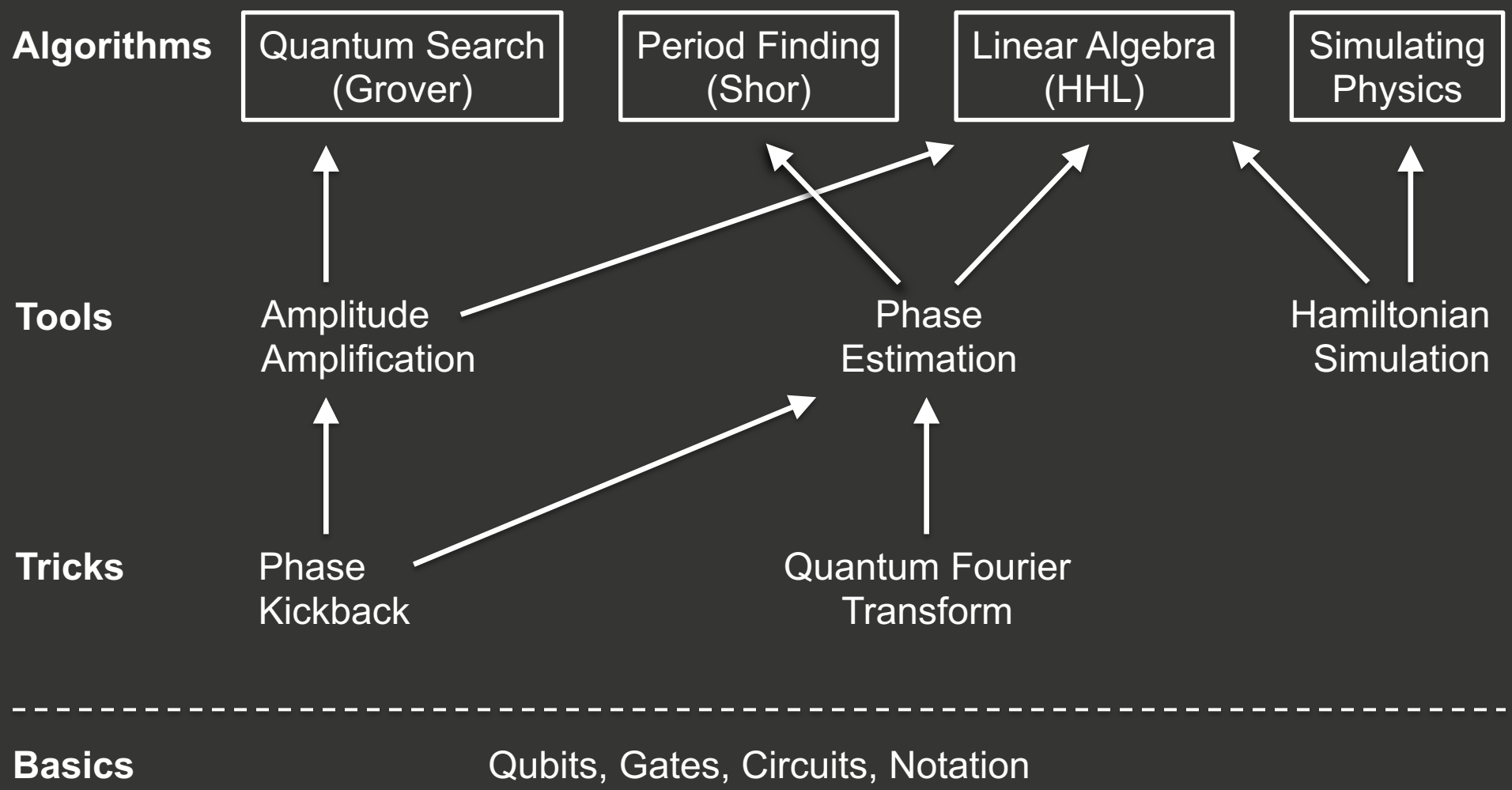
Algorithms
Quantum Search (Grover)
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Amplitude Amplification
Phase Estimation
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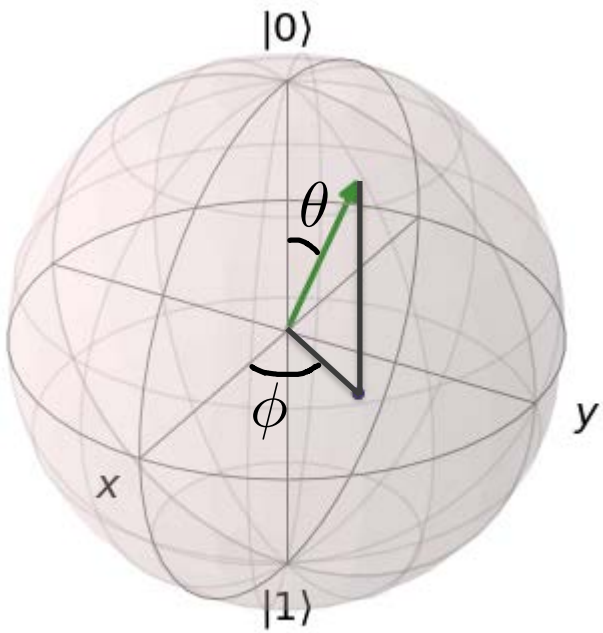
Basics
Qubits, Gates, Circuits, Notation

Theory (Slides) and Practice (Quirk)

https://cnls.lanl.gov/~baertschi/QCA/

From Bits to Qubits

Bits: Either 0 or 1



Qubits: $\alpha|0\rangle + \beta|1\rangle$ with $\alpha, \beta \in \mathbb{C}$

$$|\alpha|^2 + |\beta|^2 = 1$$

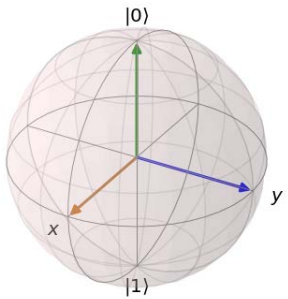
$$\Rightarrow e^{i\delta}\left(\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right)$$

global phase relative phase

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

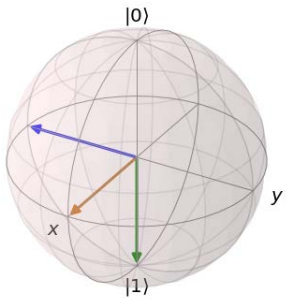
$$|+\rangle := \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad |-\rangle := \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Single Qubit Gates

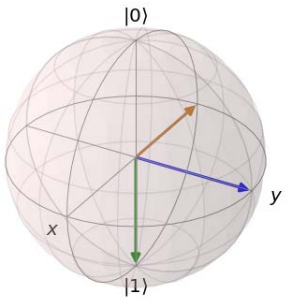


Pauli

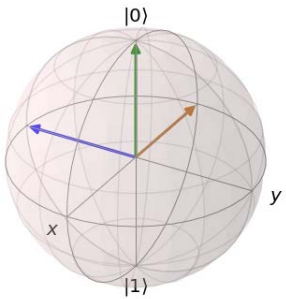
Clifford



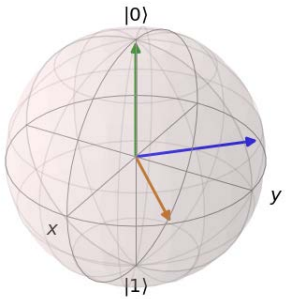
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



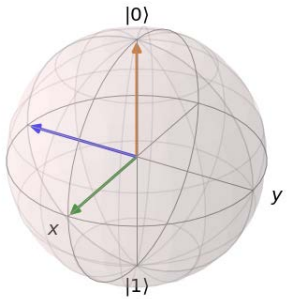
$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$



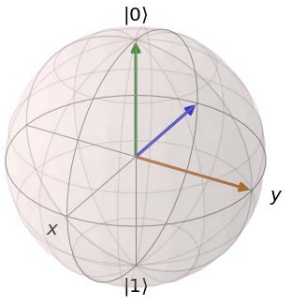
$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



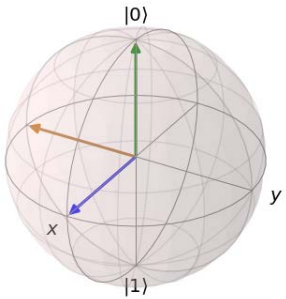
$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$



$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



$$S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$$



$$S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/2} \end{pmatrix}$$

Multiple Qubits

$$\left.\begin{array}{l}|\Phi\rangle \\ |\Psi\rangle\end{array}\right\} |\Psi\rangle \otimes |\Phi\rangle = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} \otimes \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} = \begin{pmatrix}\psi_1 \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \\ \psi_2 \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\psi_1 \cdot \phi_1 \\ \psi_1 \cdot \phi_2 \\ \psi_2 \cdot \phi_1 \\ \psi_2 \cdot \phi_2\end{pmatrix}$$

$$|\Psi\rangle = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} \quad |\Phi\rangle = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$$

Example:

$$|1\rangle \otimes |0\rangle = \begin{pmatrix}0 \\ 1\end{pmatrix} \otimes \begin{pmatrix}1 \\ 0\end{pmatrix} = \begin{pmatrix}0 \\ 0 \\ 1 \\ 0\end{pmatrix}$$

3rd Entry: $|3\rangle$ or $|10\rangle$

***n* Qubits live in a *N* = 2^n-dimensional Hilbert Space:**

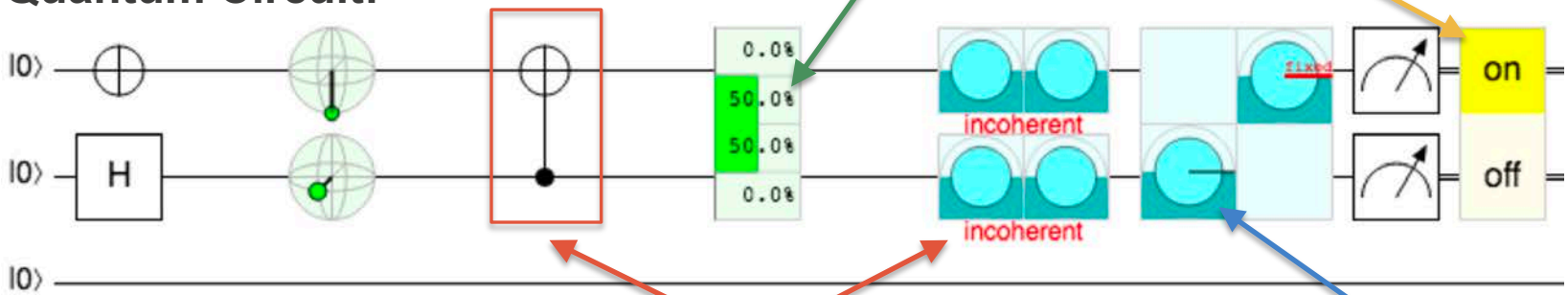
$$\alpha_0|0..00\rangle + \alpha_1|0..01\rangle + \alpha_2|0..10\rangle + \alpha_3|0..11\rangle + \ldots + \alpha_{2^n-1}|1..11\rangle$$

2-Qubit Gates and Entanglement

Probability to measure $|01\rangle$

Measurement: $|01\rangle$

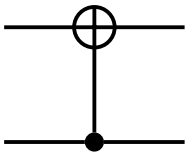
Quantum Circuit:



Entanglement

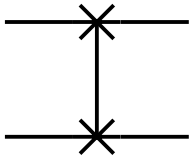
Amplitude $\alpha_3|10\rangle$ with Mag^2 = 0.5, Phase 1

Notable 2-Qubit Gates:



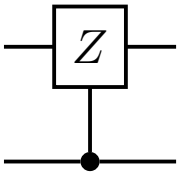
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{array}{lcl} |00\rangle & \rightarrow & |00\rangle \\ |01\rangle & \rightarrow & |01\rangle \\ |10\rangle & \rightarrow & |11\rangle \\ |11\rangle & \rightarrow & |10\rangle \end{array}$$

CNOT



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

SWAP



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Controlled-Z

Algorithms

given by quantum circuits with 1- and 2-qubit gates

Qubits:

- Total Number of Qubits
- Number of Ancillas (“extra qubits”)

Gates:

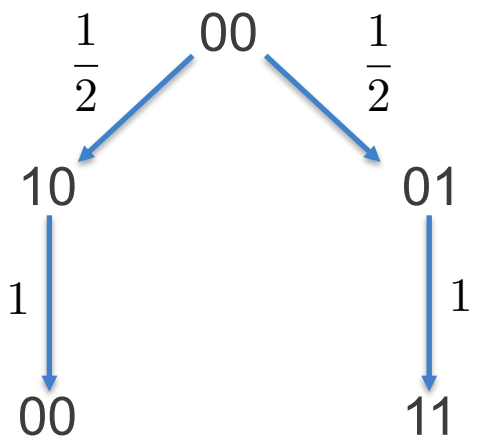
- Total Number of Gates
- Depth of Circuit
- Number of T-Gates

Measurements:

- Total Number of Measurements
- Number of Consecutive Measurements (Error Correction)

Quantum Computing in One Slide

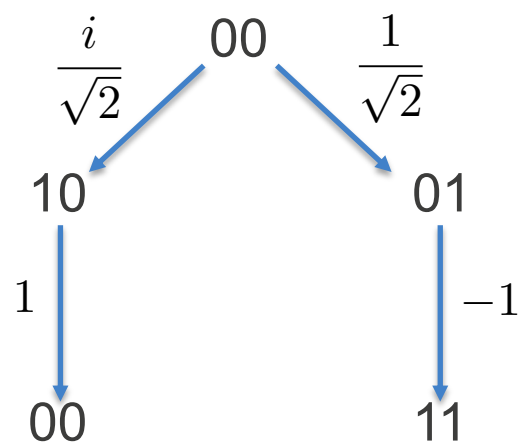
Classical Randomized Algorithm:



$$\Pr[x] = \sum_{\text{paths } p \text{ to } x} \prod_{\text{edges } e \text{ in } p} p_e$$

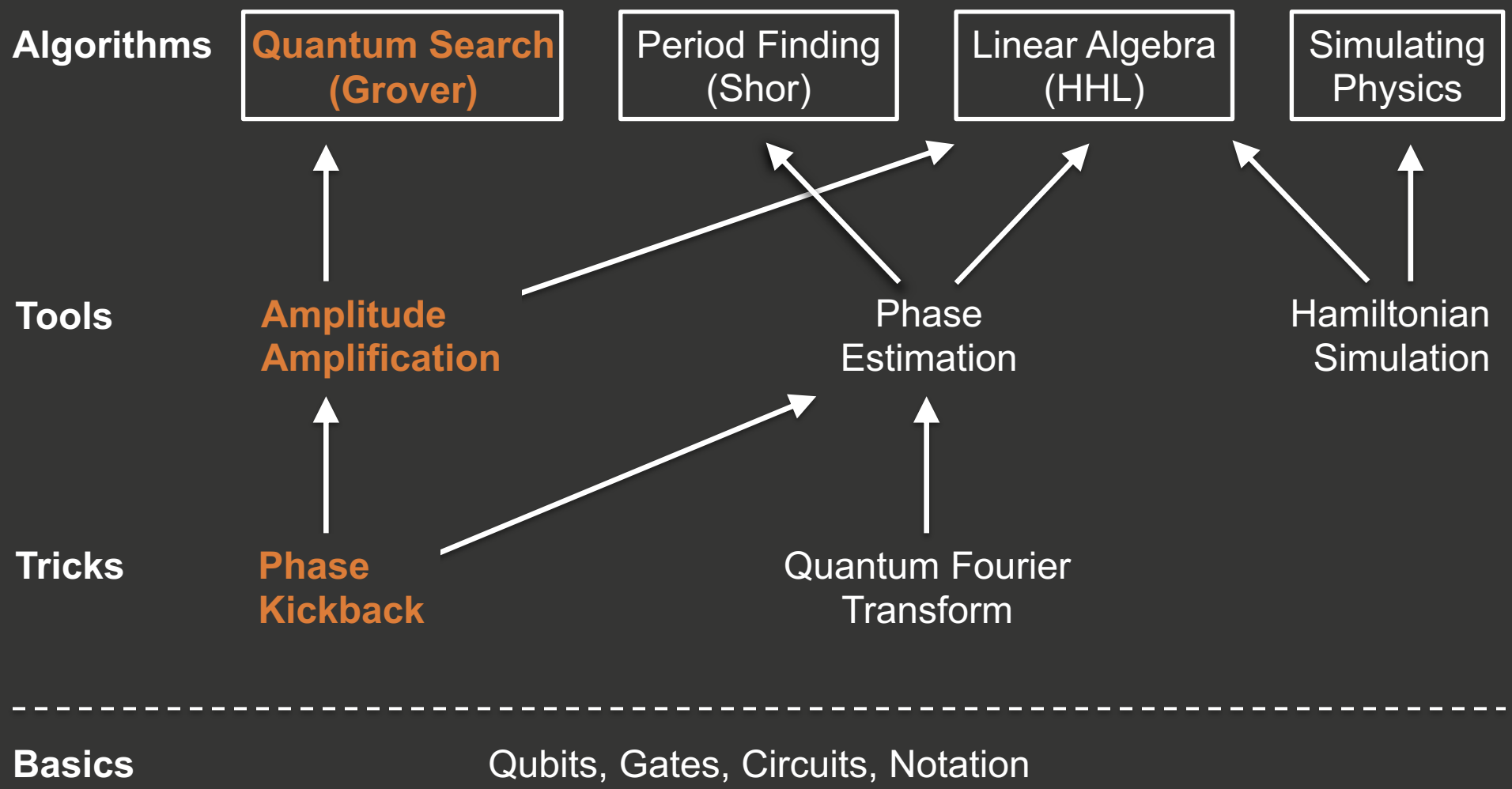
Stochastic Matrices

Quantum Algorithm:



$$\Pr[x] = \left| \sum_{\text{paths } p \text{ to } x} \prod_{\text{edges } e \text{ in } p} \alpha_e \right|^2$$

Unitary Matrices $U^{-1} = U^\dagger$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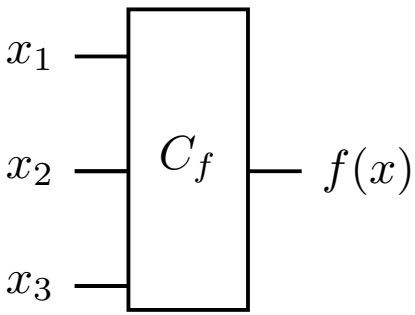
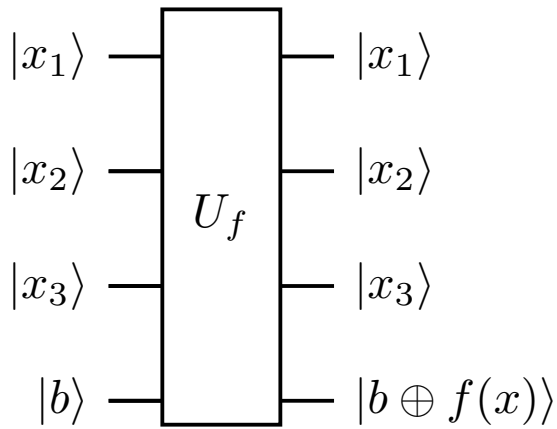
3SAT: hard to solve, easy to verify

$$f(x) = f(x_1, x_2, x_3) = (\overline{x_1}) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2)$$

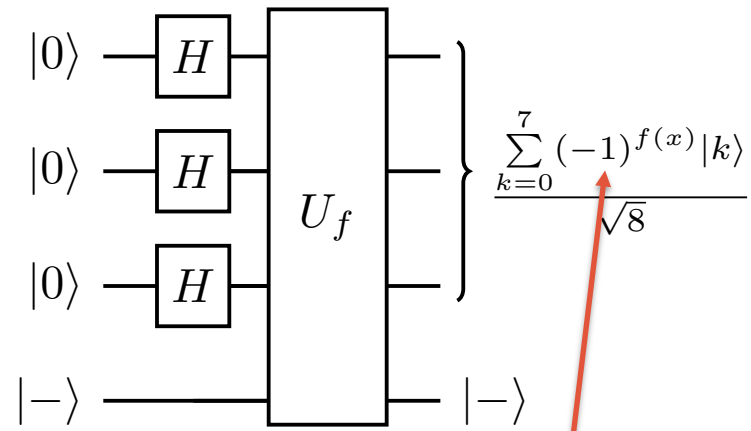
Only one solution: $(x_1, x_2, x_3) = (0, 1, 1)$



Classical Circuit (irreversible)



Quantum Circuit (reversible)



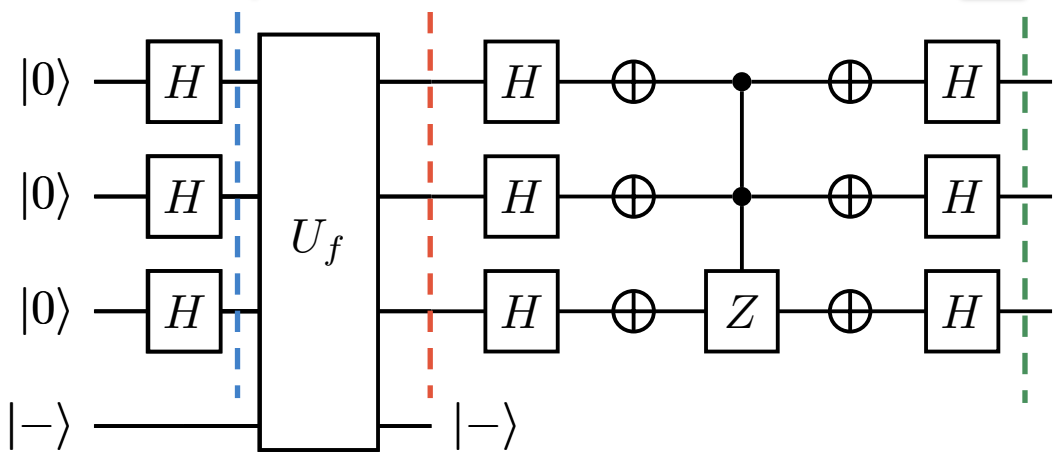
Phase Kickback

$$\frac{|0 \oplus 1\rangle - |1 \oplus 1\rangle}{\sqrt{2}} = \frac{|1\rangle - |0\rangle}{\sqrt{2}} = -\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Amplitude Amplification

Repetitions:

$$\sim \frac{\pi}{4}\sqrt{N}$$

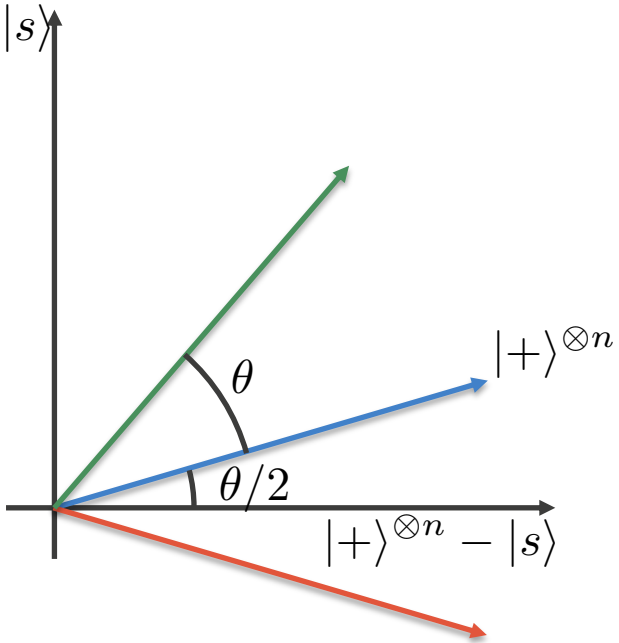


Reflection at $|+\rangle^{\otimes n} - |s\rangle$

Reflection at $|+\rangle^{\otimes n}$

Rotation Angle: $\sin(\theta/2) = \frac{1}{\sqrt{N}}$

$$\Rightarrow \theta \approx \frac{2}{\sqrt{N}}$$



Quantum Search (Grover)

Searching over all Function Inputs with $\mathcal{O}(\sqrt{N})$ Iterations

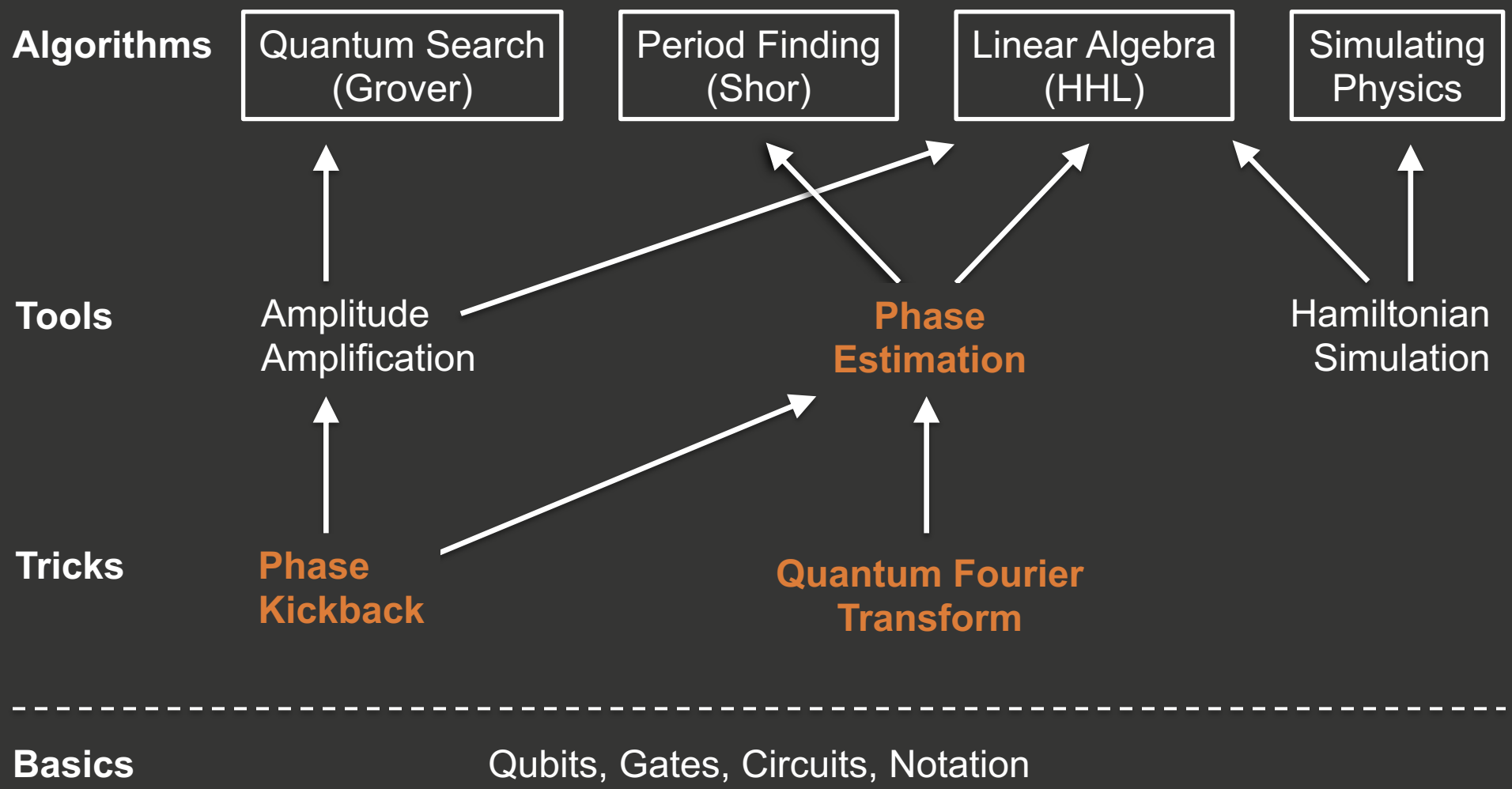
Details

- What when there are M items? $\Rightarrow \mathcal{O}(\sqrt{N/M})$ Iterations
- Grover is known to be optimal (in black-box sense).
- 3SAT Grover $\widetilde{\mathcal{O}}(\sqrt{2^n}) \approx \widetilde{\mathcal{O}}(1.41^n)$
  - vs. Schöning Randomized Alg. with $\approx \widetilde{\mathcal{O}}(1.34^n)$ rounds of success probability $\approx \frac{1}{1.34^n}$
  - Use Amplification $\Rightarrow \widetilde{\mathcal{O}}(\sqrt{1.34^n})$!

Applications

- Database Search? needs Quantum Memory…
- Quantum Speedups for Dynamic Programming
- Distributed QC
  - Sublinear Diameter Computation in CONGEST model

Quantum Speedup: Polynomial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Quantum Fourier Transform

$$QFT: |x\rangle = |x_4 x_3 x_2 x_1\rangle \rightarrow \frac{1}{\sqrt{2^4}} \sum_{k=0}^{2^4-1} e^{2\pi i k x/2^4} |k\rangle$$

$$= \frac{x_2}{2} + \frac{x_1}{4}$$

$$= \frac{1}{\sqrt{2^4}} (|0\rangle + e^{2\pi i 0.x_1}|1\rangle) \otimes (|0\rangle + e^{2\pi i 0.x_2 x_1}|1\rangle)$$

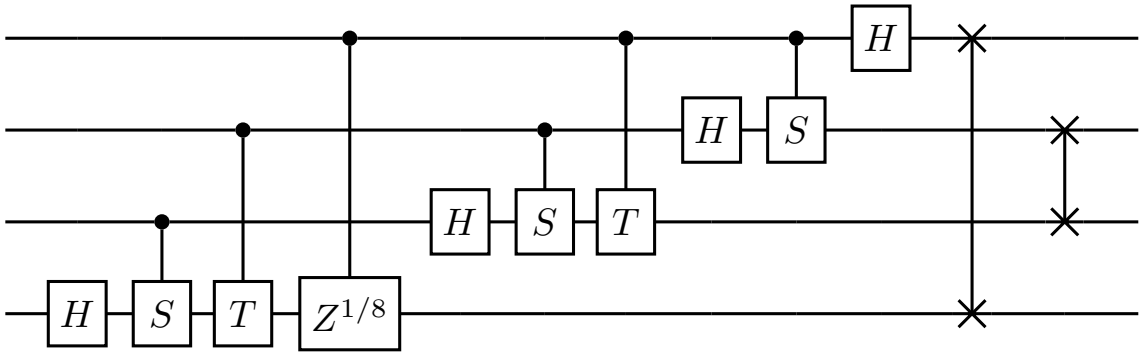
$$\otimes (|0\rangle + e^{2\pi i 0.x_3 x_2 x_1}|1\rangle) \otimes (|0\rangle + e^{2\pi i 0.x_4 x_3 x_2 x_1}|1\rangle)$$

$$\frac{1}{\sqrt{2}} \left(|0\rangle \pm e^{\pi i x_3/2} \cdot e^{\pi i x_2/4} \cdot e^{\pi i x_1/8} |1\rangle \right)$$

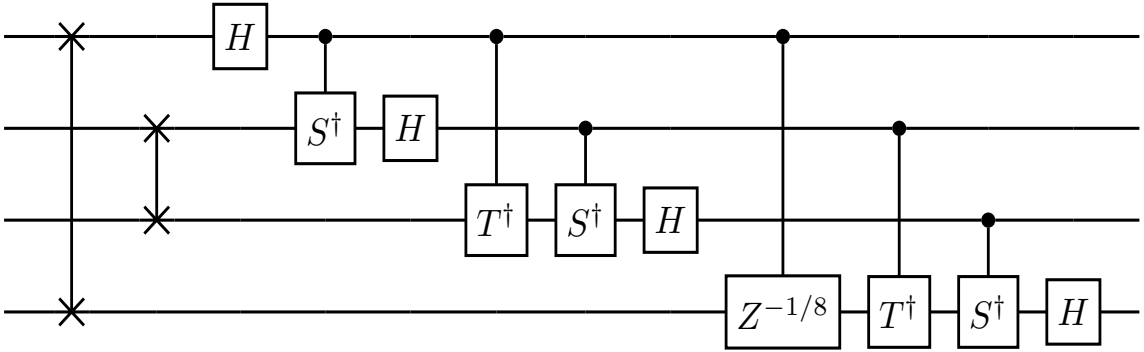
H $\quad S = Z^{1/2}$ $\quad T = Z^{1/4}$ $\quad Z^{1/8}$

QFT Circuits

$QFT:$



$QFT^\dagger:$



Quantum Speedup: Exponential

Quantum Phase Estimation

Finding the phase of a Unitary and its Eigenstate

For a Unitary U and an Eigenstate $|u\rangle$ we always have:

Phase
$\phi \in [0, 1)$

$$U|u\rangle = e^{2\pi i \phi}|u\rangle = e^{2\pi i \, \underline{\underline{0.\phi_n \phi_{n-1} \ldots \phi_2 \phi_1}}}|u\rangle$$

Terms from QFT!

Note that:

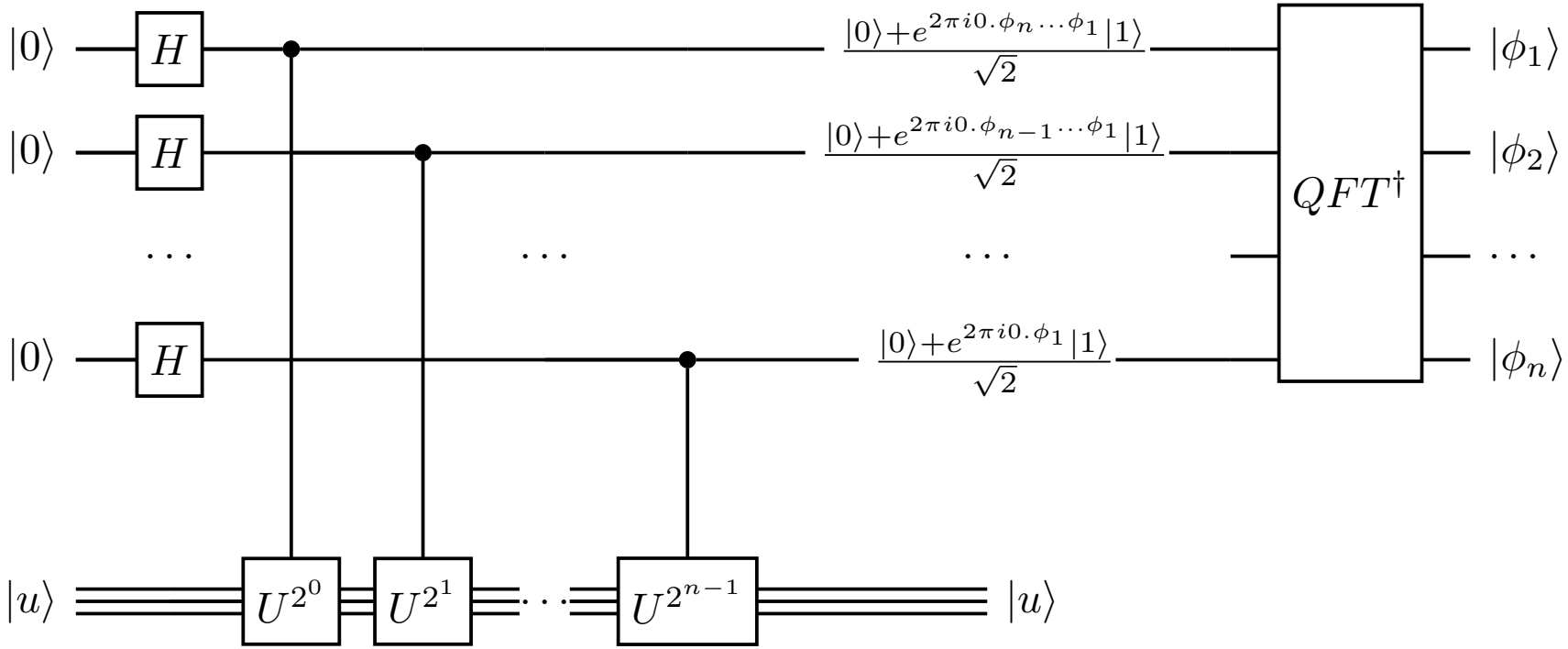
$$U^2|u\rangle = e^{2\pi i \, \phi_n . \phi_{n-1} \ldots \phi_2 \phi_1}|u\rangle = e^{2\pi i \, \underline{0.\phi_{n-1} \ldots \phi_2 \phi_1}}|u\rangle$$

$$\ldots$$

$$U^{2^{n-1}}|u\rangle = e^{2\pi i \, \phi_n \phi_{n-1} \ldots \phi_2 . \phi_1}|u\rangle = e^{2\pi i \, \underline{0.\phi_1}}|u\rangle$$

If only one could kick back the phases into an (ancilla) register and do inverse Quantum Fourier Transform on it…

QPE Circuit


$|0\rangle$
$|0\rangle$
$|0\rangle$
H
H
H
$\frac{|0\rangle+e^{2\pi i 0.\phi_n\cdots\phi_1}|1\rangle}{\sqrt{2}}$
$\frac{|0\rangle+e^{2\pi i 0.\phi_{n-1}\cdots\phi_1}|1\rangle}{\sqrt{2}}$
$\frac{|0\rangle+e^{2\pi i 0.\phi_1}|1\rangle}{\sqrt{2}}$
$QFT^\dagger$
$|\phi_1\rangle$
$|\phi_2\rangle$
$|\phi_n\rangle$
$|u\rangle$
U^{2^0}
U^{2^1}
$U^{2^{n-1}}$
$|u\rangle$

Quantum Computing Algorithms
Algorithms

LLNL CASIS Quantum Sensing and Information Processing Series

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July 31 / August 1, 2019

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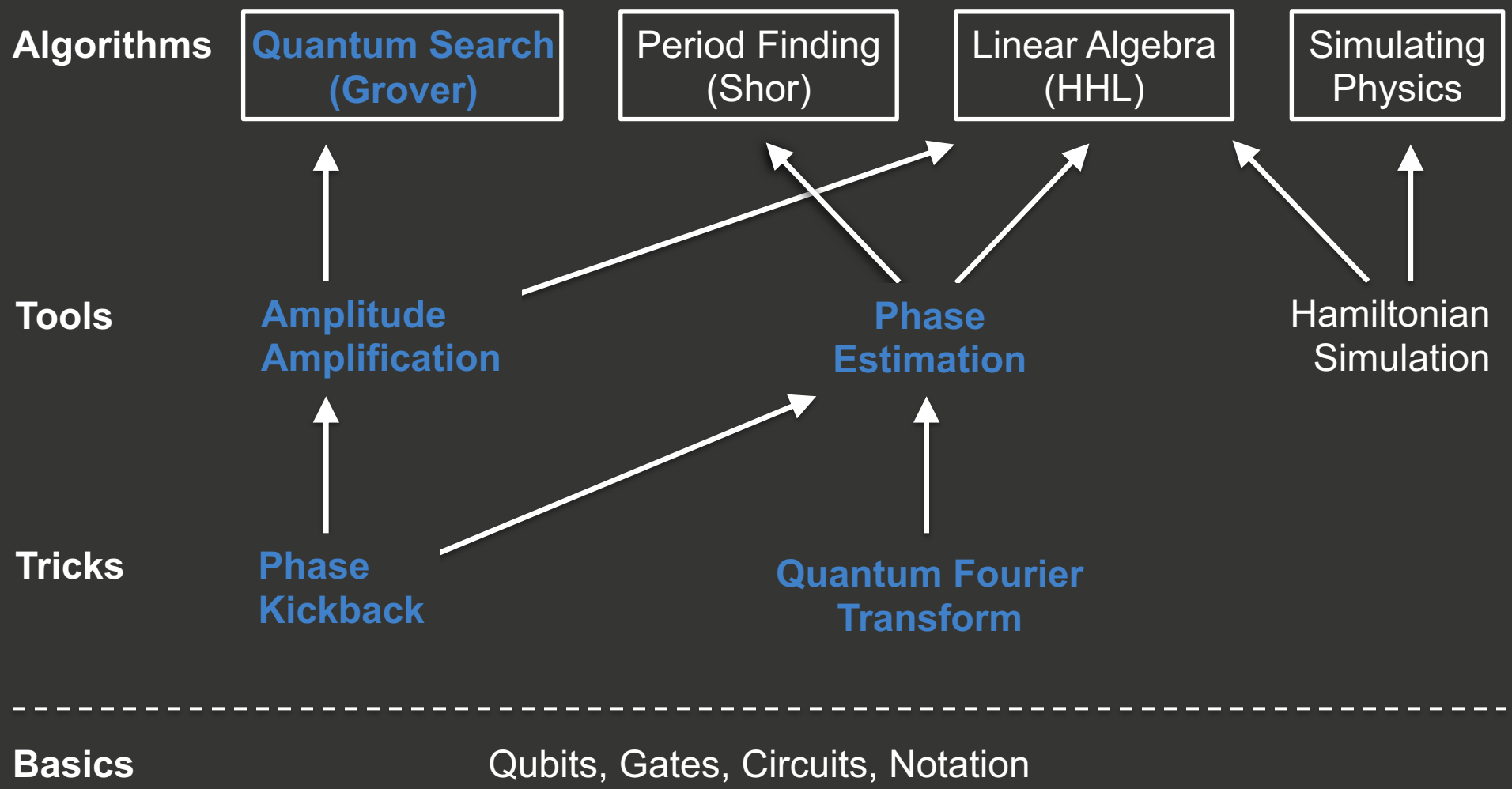
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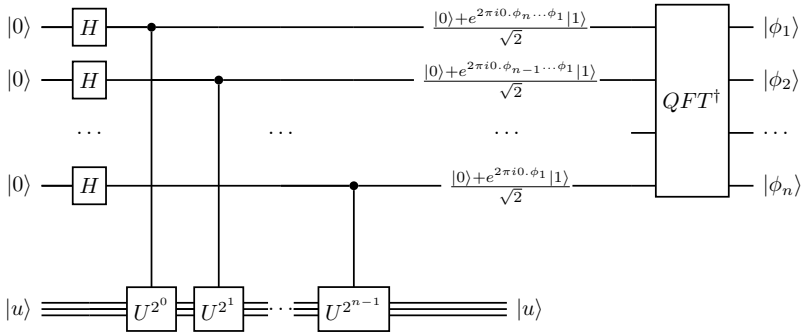
Recap

Amplitude Amplification

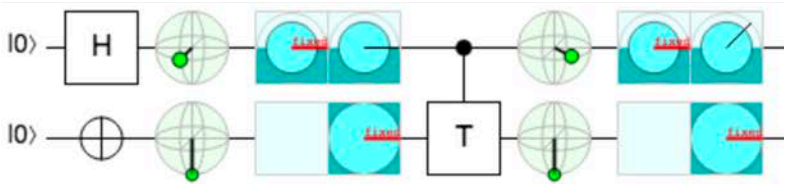
Given algorithm A with success probability *p*, one can do

- **Classical**
  Repeating A for 1/*p* times
- **Quantum**
  Boost amplitudes over √1/p rounds

Phase Estimation



Phase Kickback



Quantum Fourier Transform

QFT

$$\bigotimes_{k=n}^{1} |\phi_k\rangle \quad \bigotimes_{k=1}^{n} \left(|0\rangle + e^{2\pi i\, 0.\phi_k \ldots \phi_1}\, |1\rangle\right)$$

Inverse QFT

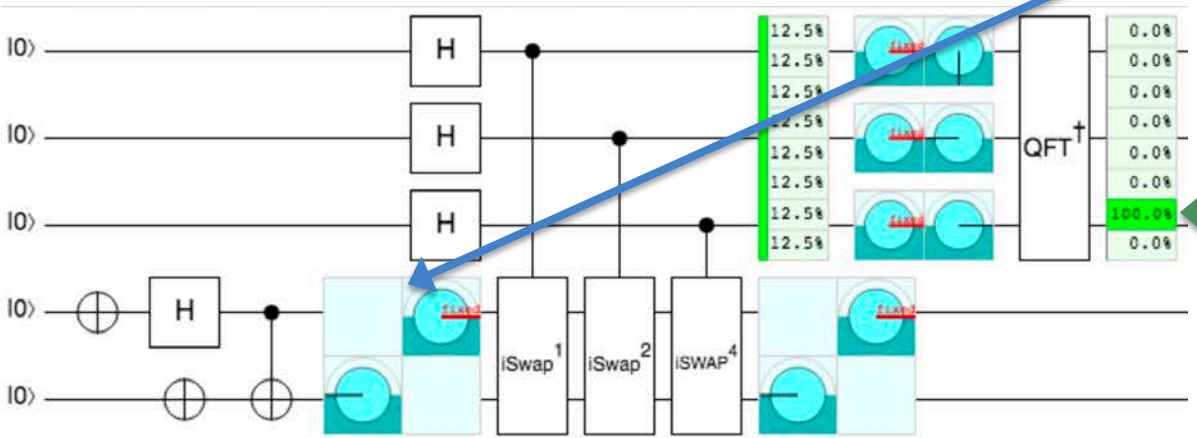
Recap Phase Estimation

Estimating Phases for the iSWAP gate:

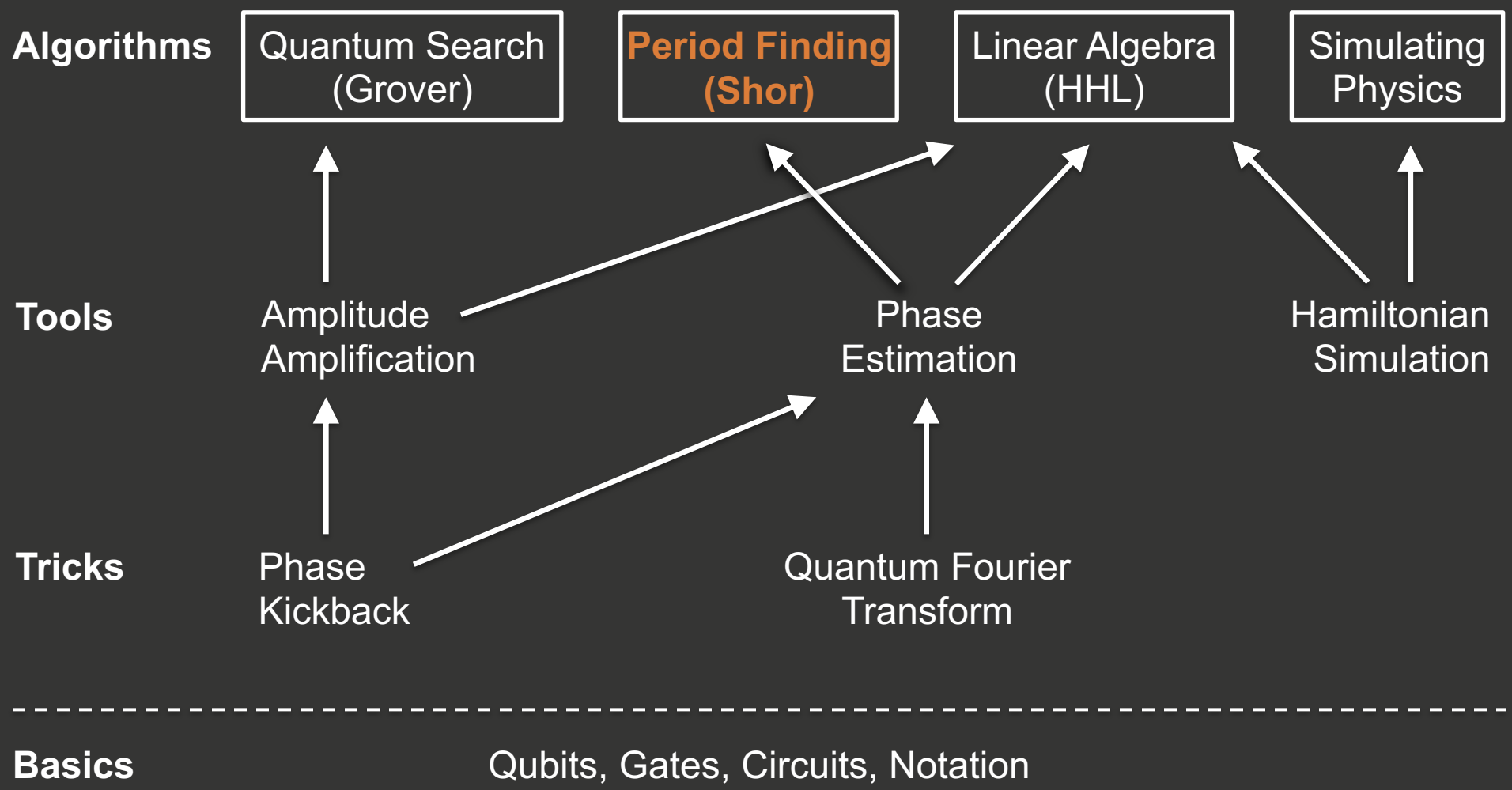
$$\begin{array}{cc} & \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \end{array}$$

With Eigenstates and corresponding Eigenvalues:

$\|00\rangle$	$\frac{\|01\rangle + \|10\rangle}{\sqrt{2}}$	$\frac{\|01\rangle - \|10\rangle}{\sqrt{2}}$	$\|11\rangle$
1	i	$-i$	1



$$e^{2\pi i\, 0.110}$$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Integer Factorization (Shor)

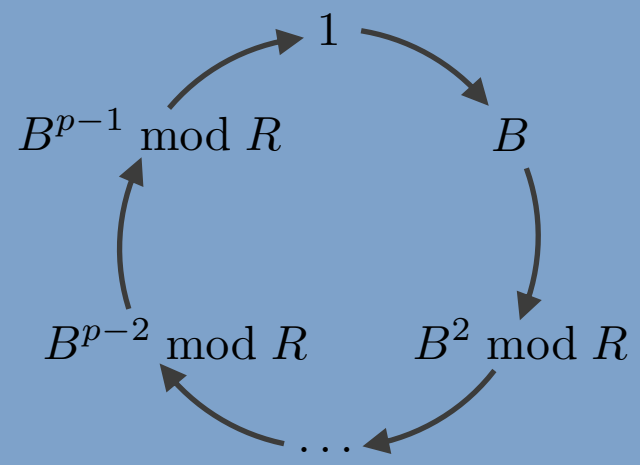
Given *n*-bit (composite) integer *R*, find a proper factor *f*

Classical Part:
Pre- / Post-Processing, Loop

1. Discard "simple" cases:
   R even, *R* prime power, (*R* prime)
2. Randomly choose 0 < *B* < *R*
   - check $f = \gcd(B, R)$
3. Compute period *p* of *B*
4. If *p* is even:
   - check $f = \gcd(B^{p/2} + 1, R)$
5. Repeat if necessary

Quantum Part:
Find Period *p* of

$$B^x \equiv 1 \pmod{R}$$



Period Finding I (Shor)

If *B* has period *p* modulo *R*, the unitary operator $\times B \pmod{R}$ has:

- *p* Eigenstates

$$|u_s\rangle = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} e^{-2\pi i k s/p} \left| B^k \bmod R \right\rangle$$

- with Eigenvalues $e^{2\pi i \underline{s/p}}$ Phase Estimation!

Need phase s/p to determine the period *p*,

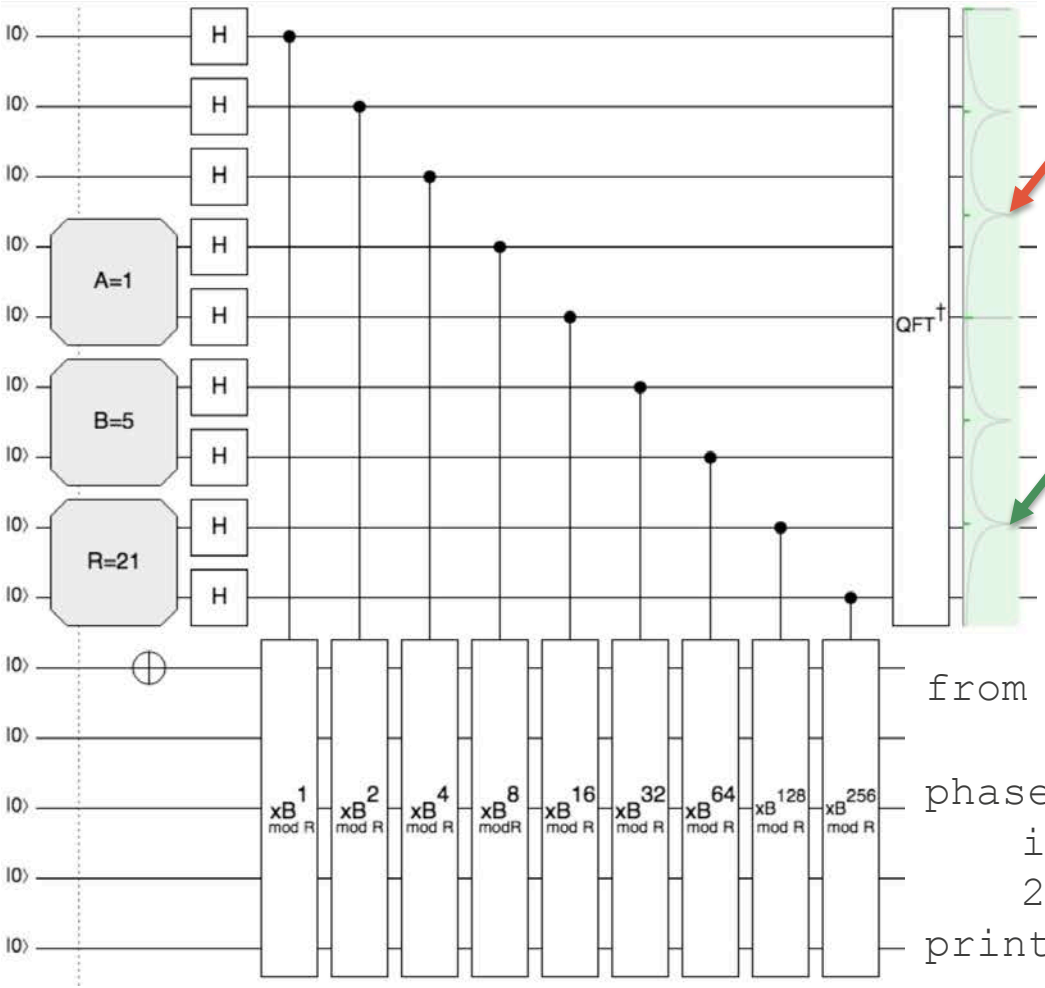
but need the period *p* to create $|u_s\rangle$.

Start in $|1\rangle = \frac{1}{\sqrt{p}} \sum_{s=0}^{p-1} |u_s\rangle$ **!!**

$5^k \bmod 21:$

$$|5\rangle \to |4\rangle \to |20\rangle \to |16\rangle \to |17\rangle \to |1\rangle$$

Period Finding II (Shor)



Phase $s/p \approx 2/6 = 1/3$
gives incorrect period
⇒ fail ⇒ repeat

Phase $427/512 \approx 5/6$

Use continued fractions
to find the “real fraction”

```
from fractions import Fraction

phase = Fraction(
    int(“110101011”, 2),
    2**9)
print(phase.limit_denominator(21))
```

Extra square roots (Shor)

If *R* is composite and not a prime power,
the number 1 has at least four square roots modulo *R* (not only 1, -1):

$$B^p \equiv 1 \pmod{R}$$
$$B^{p/2} \cdot B^{p/2} \equiv 1 \pmod{R}$$

$$(B^{p/2} - 1) \cdot (B^{p/2} + 1) \equiv 0 \pmod{R}$$

Want / hope for: $B^{p/2} + 1 \not\equiv 0 \pmod{R}$

Integer Factorization (Shor)

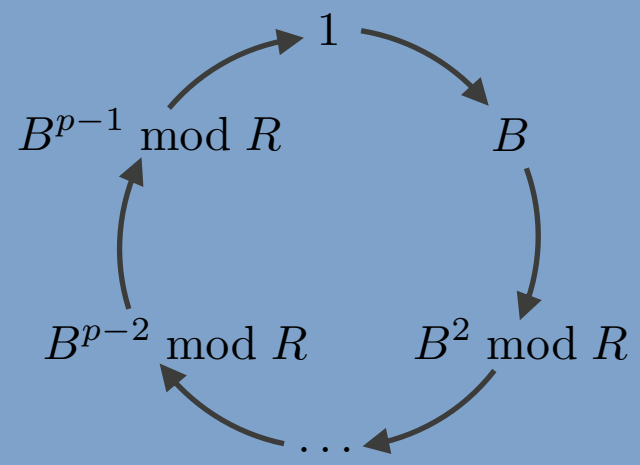
Given *n*-bit integer *R*, find a proper factor *f*

Classical Part:
Pre- / Post-Processing, Loop

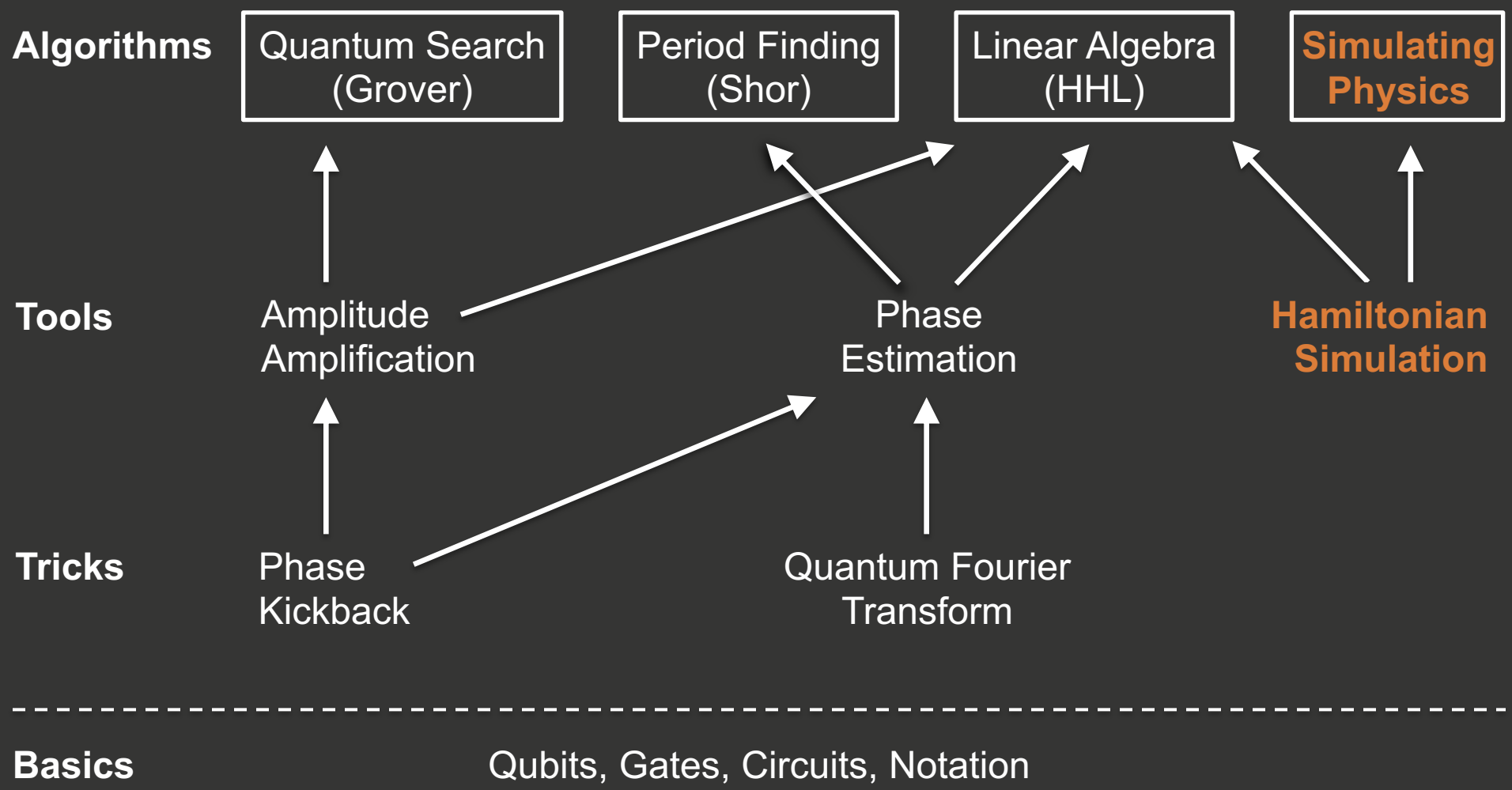
1. Discard "simple" cases:
   R even, *R* prime, *R* prime power
2. Randomly choose 0 < *B* < *R*
   - check $f = \gcd(B, R)$
3. Compute period *p* of *B*
4. If *p* is even:
   - check $f = \gcd(B^{p/2} + 1, R)$
5. Repeat if necessary (~ prob. 0.75)

Quantum Part:
Find Period *p* of

$$B^x \equiv 1 \pmod{R}$$



Everything $poly(n)$ vs. classical $e^{\sqrt[3]{n}}$: **Quantum Speedup: Superpolynomial**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Hamiltonian Simulation
Problem Definition

Schrödinger Equation:

$$i\frac{d\,|\Psi(t)\rangle}{dt} = H\,|\Psi(t)\rangle$$

Time-independent Hamiltonian:

$$|\Psi(t)\rangle = e^{-itH}\,|\Psi(0)\rangle$$

Task:

Find a quantum circuit, that simulates e^{-itH} as close as possible!

Hamiltonian Simulation

Two notable Problem Types

From Physics

- *H* is a sum of Pauli Tensor Product terms:

$$H = \sum_{j=1}^{m} H_j$$

- with, e.g., acting on qubits 3, 2, 1, respectively.

$$H_j = Z_3 \otimes Y_2 \otimes X_1$$

- Structure from Physics through Jordan-Wigner transform.

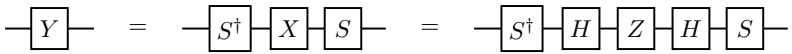
From Maths & CS

- *H* is a *s*-sparse (at most *s* non-zero entries per row and per column)
- Given row *i* and number *x*, there must be a fast way ("Oracle") to get
  - the index *j* and
  - the matrix entry $H_{i,j}$

  of xth non-zero entry in row *i.*

Pauli Tensor Products
can be simulated exactly by **Diagonalization**

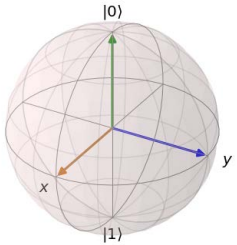
Axis Change



$$Y = S \cdot H \cdot Z \cdot H \cdot S^\dagger$$

$$\Rightarrow e^{-itY} = e^{-it \cdot S \cdot H \cdot Z \cdot H \cdot S^\dagger}$$

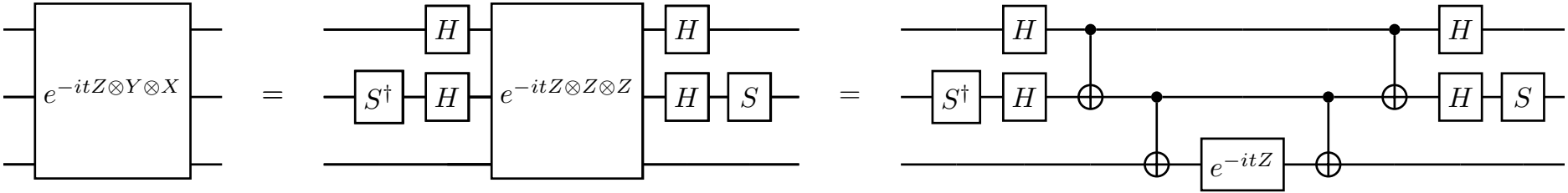
$$= S \cdot H \cdot e^{-it \cdot Z} \cdot H \cdot S^\dagger$$



Parity

$$Z \otimes Z \otimes Z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Diagonalization & Break down



Sum of (non-commuting) H_j

$$H = \sum_j H_j, \quad \text{e.g.} \quad H = X + Z$$

Problem

If the H_j do not commute,

$$X \cdot Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = Z \cdot X$$

then the exponential does not factor:

$$e^{-it\cdot(X+Z)} \neq e^{-itX} \cdot e^{-itZ}$$

Suzuki-Trotter

Small alternating H_j steps

$$H = \sum_{j=1}^{m} H_j = X + Z$$

Approximate by:

$$U_1(t,r) = \left(\prod_{j=1}^{m} e^{-i\frac{t}{r}H_j} \right)^r$$

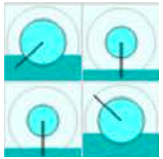
$$U_2(t,r) = \left(\prod_{j=1}^{m} e^{-i\frac{t}{2r}H_j} \prod_{j=m}^{1} e^{-i\frac{t}{2r}H_j} \right)^r$$

$$U_{2k}(t,r) = \left([U_{2k-2}(tp_k,r)]^2 U_{2k-2}((1-4p_k),r)[U_{2k-2}(tp_k,r)]^2 \right)^r$$

$$\text{with } p_k = 1/(4 - 4^{1/(2k-1)})$$

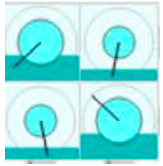
Example, $t = \pi/2$:

- Exact: numpy



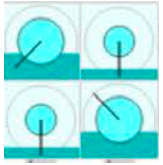
$e^{-i\pi/2(X+Z)}$

- Trotter: 16 gates



$U_1(\pi/2, 8)$

- Suzuki: 9 gates



$U_2(\pi/2, 4)$

Hamiltonian Simulation Results

Number of "easy" exponentials for error $\leq \varepsilon$ and time *t*

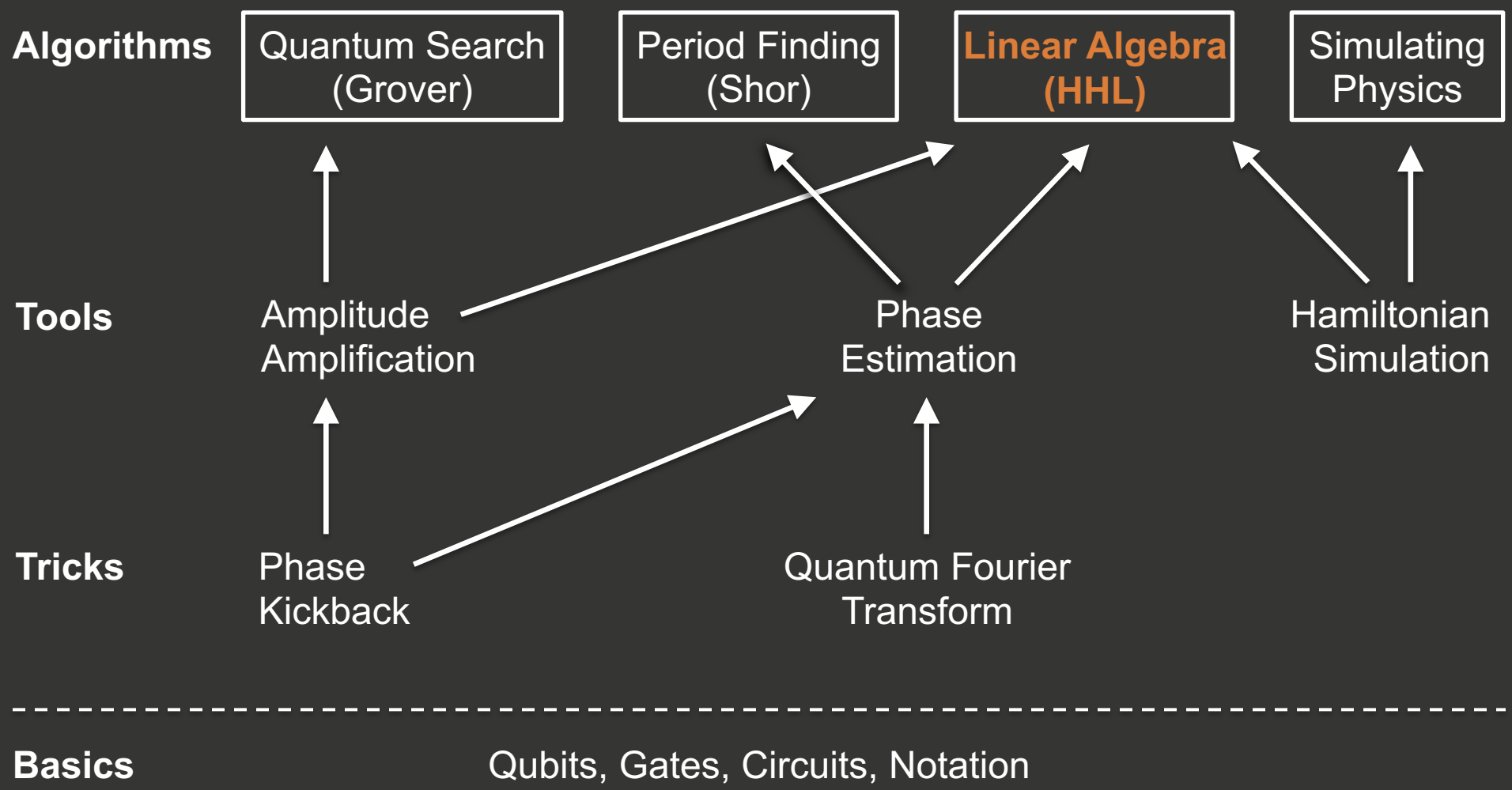
- From Physics: Sum of Pauli Tensor Product terms

$$N_{exp} \leq 5^{2k} 2m (m \cdot t \cdot ||H||)^{1+1/2k} / \varepsilon^{1/2k} \approx \widetilde{\mathcal{O}}(m^2 \cdot t \cdot ||H||)$$

- From Mathematics:
  Decompose *s*-sparse *H* into Sum of $m = 6s^2$ 1-sparse Hamiltonians, then use techniques from above:

$$N_{exp} \approx \widetilde{\mathcal{O}}(s^4 \cdot t \cdot ||H||), \qquad \text{plus } \widetilde{\mathcal{O}}(\log N s^4 \cdot t \cdot ||H||) \text{ extra gates.}$$

Quantum Speedup: Exponential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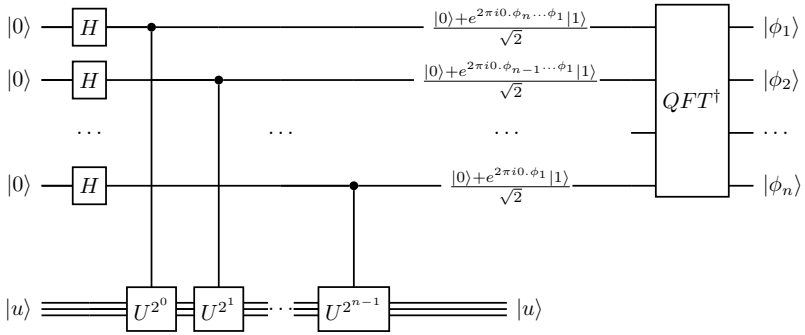
Recall Recap

Amplitude Amplification

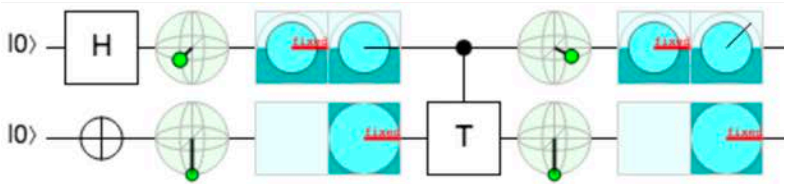
Given algorithm A with success probability *p*, one can do

- **Classical**
  Repeating A for 1/*p* times
- **Quantum**
  Boost amplitudes over √1/p rounds

Phase Estimation



Phase Kickback



Quantum Fourier Transform

QFT

$$\bigotimes_{k=n}^{1} |\phi_k\rangle \quad \bigotimes_{k=1}^{n} \left(|0\rangle + e^{2\pi i\, 0.\phi_k \ldots \phi_1}\, |1\rangle\right)$$

Inverse QFT

Linear System Problems

$$A \cdot \vec{x} = \vec{b}$$

Classical and Quantum (Harrow, Hassidim, Lloyd)

Classical problem

Conjugate gradient method: $\mathcal{O}(Ns\sqrt{\kappa}\log(1/\varepsilon))$

NxN Matrix *A*:

- Condition number κ
- Sparsity *s*
- *A* positive semidefinit

Solves $\vec{x} = A^{-1} \cdot \vec{b}$

Quantum problem

HHL: $\mathcal{O}(\kappa T_B + \log(N) T_A s^4 \kappa^2/\varepsilon)$

Additionally:

- Eigenvalues in [1/κ, 1]
- $|b\rangle = \vec{b}/||\vec{b}||$ preparable in time T_B
- Oracle access to *A* in time T_A
- *A* Hermitian, $A = A^\dagger$
- **no interest in all entries of $\vec{x}$**

Solves $|x\rangle = \dfrac{A^{-1}|b\rangle}{||A^{-1}|b\rangle||}$

Quantum Speedup: depends…

Quantum Linear System Problem (HHL)

Example

$$A \cdot \vec{x} = \vec{b}: \qquad \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\vec{x} = A^{-1} \cdot \vec{b} = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$|x\rangle = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Eigenstates	**Eigenvalue**
$\lvert+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	**1**
$\lvert-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	**0.5**

$$|b\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

Quantum Linear System Problem (HHL)
Helpful Facts

Let *A* have Eigenstates $|u_j\rangle$

with Eigenvalues $\lambda_j \in [1/\kappa, 1]$:

- We can write |b⟩ as
  $\sum \beta_j |u_j\rangle$
- We can write |x⟩ as
  $\approx \sum \frac{\beta_j}{\lambda_j} |u_j\rangle$

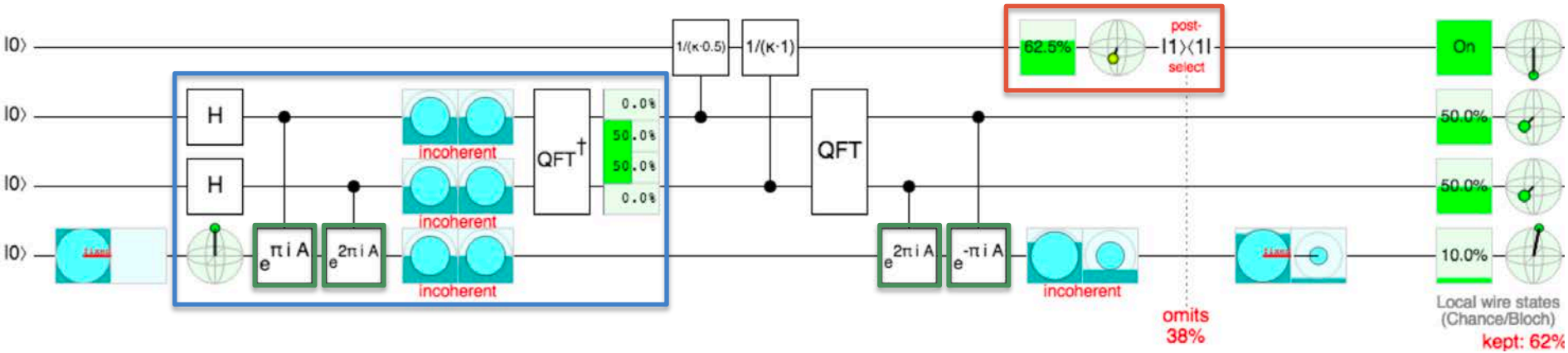
Need to find the Eigenvalues!

- $e^{it\cdot A}$ has Eigenvalues $e^{it\cdot\lambda_j}$
  in particular
  $$e^{i\pi A}|u_j\rangle = e^{i\pi\lambda_j}|u_j\rangle = e^{2\pi i\cdot\lambda_j/2}|u_j\rangle$$
  has a phase of
  $$\lambda_j/2 \in [1/(2\kappa), 1/2] \subset [0, 1)$$

Using Hamiltonian Simulation we can do Phase Estimation!

Quantum Linear System Problem (HHL)

Hamiltonian Simulation | Phase Estimation | Amplitude Amplification

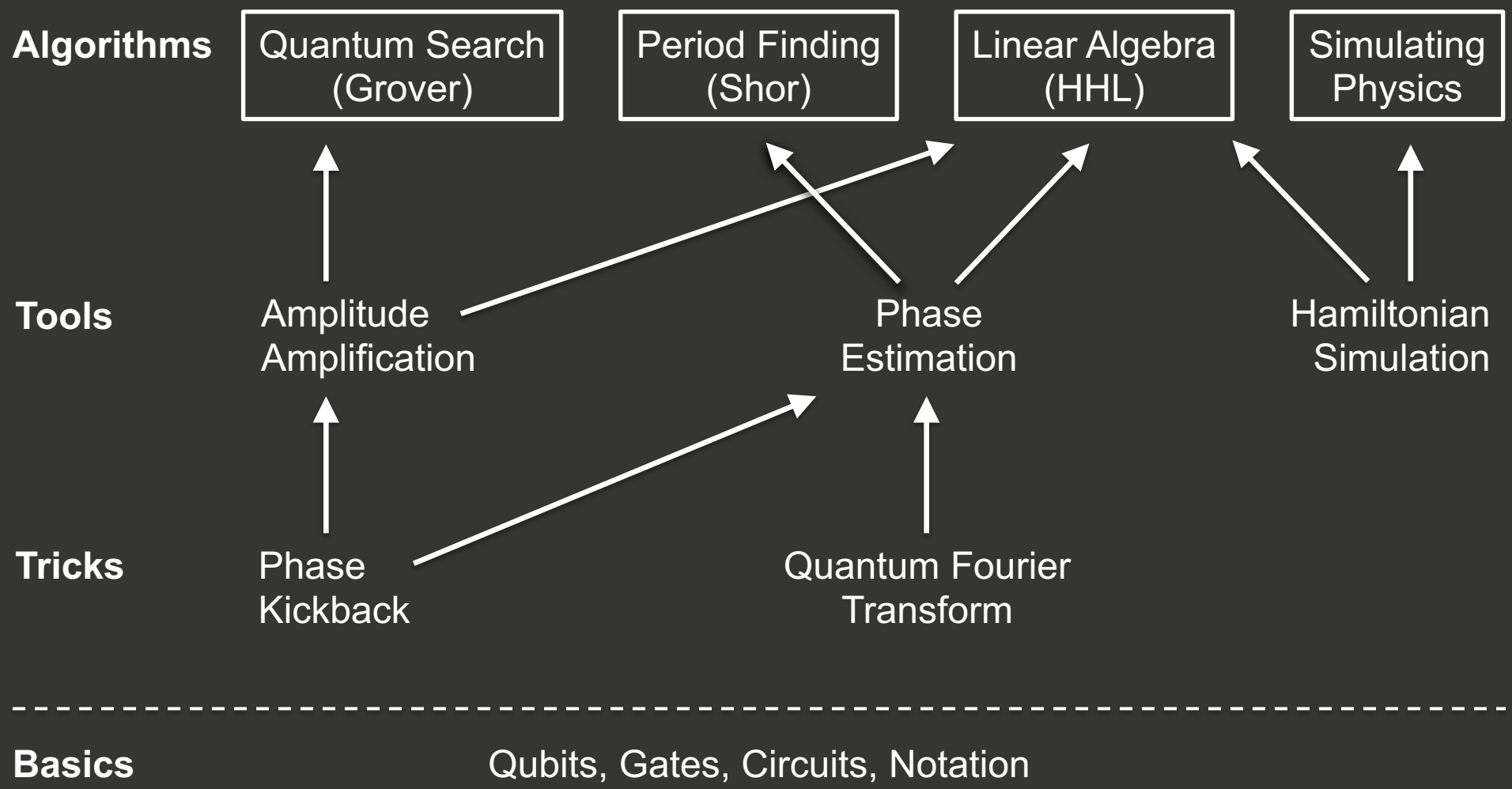


$$|b\rangle = \sum \beta_j |u_j\rangle |0\rangle_r |0\rangle_a \rightarrow \sum \beta_j |u_j\rangle |\lambda_j\rangle_r |0\rangle_a \qquad \text{Phase Estimation}$$

$$\rightarrow \sum \beta_j |u_j\rangle |\lambda_j\rangle_r \left(\sqrt{1-\frac{1}{\kappa^2\lambda_j^2}}\,|0\rangle + \frac{1}{\kappa\lambda_j}|1\rangle\right)_a \qquad \text{Conditional Rotation}$$

$$\rightarrow \sum \beta_j |u_j\rangle |0\rangle_r \left(\sqrt{1-\frac{1}{\kappa^2\lambda_j^2}}\,|0\rangle + \frac{1}{\kappa\lambda_j}|1\rangle\right)_a \qquad \text{uncompute}$$

$$\rightarrow \sqrt{\frac{\kappa^2}{\sum \beta_j^2/\lambda_j^2}} \sum \frac{\beta_j}{\lambda_j} |u_j\rangle |0\rangle_r |1\rangle_a \approx \sum \frac{\beta_j}{\lambda_j} |u_j\rangle \qquad \text{if we measured } |1\rangle_a$$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